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Renormalized $(1/\sigma)$ expansion for lattice animals and localization

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The use of the $(1/\sigma)$ expansion to calculate the thermodynamic properties of systems such as the Ising model or percolation whose diagrammatic expansion contains only diagrams with no free ends is reviewed. Here $\sigma = z-1$, where z is the coordination number of the lattice. For more general problems we formulate a self-consistency condition for a site potential h, so that diagrams with free ends are eliminated. Construction of h gives the leading order in $(1/\sigma)$ solution and is exact for the Cayley tree. We obtain correction terms by using a bond renormalized interaction so that to order $(1/\sigma)^5$ we need only consider two-site problems. Results are given for (1) K_c, the critical fugacity for animals, when either H, the fugacity for free ends, or Q, the density of free ends, is fixed, and (2) (t/E_c), where E_c is the mobility energy and t is the magnitude of the hopping matrix element whose sign is random. At $d=8$ our results appear to be accurate to within about 0.01% for both animals and localization. We also obtain an expansion for $Q(K_c)/(zK_c)$ whose divergence near spatial dimensionality $d=4$ supports the idea that the order-parameter exponent β for lattice animals passes through zero at $d=4$.

I. INTRODUCTION

Although the development of the renormalization-group approach has revolutionized our concept of critical phenomena,[1] there still remain problems where accurate numerical estimates of critical exponents are desired. Of the numerical approaches, perhaps the most direct is that based on analysis or extrapolation of power-series expansions. Except for special cases, this approach is as follows: One starts from the exact solution (which is normally trivial) for some coupling constant λ equal to zero. It is important that $\lambda=0$ corresponds to independent degrees of freedom. Then a cumulant expansion for desired quantities, e.g., correlation functions, are developed in powers of λ, the coefficient of λ^n depending on the solution to the problem restricted to systems having n bonds, i.e., n pairwise interactions. The high-temperature expansion corresponds to the choice of $\lambda = 1/kT \equiv \beta$. In the percolation problem one sets $\lambda = p$, where p is the probability that a bond is occupied. Other problems which may be approached in this way are polymer formation, where λ is the monomer fugacity, Anderson localization, with $\lambda = 1/E$, where E is the energy, and the Ising model in a transverse field H, where $\lambda = 1/H$. Normally one seeks to determine a critical exponent x associated with the power series $f(\lambda)$,

$$f(\lambda) = (\lambda_c - \lambda)^x , \tag{1}$$

where λ_c is the critical value of the coupling constant. This determination usually involves a Dlog-Pade analysis or a ratio analysis. Usually the uncertainty in x is caused in large part by the uncertainty in determining λ_c. This is because the results of the various Pade approximants often define a curve $x(\lambda)$ in the x vs λ plane. When λ_c is known, a very accurate value of x often results.[2] Some years ago, Fisher and Gaunt[3] (FG) showed how one could determine λ_c for certain models as a series in $1/\sigma$, where $(\sigma+1)=z$ is the coordination number of the lattice. This development was well suited for investigations of the critical behavior at high spatial dimensionality d, since large d corresponds to small $1/\sigma$. Thus, power-series tests of renormalization-group predictions concerning the upper critical dimension d^*, below which mean-field theory becomes invalid, are facilitated by reliable estimates for λ_c. There remained a class of problems which we will describe presently to which the technique of FG is unwieldy. It is the purpose of the present paper to present a technique which enables a program similar to that of FG to be performed for a wide range of hitherto difficult cases.

Briefly this paper is organized as follows. In Sec. II we formulate an effective single-site potential which eliminates free ends from the diagrammatic expansion. In Sec. III we present the models for polymer statistics and localization and formulate "partition functions" from which their properties may be obtained. In Sec. IV we evaluate the single-site partition function. This provides the exact solution for the Cayley tree, since on it no diagrams without free ends can be formed. Section V contains the construction of the series in powers of $1/\sigma$ for K_c/K_c^0, where K_c^0 is the critical value of the monomer fugacity K for the Cayley tree. For the usual case when H, the fugacity for free ends, is held constant we give results to order $(1/\sigma)^5$, whereas for the case where Q, the density of free ends, is held fixed we give results to order $(1/\sigma)^4$. Here we also obtain an expansion in powers of $1/\sigma$ for $Q(K_c)/(zK_c)$. This quantity is an interesting one to calculate, since it diverges when the order-parameter exponent becomes negative. In Sec. VI we present the expansion for (t/E_c) for a model of localization[4] in powers of $(1/\sigma)$ up to order $(1/\sigma)^5$, where E_c is the mobility energy and t is the magnitude of the hopping matrix element whose sign is random. Our conclusions are stated in Sec. VII. A discussion of the relation, via cumulants, between the dilute polymer or animals problem and percolation is given in Appendix A. As a check on our calculations we rederive in Appendix E the results of FG for the susceptibility of the Ising model to order $(1/\sigma)^5$.

II. CONSTRUCTION OF A RENORMALIZED EXPANSION

It is helpful to review the motivation of the $1/\sigma$ expansion developed by FG. We confine our attention to models involving nearest-neighbor interactions in which case bonds are allowed only between neighboring sites. One can expand $f(\lambda)$ as

$$f(\lambda)=\sum_{\Gamma} w(\Gamma)f^c(\lambda;\Gamma)\,, \qquad (2)$$

where the sum is carried over all diagrams Γ which represent a set of bonds. Here $w(\Gamma)$ is the weak embedding constant of the graph Γ, defined so that $Nw(\Gamma)$ is the number of ways the diagram Γ can be placed on the lattice of N sites in the limit $N\rightarrow\infty$ and $f^c(\lambda;\Gamma)$ is the cumulant value of $f(\lambda)$ for the diagram Γ, defined by

$$f^c(\lambda,\Gamma)=f(\lambda,\Gamma)-\sum_{\gamma\in\Gamma} f^c(\lambda;\gamma)\,, \qquad (3)$$

where $f(\lambda;\Gamma)$ is the exact value of $f(\lambda)$ for the system of interactions represented by the diagram

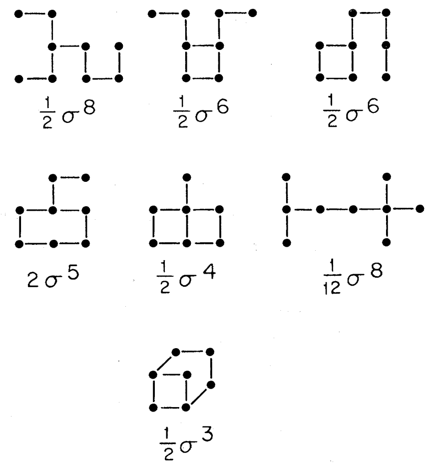

FIG. 1. Some diagrams permitted on the hypercubic lattice. The leading term for large σ for $w(\Gamma)$ is given under the diagram. Loops of length $2s$ give rise to a relative factor σ^{-s}. All diagrams with no loops have $w(\Gamma)\sim\sigma^{n_b(\Gamma)}$.

Γ and γ is summed over the subsets of Γ. It can be shown that $f^c(\lambda;\Gamma)$ involves terms of order λ^m with $m\geq n_b(\Gamma)$, where $n_b(\Gamma)$ is the number of bonds in the diagram Γ. To obtain $f(\lambda)$ correctly to order λ^n, it is thus necessary to carry the sum in Eq. (2) over all diagrams with n or less bonds. The asymptotic behavior at large σ of $w(\Gamma)$ for the d-dimensional hypercubic lattice for some diagrams is shown in Fig. 1. One can easily verify that $w(\Gamma)$ is of order σ^n for a diagram with n bonds and no loops. In general, if such a diagram contains a loop of $2s$ bonds, $w(\Gamma)$ will be of order σ^{n-s}. Thus to leading order in $(1/\sigma)$ one considers only diagrams with no loops. In essence therefore, one obtains the solution for the Cayley tree of coordination number $z=(\sigma+1)=2d$. In fact, for the problems considered by FG the task is even simpler. Consider the case when $f(\lambda)$ is the susceptibility $\chi(J/kT)$ for the Ising model for which

$$H=-J\sum_{\langle i,j\rangle}\sigma_i\sigma_j,\quad \sigma_i=\pm 1 \qquad (4)$$

where $\langle i,j\rangle$ means that the sum is over pairs of nearest-neighboring sites i and j. As usual we write

$$\chi=\sum_{i,j}\chi(i,j) \qquad (5)$$

with

$$\chi(i,j)=\mathrm{Tr}e^{-\beta H}\sigma_i\sigma_j/\mathrm{Tr}e^{-\beta H}\,. \qquad (6)$$

It is easily shown that the cumulant $\chi^c(i,j;\Gamma)$ vanishes if Γ has any free ends other than i or j. A free end is a site intersected by a single bond. Since to leading order in $(1/\sigma)$ we are restricted to tree graphs, it is clear that i and j must be the end points of a chain of length n. For the hypercubic lattice $w(\Gamma)$ for such a chain is $\frac{1}{2}\sigma^n$ to leading order in $(1/\sigma)$. Thus we write[5]

$$N^{-1}\chi = \sum_{n=0}^{\infty} [\sigma \tanh(\beta J)]^n \tag{7a}$$

$$= [1-\sigma \tanh(\beta J)]^{-1} , \tag{7b}$$

which gives a critical value of J,

$$\beta J_c \sim \sigma^{-1} , \tag{8}$$

and an exponent γ for χ of unity.

