Effects of additive noise on a nonlinear oscillator exhibiting period doubling and chaotic behavior

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We report detailed effects of additive random noise on a driven nonlinear oscillator in the periodic, chaotic, and window regimes. We observe simultaneously the power spectral density, probability density, and bifurcation diagram at varying noise levels, finding semiquantitative agreement with the logistic model and universal predictions.

A previous paper,[1] denoted by I, reported observation of universal chaotic behavior for a driven nonlinear semiconductor oscillator which shows successive period-doubling bifurcations as a route to chaos.[2] Semiquantitative agreement was observed for several universal numbers computed from simple nonlinear finite difference equations of the Feigenbaum universality class, which includes the logistic equation

$$x_{n+1}=\lambda x_n(1-x_n)\,. \tag{1}$$

The bifurcation diagram of Fig. 1 is a plot of the iterated values $\{x_n\}$ versus the control parameter λ, showing period doubling, onset of chaos at λ_c, band merging, and periodic windows. The chaos is entirely a result of computations from Eq. (1) and is describable as deterministic chaos. Since real physical systems contain fluctuations of a stochastic nature, e.g., thermal and electrical noise, it is of considerable interest to study the effect of added random noise to Eq. (1). This has been reported by several authors[3–7] who compute the effect of noise on the stability and observability of period doubling, on windows, on single-band chaos, and on the Lyapunov exponent. The purpose of this paper is to report the first detailed measurements of added noise to a chaotic physical system following the period-doubling route.[8] We find for the oscillator of I close agreement with the logistic model and universal predictions.

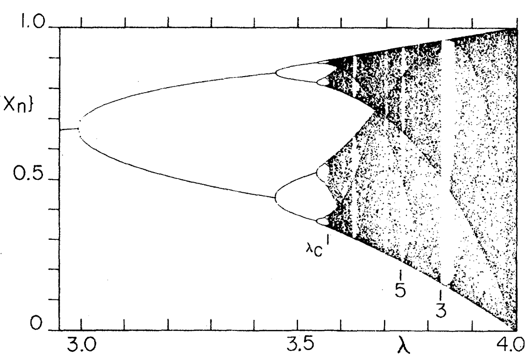

FIG. 1. Computed bifurcation diagram $\{x_n\}$ vs λ for Eq. (1), showing successive bifurcations, chaos at λ_c, and periodic windows 5 and 3.

The theoretical models[3–7] add to Eq. (1) a term p_n and iterate by computer the equation

$$x_{n+1}=\lambda x_n(1-x_n)+p_n\,, \tag{2}$$

where p_n is either a Gaussian or a uniform pseudorandom distribution with standard deviation σ and mean value zero. In our experiments a random noise voltage is added parametrically to the driving voltage, for which the equation

$$x_{n+1}=(\lambda+q_n)x_n(1-x_n) \tag{3}$$

is more appropriate. Here q_n is a random variable with standard deviation σ_q and mean value zero representing the added noise voltage. Crutchfield, Farmer, and Huberman[9] derive a relationship between σ and σ_q, our experimentally measured noise parameter. The computations are displayed for various values of σ as: Plots of a power spectral density $S(f)$ (Refs. 3 and 7); plots of the probability density $P(\{x_n\})=P(x)$, which is a vertical section through the bifurcation diagram at constant λ (Refs. 6 and 7); and noisy bifurcation diagrams.[6] We review the predicted behavior of $P(x)$ for the three cases which we experimentally investigate. (i) For $\sigma=0$ and $\lambda_c > \lambda=$ value midway into period 2^k, $P(x)$ consist of 2^k singularities. As the noise is increased to value σ, the singularities broaden and merge first to 2^{k-1} peaks, with $P(x)=0$ between the peaks. The system has become *semiperiodic*.[7,10] If the noise is then further increased to $\kappa\sigma$, the 2^{k-1} peaks merge to 2^{k-2} peaks with $P(x)=0$ between peaks, where $\kappa=6.619\ldots$ is a universal number, first computed by Crutchfield, Nauenberg, and Rudnick.[4] This process continues as σ is further increased until all peaks merge to one band and the system becomes *aperiodic*. (ii) For $\sigma=0$ and $\lambda_c < \lambda=$ value for a window of period q, $P(x)$ consist of q singularities. As σ is increased, the singularities broaden very slightly before $P(x)$ is filled in between all peaks, i.e., $P(x) \neq 0$ throughout its domain, and the system is *aperiodic*. (iii) For $\sigma = 0$ and $\lambda_c < \lambda =$ value for a one-band attractor, $P(x)$ consists of a high base line with structures and singularities corresponding to mappings of the critical point $x_c = \frac{1}{2}$. As σ is increased, the singularities disappear but the gross features remain.