For the percolation problem the pair-connectedness susceptibility $\chi(i,j)$ is defined to be the probability i and j are connected by occupied bonds. Again $\chi^c(i,j;\Gamma)$ vanishes if any site other than i or j is a free end. Thus for large σ we perform a sum over chains as in Eq. (7) and find[5]

$$N^{-1}\chi = \sum_{n=0}^{\infty} (\sigma p)^n \tag{9a}$$

$$= (1-\sigma p)^{-1} . \tag{9b}$$

For problems such as these it is straightforward to include higher-order terms in $(1/\sigma)$ by considering graphs with successively more loops. An analysis of this type enabled FG to give results to order $(1/\sigma)^5$ for $\tanh(\beta J_c)$ and Gaunt and Ruskin[6] to give results to order $(1/\sigma)^4$ for p_c, the critical concentration for percolation.

We now consider problems where the solution to leading order in $(1/\sigma)$ is not restricted to linear graphs but involves tree graphs with arbitrarily many free ends. Examples of this class of problem are (1) the counting of polymer conformations (also called lattice animals) and (2) the construction of a generating function to describe an electron in a random potential. Clearly, if we could eliminate free ends the evaluation of the susceptibility for large σ would reduce to a one-dimensional problem and the partition function would involve only a single-site calculation. Furthermore, these quantities could then be exactly calculated for the Cayley tree. Our procedure for eliminating free ends is as follows. We assume that the partition function is of the form

$$Z = \mathrm{Tr} \prod_i \rho_i \prod_{\langle ij \rangle} f_{ij} , \tag{10}$$

where ρ_i is a function only of variables associated with site i, and f_{ij} involves the interactions between sites i and j. We may rewrite Z as

$$Z = \mathrm{Tr} \prod_i (\rho_i h_i^z) \prod_{\langle ij \rangle} \left[\frac{f_{ij}}{h_i h_j} \right] , \tag{11}$$

where h_i is an arbitrary function of the variables of site i. Indeed we write

$$Z = \mathrm{Tr} \prod_i (\rho_i h_i^z) \prod_{\langle ij \rangle} \left[1 + \left[\frac{f_{ij}}{h_i h_j} - 1 \right] \right] \tag{12a}$$

$$\equiv \mathrm{Tr} \prod_i \tilde{\rho}_i \prod_{\langle ij \rangle} (1+V_{ij}) . \tag{12b}$$

From Eq. (12b) we see that the condition that diagrams with free ends not contribute to Z is simply

$$\mathrm{Tr}_j \tilde{\rho}_j V_{ij} = 0 , \tag{13}$$

where Tr_j indicates a trace only over variables of site j. In terms of h this condition is

$$\mathrm{Tr}_j \rho_j h_j^z \left[\frac{f_{ij}}{h_i h_j} - 1 \right] = 0 , \tag{14}$$

so that h_i obeys

$$h_i = \mathrm{Tr}_j \rho_j h_j^{z-1} f_{ij} / \mathrm{Tr}_j \rho_j h_j^z . \tag{15}$$

In general this equation is hard to solve. However, for the two problems we consider, this equation

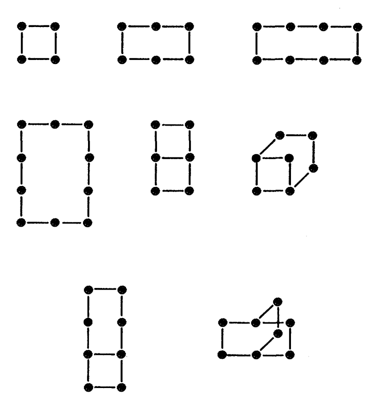

FIG. 2. In this and the next two figures we show all diagrams on the hypercubic lattice for which $w(\Gamma) \sim \sigma^m$ with $m \geq n_b(\Gamma) - 5$. In the occidental order of reading, the diagrams shown here are $\Gamma_1, \Gamma_2, \ldots, \Gamma_8$. For their values of $w(\Gamma)$ see Table II.

reduces to two coupled algebraic equations which can be solved explicitly enough to tell us all we need to know.

To summarize: We construct h_i as the solution to Eq. (15). We then obtain an expansion in powers of V_{ij} as in Eq. (12b). If V_{ij} is represented by a bond connecting sites i and j, then the construction of h ensures that only diagrams with no free ends, such as those shown in Figs. 2–4, appear in this expansion.

III. FORMULATION OF THE MODELS

A. Animals

We will consider the lattice model of polymer statistics formulated by Lubensky and Isaacson.[7] They introduce the grand partition function defined by

$$\Xi = \sum_{\{C\}} n^{N_p} H^{N_{\rm fr}} K^{N_b} \prod_{f\geq 3} w_f^{N_f} \,. \tag{16}$$

Here the sum is over all configurations C of "polymers." A polymer is a cluster of occupied bonds connecting adjacent lattice sites on a d-dimensional hypercubic lattice of N sites. For each configuration, N_b is the number of occupied bonds, $N_{\rm fr}$ the number of free ends, N_f the number of f vertices (sites connected to f occupied bonds), and N_p the number of polymers. The associated fugacities are K, H, w_f, and n, respectively. The average polymer density c_p, monomer density c, density of free ends $c_{\rm fr}$, and density of f vertices c_f are given by

$$c_p = \frac{n}{N}\frac{\partial \ln \Xi}{\partial n} \,, \tag{17a}$$

$$c = \frac{K}{N}\frac{\partial \ln \Xi}{\partial K} \,, \tag{17b}$$

$$c_{\rm fr} = \frac{H}{N}\frac{\partial \ln \Xi}{\partial H} \,, \tag{17c}$$

$$c_f = \frac{w_f}{N}\frac{\partial \ln \Xi}{\partial w_f} \,, \tag{17d}$$

where the differentiation is carried out with all but one fugacity held constant. In the limit $n \rightarrow 0$ we write

$$\lim_{n\rightarrow 0} \frac{1}{Nn} \ln \Xi = -F(K,H,\{w_f\}) \tag{18}$$

and in this limit

$$\frac{c}{c_p} = \frac{K}{F}\frac{\partial F}{\partial K} \,, \tag{19a}$$

$$\frac{c_{\rm fr}}{c_p} = \frac{H}{F}\frac{\partial F}{\partial H} \,, \tag{19b}$$

$$\frac{c_f}{c_p} = \frac{w_f}{F}\frac{\partial F}{\partial w_f} \,, \tag{19c}$$

where

$$-F = \sum_{N_b, N_{\rm fr}, \{N_f\}} A(N_b, N_{\rm fr}, \{N_f\}) \times K^{N_b} H^{N_{\rm fr}} \prod_{f\geq 3} w_f^{N_f} \,, \tag{20}$$

where $A(N_b,N_{\rm fr},\{N_f\})$ is the number of clusters per site which can be formed containing N_b bonds, $N_{\rm fr}$ free ends, and N_f f vertices for $f\geq 3$. When $H = w_3 = w_4 = \cdots = 1$, then F becomes the generating function for lattice animals. For $n=1$, we expect Ξ to describe the percolation problem.

Since the singular behavior in (c/c_p), $(c_{\rm fr}/c_p)$, and (c_f/c_p) at the critical point is dominated by the singularities in the derivatives of F, we define

$$\langle N_b \rangle = -\frac{\partial F}{\partial K} \tag{21a}$$

$$= \sum N_b A(N_b, N_{\rm fr}, \{N_f\}) K^{N_b - 1} H^{N_{\rm fr}} \prod_{f \geq 3} w_f^{N_f} \,, \tag{21b}$$

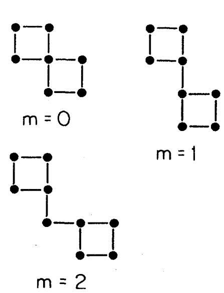

FIG. 3. Continuation of Fig. 2. Here we show $\Gamma_9^{(m)}$.