Our experimental arrangement, detailed in I, consists of a series inductance, resistance, capacitance (LRC) circuit, the capacitance C being a nonlinear varactor diode. This circuit is driven by the linear superposition of two voltage sources: $V_0\cos(2\pi f t)$ from a driving oscillator; and a wide-band random noise source of rms value V_n; V_0 and V_n are controlled by precision attenuators. In taking data, f is fixed at $f_{res} \approx 96$ kHz, and V_0 [corresponding to λ in Eq. (1)] is set to the desired point in the bifurcation diagram. The varactor diode voltage $V_c(t)$ (corresponding to $\{x_n\}$) is observed by a power spectrum analyzer, yielding $S(f)$; by a pulse-height analyzer which measures the probability density $P(V_c)$, corresponding to $P(x)$; and by a bifurcation spectrometer, which plots $\{V_c\}$ vs V_0, analogous to Fig. 1. During the conducting half cycle, V_c is clamped toward the zero line; on the reverse half cycle, V_c has a set of values which correspond to the top half of the bifurcation diagram and to the top half of the probability density $P(x)$. Data are taken for various values of noise voltage V_n, and recorded as $\sigma_m = \sigma_q/8 = V_n/8V_{cm}$, which is comparable to the value of σ assumed in computed predictions. Here V_{cm} is the maximum varactor voltage at V_0 required to reach $\lambda \gtrsim 3.8$, corresponding to a range of order unity for $\{x_n\}$ on the bifurcation diagram.

Figure 2 shows $\log_{10}P(x)$ observed for the nonlinear oscillator driven into subharmonic bifurcation at $f/16$. As noise is added, the behavior is close to that predicted. In Fig. 2(a), $\sigma_m = 0$; the system is essentially periodic with sharp peaks as expected; it has a small random noise from the driving oscillator and the diode current which limits the observable bifurcation to $f/32$ and prevents the peaks from being singularities. In Fig. 2(b), $\sigma_m = 1.4 \times 10^{-4}$; the system has semiperiodicity 8. In Fig. 2(c), $\sigma_m = 8.7 \times 10^{-4}$; increasing the noise voltage by the factor 6.3 reduces the semiperiodicity to 4. Figure 2(d)

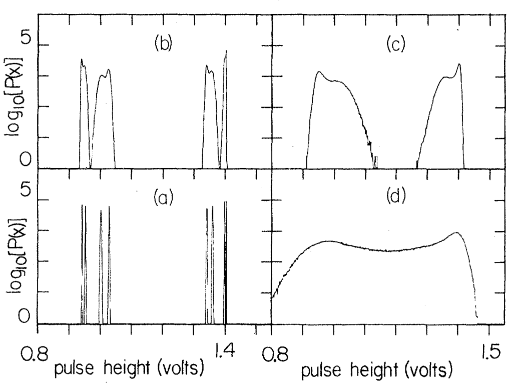

FIG. 2. Observed $\log_{10}P(V_v)$ vs V_c for varactor voltage V_c of nonlinear oscillator, bifurcated to subharmonic $f/16$: (a) added noise voltage $\sigma_m = 0$; (b) $\sigma_m = 1.4 \times 10^{-4}$; (c) $\sigma_m = 8.7 \times 10^{-4}$; (d) $\sigma_m = 5.5 \times 10^{-3}$.

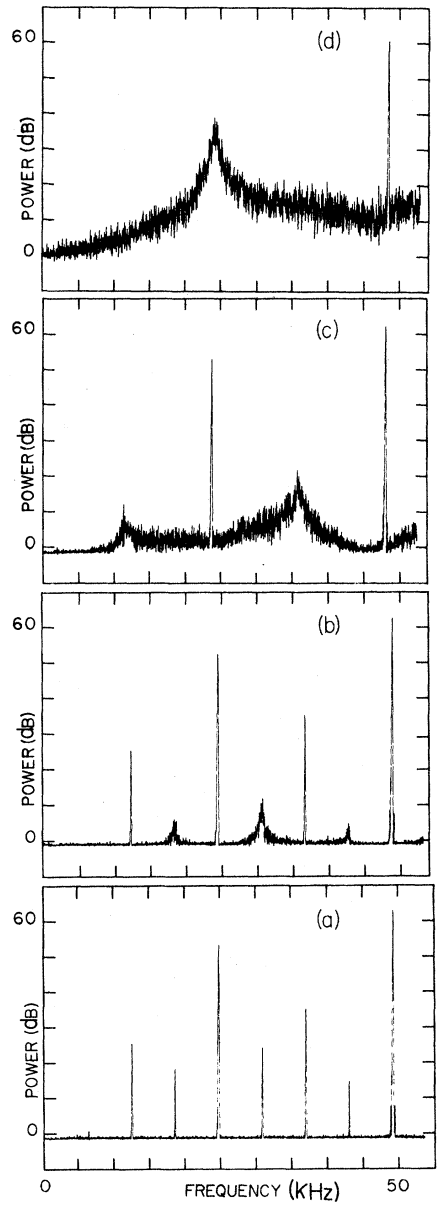

FIG. 3. Observed power spectral density $S(f)$ under same conditions as Fig. 2, with driving frequency $f = 96$ kHz: (a) no added noise, sharp subharmonics $f/2$ to $f/16$ displayed; (b) added noise $\sigma_m = 1.4 \times 10^{-4}$, removes sharp $f/16$ components; (c) $\sigma_m = 8.7 \times 10^{-4}$, removes sharp $f/8$ components; (d) $\sigma_m = 5.5 \times 10^{-3}$, removes sharp $f/4$ component.

FIG. 4. Observed upper branch of bifurcation diagram, $\{V_c\}$ vertical vs V_0 horizontal (cf. Fig. 1, $3.44 < \lambda < 3.6$): (a) added noise $\sigma_m = 1.4 \times 10^{-4}$; (b) $\sigma_m = 8.7 \times 10^{-4}$.

shows that another increase in σ_m by a factor of 6.3 reduces the semiperiodicity to 2. These features are similar to Fig. 20 of Ref. 6. Figures 3(a)–3(d) show the observed power spectral density $S(f)$ measured simultaneously under the same conditions as Figs. 2(a)–2(d). It is clear that successive factors of 6.3 in added noise voltage eliminate the sharp spectral components, reducing the period from $8 \rightarrow 4 \rightarrow 2$. A series of 15 measurements like those of Figs. 2 and 3 yield the average value $\kappa = 6.44 \pm 0.24$. Figure 4(a) shows the observed bifurcation diagram with added noise to reduce it to semiperiodicity 8. In Fig. 4(b) the noise voltage is increased by a factor 6.3, reducing the semiperiodicity to 4. These data confirm the computed diagram, Fig. 7 of Ref. 6.

To test prediction (ii), Fig. 5(a) shows $\log_{10}P(x)$ measured for $\sigma_m = 0$ and V_0 set just after the beginning of the period-5 window; sharp peaks are observed. In Fig. 5(b) the addition of small noise $\sigma_m = 1.1 \times 10^{-4}$ raises $P(x) \neq 0$ at all points in the domain of x: The system has become aperiodic. The data have a close resemblance to Figs. 8 and 9 of Meyer-Kress and Haken,[7] computed for the period-3 window. To test prediction (iii), Fig. 5(c) shows $\log_{10}P(x)$ observed at a one-band attractor at $\lambda \approx 3.7$ without added noise. Addition of noise, $\sigma_m = 1 \times 10^{-3}$, in Fig. 5(d), washes out the peaks but does not change the gross features. These results are quite similar to Figs. 11 and 12 of Ref. 7.

We have made preliminary observations of the effect of adding a sinusoidal rather than a random noise perturbation.[7] If the system is at the period-3 window, it becomes chaotic through bifurcations as an additive voltage at $f/2$ is increased. For the system chaotic at a one-band attractor after the $f/3$ window, $\lambda \approx 3.9$, a small additive voltage at $f/3$ induces periodic behavior at $f/12$, $f/6$, and $f/3$ as the additive voltage is increased. If the system is chaotic at a one-band attractor near $\lambda = 3.7$, adding a voltage at $f/2$ induces periodic behavior at $f/16$, $f/8$, $f/4$, and $f/2$ as the additive voltage is increased.

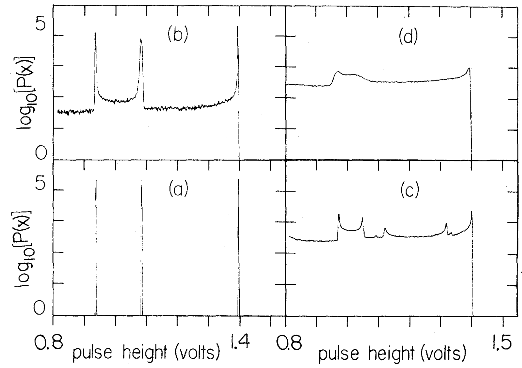

FIG. 5. Observed $\log_{10}P(V_c)$ vs V_c: (a) at onset of period-5 window with no added noise; (b) added noise $\sigma_m = 1.1 \times 10^{-4}$; (c) chaotic region ($\lambda \approx 3.7$) with no added noise $\sigma_m = 1 \times 10^{-3}$.

In conclusion, we observe that increasing the additive noise voltage by the factor $\kappa = 6.44 \pm 0.24$ to a driven nonlinear LRC oscillator produces a transitions to half the semiperiodicity if the system is periodic. Additive noise produces sudden aperiodicity at windows, but has only a small effect on chaotic behavior outside the windows. Furthermore, if the system is chaotic, it is not stable against a sinusoidal perturbation at a subharmonic frequency, which induces periodic behavior. These experimental findings are in agreement with the computed predicted behavior and are further detailed evidence for universal chaotic behavior.

ACKNOWLEDGMENTS

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