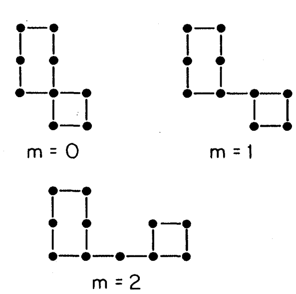

FIG. 4. Continuation of Fig. 2. Here we show $\Gamma_{10}^{(m)}$.

$$\langle N_{\rm fr}\rangle = -\frac{\partial F}{\partial H} \tag{21c}$$

$$= \sum N_{\rm fr} A(N_b, N_{\rm fr}, \{N_f\}) K^{N_b} H^{N_{\rm fr}-1} \prod_{f\geq 3} w_f^{N_f} , \tag{21d}$$

$$\langle N_{f'}\rangle = -\frac{\partial F}{\partial w_{f'}} \tag{21e}$$

$$= \sum N_{f'} A(N_b, N_{\rm fr}, \{N_f\}) K^{N_b} H^{N_{\rm fr}} w_{f'}^{-1} \prod_{f\geq 3} w_f^{N_f} , \tag{21f}$$

and the susceptibilities

$$\chi_{uv} = -\frac{\partial^2 F}{\partial u \partial v} , \tag{22}$$

where u and v assume the values H, K, and w_f.

We now construct a partition function to reproduce Ξ. Consider the quantity

$$\mathrm{Tr} \prod_{\langle ij\rangle} \left[1 + K \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha \right] , \tag{23}$$

where the single site operators S_i^α obey the trace rules

$$\mathrm{Tr}\, S_i^\alpha S_i^\beta f(\vec{S}_i) = 0, \quad \alpha \neq \beta \tag{24a}$$

$$\mathrm{Tr}\, (S_i^\alpha)^2 = \mathrm{Tr}\, 1 = 1 , \tag{24b}$$

$$\mathrm{Tr}\, S_i^\alpha = H , \tag{24c}$$

$$\mathrm{Tr}\, (S_i^\alpha)^f = w_f, \quad 3 \leq f \leq z \tag{24d}$$

$$\mathrm{Tr}\, (S_i^\alpha)^f = 0, \quad f > z \tag{24e}$$

where f is an arbitrary function. It is easily seen by expanding Eq. (23) in powers of K that

$$\Xi = \mathrm{Tr} \prod_{\langle ij\rangle} \left[1 + K \sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha \right] \tag{25}$$

so that Ξ is of the form required by Eq. (10).

B. Localization

To study the localization of electronic wave functions in a random potential we consider the following model[4]:

$$H = \sum_{\langle ij\rangle} t_{ij} (c_i^\dagger c_j + c_j^\dagger c_i) , \tag{26}$$

where $c_i^\dagger$ (c_i) creates (destroys) an electron on site i. The hopping matrix elements t_{ij} are taken to be independent random variables which assume the values $+t$ and $-t$ each with probability $\frac{1}{2}$. We study the series in (t/E) for the averaged Green's function $g(E)$ defined via

$$g(E) = [G_{ii}(E)]_t , \tag{27}$$

where $[\ ]_t$ indicates an average over configurations (of the random hopping matrix elements) and $G_{ij}(E)$ is defined as

$$G_{ij}(E) = \sum_n \psi_n(i)\psi_n(j)/(E - E_n) , \tag{28}$$

where ψ_n is the eigenfunction with energy E_n for the random Hamiltonian of Eq. (26).

Consider the configuration-dependent partition function

$$z(\{t\}) = \int \prod_i (e^{-E\psi_i^2/2} d\psi_i) \prod_{\langle ij\rangle} e^{t_{ij}\psi_i\psi_j} . \tag{29}$$

One easily sees that

$$z(\{t\}) = (2\pi)^{N/2} \det(E\underline{I} - \underline{t})^{-1/2} , \tag{30}$$

where $\underline{I}$ is the unit matrix. Also we may write

$$\frac{1}{z}\frac{\partial z}{\partial E} = -\tfrac{1}{2}\, \mathrm{Tr}(E\underline{I} - \underline{t})^{-1} \tag{31a}$$

$$= -\tfrac{1}{2} \sum_i G_{ii}(E) . \tag{31b}$$

To average over randomness we use the replica trick.[8] That is, we set

$$Z_L^{(n)} = \int \prod_i \prod_{\alpha=1}^{n} (e^{-E\psi_{i\alpha}^2/2} d\psi_{i\alpha}) \prod_{\langle ij\rangle} \left[\prod_{\alpha=1}^{n} e^{t_{ij}\psi_{i\alpha}\psi_{j\alpha}} \right]_t \tag{32a}$$

$$= [z(\{t\})^n]_t . \tag{32b}$$

In the limit $n \to 0$ we thus have

$$Z_L^{(n)} = 1 + n[\ln z(\{t\})]_t , \tag{33}$$

so that

$$\frac{\partial Z_L^{(n)}}{\partial E} = -\tfrac{1}{2} n \sum_i [G_{ii}(E)]_t \tag{34a}$$

$$= -\tfrac{1}{2} n N g(E) . \tag{34b}$$

If we set $Z_L^{(n)}=1-nNF+O(n^2)$, we have

$$g(E)=2\frac{\partial F}{\partial E} \tag{35}$$

Thus we can generate the expansion for $g(E)$ in powers of (t/E) from

$$Z_L^{(n)}=\int \prod_{i\alpha}(e^{-E\psi_{i\alpha}^2/2}d\psi_{i\alpha})\prod_{\langle ij\rangle}\cosh\left[t\prod_{\alpha=1}^{n}\psi_{i\alpha}\psi_{j\alpha}\right], \tag{36}$$

which is of the form discussed in Eq. (10).

IV. ELIMINATION OF FREE ENDS: SOLUTION TO LEADING ORDER IN $1/\sigma$

A. Animals

For animals Eq. (15) is

$$h_i=\mathrm{Tr}_j\left[1+K\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha\right]h_j^{z-1}/\mathrm{Tr}_j h_j^z , \tag{37}$$

whose solution is

$$h_i=A+B\sum_\alpha S_i^\alpha , \tag{38}$$

where A and B are the solutions of

$$A=\frac{\mathrm{Tr}_j\left[A+B\sum_{\alpha=1}^{n}S_j^\alpha\right]^{z-1}}{\mathrm{Tr}_j\left[A+B\sum_{\alpha=1}^{n}S_j^\alpha\right]^{z}}, \tag{39a}$$

$$B=K\frac{\mathrm{Tr}_j S_j^\alpha(A+BS_j^\alpha)^{z-1}}{\mathrm{Tr}_j(A+B\sum_{\alpha=1}^{n}S_j^\alpha)^z} . \tag{39b}$$

Since we are interested in the limit $n\to 0$, we solve for A and B to the following accuracy in n:

$$A=1+an , \tag{40a}$$

$$B=b , \tag{40b}$$

where a and b are determined by

$$a=-\frac{b^2}{2K} , \tag{41a}$$

$$b=K\,\mathrm{Tr}_j S_j^\alpha(1+bS_j^\alpha)^{z-1} . \tag{41b}$$

Consider now the expansion of Eq. (12b) in powers of V_{ij} for a Cayley tree. All terms other than the leading terms involve free ends and therefore vanish according to the construction. Thus, to order n the solution of Eq. (40) yields

$$\Xi=\left[\mathrm{Tr}_i\left[1+an+b\sum_{\alpha=1}^{n}S_i^\alpha\right]^z\right]^N , \tag{42}$$

so that

$$F=1+\frac{zb^2}{2K}-\mathrm{Tr}_i(1+bS_i^\alpha)^z \tag{43a}$$

$$=-zbH+\tfrac{1}{2}rz^2b^2-\sum_{f\geq 3}\frac{z!}{f!(z-f)!}z^{-f}w_f(bz)^f , \tag{43b}$$

where $r=(zK)^{-1}-1+z^{-1}$. For $w_f=1$ we have

$$F=\frac{zb^2}{2K}-zbH+1+zb-(1+b)^z , \tag{43c}$$

and in this case b satisfies

$$b=K[H-1+(1+b)^\sigma] . \tag{44}$$

The form, Eq. (43b), is essentially equivalent to the mean-field result of Lubensky and Isaacson[7]:

$$F=-QH+\tfrac{1}{2}rQ^2-\sum_{f\geq 3}\frac{1}{f!}w_fQ^f , \tag{45}$$

in which Q is determined by $\partial F/\partial Q=0$. Note that the constraint of Eq. (41b) can be written as

$$\left.\frac{\partial F}{\partial b}\right|_{K,H,\{w_f\}}=0 , \tag{46}$$

so that the identification $Q=zb$ is indicated.

If we regard F as a function of K, H, $\{w_f\}$, and b, we may write Eq. (21c) as

$$\langle N_{\mathrm{fr}}\rangle=-\frac{1}{N}\frac{\partial F}{\partial b}\frac{\partial b}{\partial H}\Bigg|_{K,\{w_f\}}-\frac{1}{N}\frac{\partial F}{\partial H} \tag{47a}$$

$$=zb , \tag{47b}$$

where we have used Eq. (46). Likewise we write

$$\langle N_b\rangle=-\frac{1}{N}\frac{\partial F}{\partial K}=\frac{zb^2}{2K} , \tag{48a}$$

$$\langle N_f\rangle=-\frac{1}{N}\frac{\partial F}{\partial w_f}=b^f\frac{z!}{f!(z-f)!} . \tag{48b}$$

Furthermore we have

$$\chi_{HH} = \frac{\partial \langle N_{\text{fr}} \rangle}{\partial H} = z \frac{\partial b}{\partial H}\bigg|_{K,\{w_f\}} , \tag{49a}$$

$$\chi_{KK} = \frac{\partial \langle N_b \rangle}{\partial K} = -\frac{zb^2}{K^3} + \frac{zb}{K^2}\frac{\partial b}{\partial K}\bigg|_{H,\{w_f\}} , \tag{49b}$$

$$\chi_{w_{f'},w_{f'}} = \frac{\partial \langle N_{f'} \rangle}{\partial w_{f'}} = \frac{z!}{(f'-1)!(z-f')!} b^{f'-1} \frac{\partial b}{\partial w_{f'}}\bigg|_{K,H,\{w_f:f\neq f'\}} . \tag{49c}$$

We may write the results of Eq. (49) in terms of a reduced susceptibility χ_0 defined by

$$z\frac{\partial b}{\partial H}\bigg|_{K,\{w_f\}} = \chi_0 . \tag{50}$$

Using Eq. (46) we have that

$$\chi_0^{-1} = \frac{1}{z^2}\frac{\partial^2 F}{\partial b^2} \tag{51a}$$

$$= \frac{1 - K(z-1)\mathrm{Tr}_j (S_j^\alpha)^2 (1+bS_j^\alpha)^{z-2}}{zK} , \tag{51b}$$

which for $w_f = 1$ for $3 \le f \le z$ is

$$\chi_0^{-1} = [1-(z-1)K(1+b)^{z-2}]/(zK) . \tag{51c}$$

We also find

$$\frac{\partial b}{\partial K}\bigg|_{H,\{w_f\}} = \frac{zb}{K^2}\chi_0 \tag{52}$$

and

$$\frac{\partial b}{\partial w_{f'}}\bigg|_{K,H,\{w_f:f\neq f'\}} = b^{f'-1}\frac{(z-1)!}{(f'-1)!(z-f')!}\chi_0 . \tag{53}$$

To summarize: In mean-field theory the various susceptibilities χ_{HH}, χ_{KK}, and χ_{w_f,w_f} are proportional to χ_0 which diverges at the critical point which occurs at $K = K_c$ and $b = b_c$ for fixed H and $\{w_f\}$ as determined by

$$\frac{\partial F}{\partial b} = 0 , \tag{54a}$$

$$\frac{\partial^2 F}{\partial b^2} = 0 . \tag{54b}$$

To analyze the behavior near the critical point we write $b = b_c + \Delta b$ and $K = K_c + \Delta K$. Expanding Eq. (54a) in powers of Δb and ΔK and taking note of Eq. (54b) we obtain

$$\frac{1}{2}\frac{\partial^3 F}{\partial b^3}(\Delta b)^2 + \Delta K \frac{\partial^2 F}{\partial b \partial K} = 0 , \tag{55}$$

from which it follows that

$$\Delta b \sim (-\Delta K)^{1/2} . \tag{56}$$

Thus the result of Eq. (56) lead to the asymptotic forms for $\Delta K < 0$,

$$\langle N_b \rangle \sim \langle N_{\text{fr}} \rangle \sim \langle N_f \rangle \sim -\,|\Delta K|^{1/2} , \tag{57a}$$

$$\chi_{HH} \sim \chi_{KK} \sim \chi w_f w_f \sim -\,|\Delta K|^{-1/2} . \tag{57b}$$

B. Localization

Equation (15) for our localization model is

$$h_i = \frac{\int \prod_{\alpha=1}^{n} (e^{-E\psi_{j\alpha}^2/2} d\psi_{j\alpha}) h_j^{z-1} \cosh\left[t \sum_{\alpha=1}^{n} \psi_{i\alpha}\psi_{j\alpha}\right]}{\int \prod_{\alpha=1}^{n} (e^{-E\psi_{j\alpha}^2/2} d\psi_{j\alpha}) h_j^{z}} . \tag{58}$$

We write the solution for h_i in the form

$$h_i = C \exp\left[\frac{\Delta}{2}\sum_\alpha \psi_{i\alpha}^2\right] \tag{59}$$

and substitution into Eq. (58) leads to

$$C = \left[\frac{E - z\Delta}{E - \sigma\Delta}\right]^{n/4} , \tag{60a}$$

$$\Delta = t^2/(E - \sigma\Delta) , \tag{60b}$$

so that

$$\Delta = \frac{E}{2\sigma}[1-(1-4\sigma t^2/E^2)^{1/2}] . \tag{61}$$

Since free ends are eliminated, we obtain the exact solution for $Z_L^{(n)}$ for the Cayley tree[9,10]:

$$Z_L^{(n)} = \int \prod_i \left(C^z \prod_\alpha e^{-\psi_{i\alpha}^2(E - z\Delta)/2} d\psi_{i\alpha}\right) , \tag{62a}$$

$$= C^{Nz}[2\pi/(E - z\Delta)]^{Nn/2} , \tag{62b}$$

$$\sim 1 + Nn\left[\frac{z-2}{4}\ln(E - z\Delta) - \frac{z}{4}\ln(E - \sigma\Delta) + \tfrac{1}{2}\ln(2\pi)\right] , \tag{62c}$$

so that

$$\frac{\partial Z_L^{(n)}}{\partial E}=\frac{Nn}{4}\left[(z-2)\frac{1-z\frac{\partial\Delta}{\partial E}}{E-z\Delta}z\frac{1-\sigma\frac{\partial\Delta}{\partial E}}{E-\sigma\Delta}\right] . \quad (63)$$

After some algebra we obtain the averaged Green's function as

$$g(E)=\frac{E}{2}\left[\frac{(\sigma+1)(1-4\sigma t^2/E^2)^{1/2}-(\sigma-1)}{E^2-z^2t^2}\right] , \quad (64)$$

the imaginary part of which gives the density of states as

$$\rho(E)=\begin{cases}\dfrac{(\sigma+1)(E_c^2-E^2)^{1/2}}{2\pi(z^2t^2-E^2)}, & E<E_c \quad (65a)\\ 0, & E>E_c \quad (65b)\end{cases}$$

where $E_c=2(\sigma t)^{1/2}$. Note that $E_c\sim\sigma^{1/2}$, so that the singularity we are observing does not correspond to the exact band edge of order σ for the hypercubic lattice. Thus we hope that the singularity we are studying for large σ corresponds to the mobility edge. It is usually argued[4] that such a singularity cannot be seen in the averaged single-particle Green's function $g(E)$. However, a mean-field and ϵ-expansion treatment[10] of the localization problem indicates that singularities may also occur in $g(E)$. In this picture the exact band edge may be analogous to the Griffith's singularity[11] in dilute magnets and hence although such a singularity does exist, it may not have a detectible effect on the series in (t/E).[12]

V. THE $1/\sigma$ EXPANSION FOR K_c FOR ANIMALS

We obtain the $1/\sigma$ expansion for K_c when H is constant in part A of this section and when Q/zK is constant in part B.

A. Constant H

We now obtain the $1/\sigma$ expansion for K_c for animals for the case $H=1$ and $w_f=1$ for $3\le f\le z$. If we write

$$-F=\sum_n c_n K^n , \quad (66)$$

then K_c is given by

$$K_c^{-1}=\lim_{n\to\infty}(c_n)^{1/n} , \quad (67a)$$

which is often written as

$$\ln K_c=-\lim_{n\to\infty}\frac{1}{n}\ln c_n . \quad (67b)$$

If we calculate $F(K)$ to a given order in $(1/\sigma)$, then we can obtain c_n to corresponding accuracy by

$$c_n=-\frac{1}{2\pi i}\int\frac{dK}{K^{n+1}}F(K) , \quad (68)$$

where the integration is over a contour in the complex K plane surrounding the point $K=0$. Since h_i involves b, and since a closed-form solution of Eq. (44) for b in terms of K does not exist, we will use b as the independent variable via

$$K=b/(1+b)^\sigma , \quad (69)$$

in which case we have

$$c_n=-\frac{1}{2\pi i}\int\frac{db}{b^{n+1}}(1+b)^{n\sigma-1}\times[1-(\sigma-1)b]F(b) . \quad (70)$$

If we insert Eq. (69) into Eq. (43) and set $H=1$, we have

$$F=1-(1+b)^{\sigma+1}+\frac{\sigma+1}{2}b(1+b)^\sigma\equiv F_0(b) . \quad (71)$$

Using Eq. (70) we obtain the leading contribution to c_n, which we denote c_n^0:

$$c_n^0=\frac{(\sigma+1)}{(n+1)!}\frac{(\sigma n+\sigma)!}{(\sigma n+\sigma-n+1)!} , \quad (72)$$

in agreement with the exact result of Fisher and Essam.[13] This result gives

$$K_c^0=\frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}\sim\frac{1}{\sigma e} . \quad (73)$$

The program is now clear: we will evaluate the contribution to F from the diagrams with no free ends which are shown in Figs. 2–4. We then can insert this result into Eq. (70) to find the contribution $\delta c_n(\Gamma)$ and thereby obtain the corresponding correction to K_c via Eq. (67). As a check that we have really included correctly all diagrams relevant to the calculations to order $(1/\sigma)^5$, we evaluate the susceptibility of the Ising model to this order in Appendix E. We there reproduce the results of FG.

The expansion for F takes the form[14]

$$F(b)=F_0(b)+\sum_\Gamma w(\Gamma)F(\Gamma;b) \quad (74a)$$

$$\equiv F_0(b)+\sum_\Gamma\delta F(\Gamma;b) \quad (74b)$$

$$\equiv F_0(b)+f(b) , \quad (74c)$$

where

$$-F(\Gamma,b)=\lim_{n\to 0}\frac{1}{n}\left[\prod_{i\in\Gamma}\mathrm{Tr}_i h_i^z \prod_{\langle ij\rangle\in\Gamma}\left[\frac{1+K\sum_\alpha S_i^\alpha S_j^\alpha}{h_i h_j}-1\right]\right], \tag{75}$$

where

$$h_i=1+an+b\sum_\alpha S_i^\alpha\ . \tag{76}$$

We first note that $F(\Gamma,b)$ has no contribution involving a. If an occurs in a factor containing b or K, there will be an additional factor of n due to a sum over α. On the other hand, if we set $b=K=0$, then each bond would give a factor an, and the contribution would vanish for $n\to 0$. So we set

$$h_i=1+b\sum_\alpha S_i^\alpha\equiv h_{i0} \tag{77}$$

to evaluate $F(\Gamma;b)$. Thus we have

$$-F(\Gamma;b)=\lim_{n\to\infty}\left[\frac{1}{n}\prod_{i\in\Gamma}\mathrm{Tr}_i h_{i0}^{\,z-n_i(\Gamma)}\prod_{\langle ij\rangle\in\Gamma}\left[K\sum_\alpha S_i^\alpha S_j^\alpha+1-h_{i0}h_{j0}\right]\right], \tag{78}$$

where $n_i(\Gamma)$ is the number of bonds of Γ which intersect site i. We perform a bond renormalization or, as it is sometimes called, a decimation transformation[15] on Eq. (78) in the following way. We will replace a chain of $k+1$ bonds connecting sites i and j by an indirect interaction $V_{ij}^{(k)}$. In view of Eq. (78) we define

$$V_{ij}^{(k)}=\prod_{m=1}^{k}\mathrm{Tr}_m(h_m)^{z-2}\prod_{m=1}^{k-1}\left[K\sum_\alpha S_m^\alpha S_{m+1}^\alpha+1-h_{m0}h_{m+1,0}\right]$$
$$\times\left[K\sum_\alpha S_i^\alpha S_1^\alpha+1-h_{i0}h_{10}\right]\left[K\sum_\alpha S_j^\alpha S_k^\alpha+1-h_{j0}h_{k0}\right], \tag{79}$$

which we evaluate as

$$V_{ij}^{(k)}=\frac{b^{k+1}}{(1+b)^k}\left[\frac{\sum_\alpha S_i^\alpha S_j^\alpha}{(1+b)^\sigma}-\sum_\alpha(S_i^\alpha+S_j^\alpha)-(b+k)\sum_{\alpha\beta}S_i^\alpha S_j^\beta+n(1+b)^\sigma\delta_{ij}\right]. \tag{80}$$

The term in δ_{ij} is needed in order to renormalize away polygons. In terms of V we write

$$-F(\Gamma;b)=\lim_{n\to 0}\left[\frac{1}{n}\prod_i{}'\,\mathrm{Tr}_i h_{i0}^{\,z-n_i(\Gamma)}\prod_{ij}{}'\,V_{ij}^{(k)}\right], \tag{81}$$

where the prime restricts the products to sites with $n_i(\Gamma)\geq 3$. Now all our calculations of $F(\Gamma;b)$ are reduced to treating diagrams with at most two sites!

For instance, for the polygon with s sides we set $k=s-1$ and $i=j$. We thereby obtain

$$-F(\Gamma;b)=\lim_{n\to 0}\left[\frac{1}{n}\mathrm{Tr}_i\left[1+b\sum_\alpha S_i^\alpha\right]^{z-2}\frac{b^s}{(1+b)^{s-1}}\left[\frac{\sum_\alpha S_i^\alpha}{(1+b)^\sigma}-2\sum_\alpha S_i^\alpha-(b+s-1)\sum_\alpha S_i^\alpha+n(1+b)^\sigma\right]\right] \tag{82a}$$

$$=\left[\frac{b}{1+b}\right]^s[1-s(1+b)^\sigma]\ . \tag{82b}$$

TABLE I. $\delta F(\Gamma,b)$ for diagrams of Figs. 2–4.

Γ	$-\delta f(\Gamma,b)$
1	$[b/(1+b)]^4(1-4\hat{X})$[a]
2	$[b/(1+b)]^6(1-6\hat{X})$
3	$[b/(1+b)]^8(1-8\hat{X})$
4	$[b/(1+b)]^{10}(1-10\hat{X})$
5	$[b^7/(1+b)]^8[\hat{X}^{-1}(10+3b)+(30+16b+b^2)\hat{X}]$
6	$[b/(1+b)]^8[(1+b)\hat{X}]^{-1}[1-(11+3b)\hat{X}+(37+18b+b^2)\hat{X}^2]$
7[b]	$[b/(1+b)]^9[(1+b)\hat{X}]^{-1}[1-12\hat{X}+42\hat{X}^2]$
8	$[b/(1+b)]^9[(1+b)\hat{X}]^{-1}[1-3(4+b)\hat{X}+(46+20b+b^2)\hat{X}^2]$
9^m	$[b/(1+b)]^{m+8}[(1+b)\hat{X}]^{-1}[1-(5+b)\hat{X}]^2$
10^m	$[b/(1+b)]^{10+m}[(1+b)\hat{X}]^{-1}[1-(7+b)\hat{X}][1-(5+b)\hat{X}]$

[a] $\hat{X}=(1+b)^{\sigma}$.
[b] Terms higher order in b are omitted because they give contributions higher order in $1/\sigma$. The full expression is rather lengthy.

TABLE II. Weak embedding constants to order $(1/\sigma)^5$ for diagrams with no free ends.

Γ	$n_b(\Gamma)$	$w(\Gamma)/\sigma^{n_b(\Gamma)}$
1	4	$\frac{1}{8\sigma^2}-\frac{1}{8\sigma^4}$
2	6	$\frac{1}{3\sigma^3}-\frac{3}{4\sigma^4}-\frac{1}{3\sigma^5}$
3	8	$\frac{27}{16\sigma^4}-\frac{77}{8\sigma^5}$
4	10	$\frac{62}{5\sigma^5}$
5	7	$\frac{1}{4\sigma^4}-\frac{1}{2\sigma^5}$
6	8	$\frac{1}{2\sigma^5}$
7	9	$\frac{2}{\sigma^5}$
8	9	$\frac{1}{12\sigma^5}$
9^m	$8+m$	$\frac{1}{8\sigma^4}-\frac{1}{8\sigma^5}(3+\delta_{m0})$
10^m	$10+m$	$\frac{1}{\sigma^5}$

For some of the other diagrams in Figs. 2–4 the results are obtained from

$$-F(\Gamma_5;b)=\lim_{n\to 0}\frac{1}{n}\left[\prod_{i=1,2}\mathrm{Tr}_i h_{i0}^{z-3}\right]V_{12}^{(0)}(V_{12}^{(2)})^2\,, \tag{83a}$$

$$-F(\Gamma_9^{(m)};b)=\lim_{n\to 0}\frac{1}{n}\left[\prod_{i=1,2}\mathrm{Tr}_i h_{i0}^{z-3}\right]\times V_{11}^{(3)}V_{22}^{(3)}V_{12}^{(m-1)}\,. \tag{83b}$$

The evaluation of $F(\Gamma;b)$ for all diagrams shown in Figs. 2–4 are listed in Table I.

To get $\delta F(\Gamma;b)$ we need the embedding constants which are given in Table II to relative order σ^{-5} for the diagrams of interest. Combining the results of Tables I and II to get $\delta F(\Gamma;b)$ and using Eq. (70) we obtain $\delta c_n(\Gamma)$ and these are analyzed in Appendix B. For instance, we find $\delta c_n(\Gamma)$ for the square to be

$$\delta c_n(\Gamma_1)=\frac{\sigma^2-1}{2(n-4)!}\left[\frac{(\sigma n-5)!(\sigma-1)}{(\sigma n-n)!}-\frac{(\sigma n+\sigma-5)!(5\sigma-4)}{(\sigma n+\sigma-n)!}\right]. \tag{84}$$

This result is the leading approximation (in $1/\sigma$) for the correction to c_n due to all clusters consisting of a square with trees appended to its corners. More accurate enumerations of such tree dressings can be obtained by including more complicated diagrams consisting of a square with additional loops appended to it. We may check the result for $n=4$ where

$$\delta c_4(\Gamma_1)=\frac{\sigma^2-1}{8}(1-4)\ . \tag{85}$$

The factor $(1-4)$ arises as follows. The 1 counts the number of squares $(\sigma^2-1)/8$, one can place on a hypercubic lattice. These are of course not included in the Cayley-tree result. In addition, there are on the Cayley tree $4(\sigma^2-1)/8$ clusters of the type shown in Fig. 5 which have no analog on the hypercubic lattice. These clusters would be squares on the hypercubic lattice, but on the Cayley tree the points 1 and 5 do not coincide. Such explicit considerations are clearly unwieldy for more complicated diagrams.

Collecting the results for $\delta c_n(\Gamma)$ from Appendix B we obtain

$$\begin{aligned}\frac{c_n}{c_n^0}=1&+\frac{1}{\sigma^2}\left[n\left[\frac{1}{2e}-\frac{5}{2}\right]-\left[\frac{7}{4e}-\frac{45}{2}\right]\right]+\frac{1}{\sigma^3}\left[n\left[\frac{1}{4e}-7\right]-\left[\frac{99}{8e}-187\right]\right]\\&+\frac{1}{\sigma^4}\left[\frac{1}{2}n^2\left[\frac{1}{2e}-\frac{5}{2}\right]^2+n\left[-\frac{3}{4e^2}+\frac{889}{48e}-\frac{411}{4}\right]\right]\\&+\frac{1}{\sigma^5}\left[n^2\left[\frac{1}{2e}-\frac{5}{2}\right]\left[\frac{1}{4e}-7\right]+n\left[-\frac{101}{24e^2}+\frac{4377}{32e}-\frac{2708}{3}\right]\right]\ .\end{aligned} \tag{86}$$

In Eq. (86) each power of $1/\sigma$ multiplies an infinite series in descending powers of n. We have kept only such terms in these series as influence our present calculations. It is essential and constitutes a partial check of our calculations that the terms proportional n^2 are such that they eventually drop out of the calculation. Next we write

$$\lim_{n\to\infty}\frac{1}{n}\ln\frac{c_n}{c_n^0}=\frac{1}{\sigma^2}\left[\frac{1}{2e}-\frac{5}{2}\right]+\frac{1}{\sigma^3}\left[\frac{1}{4e}-7\right]+\frac{1}{\sigma^4}\left[\frac{1}{8e^2}+\frac{139}{48e}-\frac{93}{2}\right]+\frac{1}{\sigma^5}\left[\frac{29}{12e^2}-\frac{177}{32e}-\frac{833}{3}\right]\ , \tag{87}$$

from which the final result follows:

$$\frac{K_c}{K_c^0}=1+\frac{1}{\sigma^2}\left[\frac{5}{2}-\frac{1}{2e}\right]+\frac{1}{\sigma^3}\left[7-\frac{1}{4e}\right]+\frac{1}{\sigma^4}\left[\frac{397}{8}-\frac{199}{48e}\right]+\frac{1}{\sigma^5}\left[\frac{1771}{6}+\frac{45}{32e}-\frac{55}{24e^2}\right]\ . \tag{88}$$

This result disagrees with that of Gaunt and Ruskin[6] who quote results to order $(1/\sigma)^2$. This difference arises from their approximation

$$\left[\frac{n}{n+1}\right]^n\sim 1\ , \tag{89a}$$

whereas we set

$$\left[\frac{n}{n+1}\right]^n\sim\frac{1}{e}\left[1+\frac{1}{2n}\right]\ . \tag{89b}$$

Numerical results for various dimensions are shown in Table III. The importance of the higher-order terms we have calculated is apparent. Clearly near eight dimensions ($d^*=8$ according to Ref. 10) our results are very precise. For comparison, the results of Gaunt and Ruskin[6] based on a more global analysis are also given.

B. $1/\sigma$ expansion for K_c for animals at constant q

In this section we will give results for K_c for animals when $q=Q/zK$ is held constant but H is allowed to vary. We shall again restrict ourselves to the case $w_f=1$ for $3\le f\le z$. This case is important because the exponent for χ_{HH} in mean-field theory is unity if Q is held constant[16] but is $\frac{1}{2}$ if H is held constant. It is believed that holding q constant is equivalent to holding Q constant. We consider the constant q case because it is easier to construct a series for this case than for the constant Q case. Series work[17] on the constant q case may give clearer results than for the constant H case.

Since H varies, we must use the relation

$$b=K[H-1+(1+b)^{\sigma}] \tag{90}$$

to express F as a function of K and H. Then Q is obtained via

$$Q=-\frac{\partial F(H,K)}{\partial H} . \tag{91}$$

We construct F conveniently as a function of b and K. Thus we have

$$Q=-\frac{\partial F(b,K)}{\partial b}\left.\frac{\partial b}{\partial H}\right|_K , \tag{92}$$

where $\partial b/\partial H\,|_K=\chi_0(b,K)$ is given in Eq. (51c). For F we use the expansion of Eq. (74c) with $F_0(b)$ given as

$$F_0(b)=-\frac{(\sigma+1)b^2}{2K}-(1+b)^{\sigma+1}+1 \\ +(\sigma+1)b(1+b)^{\sigma} . \tag{93}$$

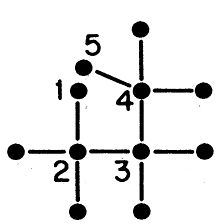

FIG. 5. Section of a Cayley tree with $z=\sigma+1=4$. In the Cayley tree there are no loops. This same section of sites on a square lattice would have a loop because sites 1 and 5 would coincide.

Combining Eqs. (91) and (93) we obtain an implicit equation for $q=Q/Kz$:

$$b=Kq+\frac{1}{(\sigma+1)^2}\chi_0(b,K)\frac{\partial f(b,K)}{\partial b} \tag{94a}$$

$$=Kq+\Psi(b,K) . \tag{94b}$$

This equation can be solved iteratively to give b as a function of K and q:

$$b=Kq+\sum_{r=1}^{\infty}\frac{1}{r!}\left.\left[\frac{\partial}{\partial b}\right]^{r-1}\Psi(b,K)^r\right|_{b=Kq} \\ \equiv\Phi(Kq,K) . \tag{95}$$

We find it necessary to work only to order f^2 to get K_c correct to order $(1/\sigma)^4$. The susceptibility χ is given by

$$\chi=\left.\frac{\partial Q}{\partial H}\right|_K \tag{96}$$

and is obtained via

$$\left.\frac{\partial F}{\partial Q}\right|_K=\left.\frac{\partial F}{\partial H}\right|_K\left.\frac{\partial H}{\partial Q}\right|_K=-\frac{Q}{\chi} , \tag{97}$$

so that

$$\chi^{-1}=-\frac{1}{Q}\left.\frac{\partial F}{\partial Q}\right|_K=-\frac{1}{(\sigma+1)^2K^2q}\left.\frac{\partial F}{\partial q}\right|_K . \tag{98}$$

We locate the critical point by setting $\chi^{-1}=0$. Thus we write

$$zK\chi^{-1}=-(zKq)^{-1}\left.\frac{\partial F}{\partial q}\right|_K=0 . \tag{99}$$

We set

$$\left.\frac{\partial F}{\partial q}\right|_K=\left[\frac{\partial F}{\partial b}\right]_K\left[\frac{\partial b}{\partial q}\right]_K , \tag{100}$$

and we use

$$\left[\frac{\partial F}{\partial b}\right]_K=-z^2b\chi_0^{-1}(b,K)+\left[\frac{\partial f}{\partial b}\right]_K \tag{101a}$$

and

$$\left.\frac{\partial b}{\partial q}\right|_K=\frac{\partial}{\partial q}\Phi(Kq,K) . \tag{101b}$$

TABLE III. Estimates for K_c for $H=1$ based on the nth order in $(1/\sigma)$ results, denoted K_{cn}.

	$d=6$	$d=7$	$d=8$	$d=9$
H_{c0}	0.035 049	0.029 438	0.025 376	0.022 299
K_{c2}	0.035 720	0.029 842	0.025 637	0.022 478
K_{c3}	0.035 902	0.029 934	0.025 689	0.022 509
K_{c4}	0.036 017	0.029 984	0.025 713	0.022 522
K_{c5}	0.036 082	0.030 007	0.025 723	0.022 527
K_c	0.036 13	0.03002	0.025 730	0.022 528
K_{c0}^{-1}	28.531	33.969	39.407	44.845
K_{c2}^{-1}	27.995	33.510	39.006	44.488
K_{c3}^{-1}	27.837	33.397	38.921	44.422
K_{c4}^{-1}	27.764	33.351	38.890	44.401
K_{c5}^{-2}	27.715	33.325	38.875	44.392
K_c^{-1}	27.69	33.31	38.865	44.385
K_c^{-1}	27.75	33.25	39.0	44.5
Ref. 6	± 1.0	± 1.5	± 2.0	± 2.5

Then, since Eq. (95) gives Φ in terms of f, Eq. (99) can be written in terms of $f(b,K)$. After some manipulation we find the result up to order f^2 to be

$$zK\chi^{-1}(q,K)=\Gamma_0(q,K)-\Sigma(q,K)=0\,, \tag{102}$$

where

$$\Gamma_0(q,K)=zK\chi_0^{-1}(Kq,K) \tag{103a}$$

and

$$\Sigma(q,K)=\frac{\partial^2}{\partial b^2}\left[-\frac{K}{\sigma+1}f(b,K) - \frac{K\chi_0(b,K)}{2(\sigma+1)^3}\left[\frac{\partial f(b,K)}{\partial b}\right]^2\right]_{b=Kq}. \tag{103b}$$

Recall that b and K are of order σ^{-1} and f is of order σ^{-2}. Consequently, $\Sigma(q,K)$ is of order σ^{-2}. In Appendix C we evaluate $f(b,K)$ from diagrams up to order σ^{-4}. There we find

$$\Sigma(q,K)=\frac{\gamma_2(q,K)}{\sigma^2}+\frac{\gamma_3(q,K)}{\sigma^3}+\frac{\gamma_4(q,K)}{\sigma^4}+\cdots\,. \tag{104}$$

This is not the final result because K and possibly q depend on σ. Therefore we write K_c in terms of an expansion relative to the value for a Cayley tree, denoted K_{c0} and defined by $\Gamma(q,K_{c0}(q))=0$, i.e.,

$$1-\sigma K_{c0}(q)[1+qK_{c0}(q)]^{\sigma-1}=0\,. \tag{105}$$

Thus we set

$$K_c(q)=K_{c0}(q)\left[1+\frac{\beta_2(q)}{\sigma^2}+\frac{\beta_3(q)}{\sigma^3}+\frac{\beta_4(q)}{\sigma^4}+\cdots\right], \tag{106}$$

where the β_n are to be determined. Then we have

$$\Sigma(q,K_c)=\frac{\gamma_2(q,K_{c0}(q))}{\sigma^2}+\frac{\gamma_3(q,K_{c0}(q))}{\sigma^3}+\frac{\gamma_4(q,K_{c0}(q))}{\sigma^4}+\frac{\beta_2K_{c0}}{\sigma^4}\left[\frac{\partial}{\partial K_c}\gamma_2(q,K_c)\right]_{K_c=K_{c0}}. \tag{107}$$

Then if we insert the expansion of Eq. (106) into $\Gamma_0(q,K)$ in Eq. (102), we determine the β_n as

$$\beta_2(q) = -2r_0D + \tfrac{9}{2}r_0q + \tfrac{3}{2} - \tfrac{1}{2}D \equiv \beta_2(r_0,q) , \tag{108a}$$

$$\beta_3(q) = \tfrac{1}{2}(-9r_0^2q^2 + 63r_0q - 21) + \tfrac{25}{2}D + r_0(2 - \tfrac{15}{2}D) \equiv \beta_3(r_0,q) , \tag{108b}$$

$$\begin{aligned}\beta_4(q) &= r_0^2(2D^3 + 4D^2 - \tfrac{9}{8}D) + r_0(D^3 + D^2 - 49D + \tfrac{5}{2} - 4r_0q) \\ &\quad + \tfrac{1}{8}D^3 + \tfrac{865}{4}D - \tfrac{1635}{8} + \tfrac{3005}{8}r_0q - \tfrac{129}{2}r_0^2q^2 + 11r_0^3q^3 \\ &\equiv \beta_4(r_0,q) ,\end{aligned} \tag{108c}$$

where $r_0 = K_{c0}(q)\sigma$ and $D = (1+r_0q)^{-1}$. Some details leading to Eq. (108) are given in Appendix D. We can now use Eq. (106) in various ways.

First of all, if q is fixed (i.e., independent of σ), we can rearrange the expansion of Eq. (106) purely in powers of σ^{-1} by expanding r_0 in powers of σ^{-1}. We write

$$r_0 = r_0^0 \left[1 + \frac{r_1}{\sigma} + \frac{r_2}{\sigma^2} + \cdots \right] \tag{109}$$

and from Eq. (105) we obtain

$$r_1 = \frac{r_0^0q(1 + \frac{1}{2}r_0^0q)}{(1 + r_0^0q)} , \tag{110a}$$

$$r_2 = \frac{(r_0^0q)^2}{(1 + r_0^0q)^3}[1 + \tfrac{5}{3}r_0^0q + \tfrac{23}{24}(r_0^0q)^2 + \tfrac{1}{6}(r_0^0q)^3] , \tag{110b}$$

where r_0^0 is the $\sigma \to \infty$ solution to Eq. (105), viz.,

$$1 = r_0^0 e^{-r_0^0} , \tag{110c}$$

where q is simply a number in the case under consideration. In this way we find that

$$K_c(q) = K_{c0}(q) \left[1 + \frac{\hat{\beta}_2(q)}{\sigma^2} + \frac{\hat{\beta}_3(q)}{\sigma^3} + \frac{\hat{\beta}_4(q)}{\sigma^4} \cdots \right] , \tag{111}$$

where

$$\hat{\beta}_2(q) = \beta_2(r_0^0,q) , \tag{112a}$$

$$\hat{\beta}_3(q) = \beta_3(r_0^0,q) + r_1 \left[r\frac{\partial \beta_2}{\partial r} \right]_{r=r_0^0} , \tag{112b}$$

$$\hat{\beta}_4(q) = \beta_4(r_0^0,q) + r_1 \left[r\frac{\partial \beta_3}{\partial r} \right]_{r=r_0^0} + r_2 \left[r\frac{\partial \beta_2}{\partial r} \right]_{r=r_0^0} + \tfrac{1}{2}r_1^2 \left[r^2\frac{\partial^2 \beta_2}{\partial r^2} \right]_{r=r_0^0} . \tag{112c}$$

The expressions for $\hat{\beta}_n$ are given in Appendix D. Values for $K_c(q)$ are given in Table IV as a function of dimensionality for an arbitrarily chosen value of q, namely $q = 1/e$. This is the value of q at the critical point for $\sigma \to \infty$. We also give results for $q=0$, although we do not know what interpretation to give to this value of q.

Secondly, we can obtain the σ^{-1} expansion for K_c when q is set equal to the value appropriate to the Cayley tree, i.e., $q = (\sigma/\sigma-1)^\sigma \equiv q^*$. In this case we write

$$q = q_0 \left[1 + \frac{q_1}{\sigma} + \frac{q_2}{\sigma^2} + \cdots \right] , \tag{113}$$

where $q_0 = e$, $q_1 = \frac{1}{2}$, and $q_2 = \frac{11}{24}$. More generally we may consider the coefficients q_n to be arbitrary parameters for the expansion of q in powers of σ^{-1}. Then we obtain

$$K_c = K_{c0}(q) \left[1 + \frac{\beta_2'}{\sigma^2} + \frac{\beta_3'}{\sigma^3} + \frac{\beta_4'}{\sigma^4} \cdots \right] , \tag{114}$$

TABLE IV. Results for $K_c(q)$ at fixed q based on results to order σ^{-n}, denoted $K_{cn}(q)$.

	$d=6$	$d=7$	$d=8$	$d=9$
$K_{c0}(q^*)$[a]	0.035 049	0.029 438	0.025 376	0.022 299
$K_{c2}(q^*)$	0.036 084	0.030 376	0.025 983	0.022 714
$K_{c3}(q^*)$	0.037 312	0.030 734	0.026 184	0.022 836
$K_{c4}(q^*)$	0.037 889	0.030 982	0.026 304	0.022 900
$K_c(q^*)$	0.0384	0.0315	0.026 48	0.022 97
$K_{c0}(e)$[b]	0.035 862	0.030 014	0.025 805	0.022 614
$K_{c2}(e)$	0.037 457	0.030 970	0.026 422	0.023 053
$K_{c3}(e)$	0.038 242	0.031 317	0.026 617	0.023 170
$K_{c4}(e)$	0.038 714	0.031 562	0.026 735	0.023 233
$K_c(e)$	0.0395	0.0320	0.026 92	0.023 30
$K_{c0}(0)$[b]	0.090 909	0.076 923	0.066 667	0.058 824
$K_{c2}(0)$	0.090 158	0.076 468	0.066 370	0.058 620
$K_{c3}(0)$	0.089 919	0.076 345	0.066 301	0.058 578
$K_{c4}(0)$	0.089 747	0.076 271	0.066 264	0.058 559
$K_c(0)$	0.0894	0.0762	0.066 23	0.058 543

[a]From Eq. (114).
[b]From Eq. (111).

where

$$\beta'_2=\hat\beta_2(q_0)\,, \tag{115a}$$

$$\beta'_3=\hat\beta_3(q_0)+q_1\left[q\frac{d\hat\beta_2(q)}{dq}\right]_{q=q_0}, \tag{115b}$$

$$\beta'_4=\hat\beta_4(q_0)+q_2\left[q\frac{d\hat\beta_2(q)}{dq}\right]_{q=q_0} +q_1\left[q\frac{d\hat\beta_3(q)}{dq}\right]_{q=q_0}+\tfrac{1}{2}q_1^2\left[q^2\frac{d^2\hat\beta_2}{dq^2}\right]_{q=q_0}. \tag{115c}$$

In evaluating the derivatives with respect to q it is necessary to keep in mind that r_0^0 depends on q via Eq. (110c). The expressions for β'_n are given in Appendix D. The convergence of the expansion can be judged from the numerical results presented in Table IV for K_c as given by Eq. (114).

Finally, we can use this formulation to get the expansion for $q_c(H=1)$, the value of q at the critical point for the case $H=1$. If we knew the value of $q_c(H=1)$, we could calculate K_c either from Eq. (88) or from Eq. (114). By equating these two results we can derive the value of $q_c(H=1)$. That is, we write

$$K_{c0}(q_c(H=1))\left[1+\frac{\beta'_2}{\sigma^2}+\frac{\beta'_3}{\sigma^3}+\frac{\beta'_4}{\sigma^4}\right] =K_{c0}(q^*)\left[1+\frac{\beta_2(H=1)}{\sigma^2}+\frac{\beta_3(H=1)}{\sigma^3}+\frac{\beta_4(H=1)}{\sigma^4}\right], \tag{116}$$

where the $\beta_n(H=1)$ are the coefficients in Eq. (88). For instance, to order σ^{-2} this relation is

$$K_{c0}(q_c(H=1))\left[1+\left[\frac{23}{4}-\frac{1}{e}\right]\frac{1}{\sigma^2}\right] =K_{c0}(q^*)\left[1+\left[\frac{5}{2}-\frac{1}{2e}\right]\frac{1}{\sigma^2}\right], \tag{117}$$

so that

$$[q(H=1)-q^*]\frac{dK_{c0}}{dq}\bigg|_{q^*} =\left[-\frac{13}{4}+\frac{1}{2e}\right]\frac{1}{\sigma^2}K_{c0}(q^*)+\cdots, \tag{118}$$

which gives

TABLE V. Results for q_c at the critical point for $H=1$.

	$d=6$	$d=7$	$d=8$	$d=9$
q_{c0}[a]	2.8531	2.8308	2.8148	2.8028
q_{c2}	2.9977	2.9335	2.8915	2.8623
q_{c3}	3.0893	2.9885	2.9271	2.8866
q_{c4}	3.1684	3.0288	2.9497	2.9002
q_{Pade}[b]	3.5082	3.1164	2.9826	2.9152

[a]Here q_{cn} denotes the result in Eq. (119b) to order σ^{-n}.
[b]From Eq. (120).

$$q_c(H=1)=q^*\left[1+\left(\frac{13}{2}-\frac{1}{e}\right)\frac{1}{\sigma^2}+\cdots\right]. \tag{119a}$$

Note that since we now know $q_c(H=1)$ to order σ^{-2}, we can compute explicit values for q_1 and q_2 in Eq. (115) and therefore also β'_3 and β'_4. To order σ^{-4} this type of calculation yields

$$q_c(H=1)=q^*\left[1+\left(\frac{13}{2}-\frac{1}{e}\right)\frac{1}{\sigma^2}+\left(44-\frac{7}{2e}\right)\frac{1}{\sigma^3}+\left(\frac{3315}{8}-\frac{541}{24e}-\frac{3}{2e^2}\right)\frac{1}{\sigma^4}\right]. \tag{119b}$$

The ratio of successive coefficients in Eq. (119b) is not too different from 7, in accord with the idea[7] that q_c diverges as $d\to4$. To give a rough numerical estimate of q_c we were forced to replace Eq. (119b) by a Pade approximation, in this case of the form

$$q_c(H=1)=q^*\left[\frac{1+\dfrac{a}{\sigma}+\dfrac{b}{\sigma^2}}{1+\dfrac{c}{\sigma}+\dfrac{d}{\sigma^2}}\right]^{1/2}, \tag{120}$$

with the constants in Eq. (120) determined from those in Eq. (119b). Using Eq. (120) we found the estimates for q_c listed in Table V.

VI. THE MOBILITY EDGE FOR THE LOCALIZATION PROBLEM

We write

$$Z=1-NnF,\quad n\to0\,. \tag{121}$$

and the series for F is of the form

$$F(t^2/E^2)=\tfrac{1}{2}\ln\left[\frac{E}{2\pi}\right]-\sum_n c_n(t^2/E^2)^n\,. \tag{122}$$

We get c_n from F by

$$c_n=-\frac{1}{2\pi i}\int\frac{1}{x^{n+1}}F(x)dx\,. \tag{123}$$

We find it convenient to work in terms of Δ. We write Eq. (60b) as

$$\varphi=\frac{1}{2\sigma}[1-(1-4\sigma x)^{1/2}]\,, \tag{124}$$

where $\varphi=\Delta/E$ and $x=t^2/E^2$. We can eliminate x using this relation in the form

$$x=\varphi(1-\sigma\varphi)\,. \tag{125}$$

Then we have

$$c_n=-\frac{1}{2\pi i}\int\frac{d\varphi}{[\varphi(1-\sigma\varphi)]^{n+1}}(1-2\sigma\varphi)F(\varphi)\,. \tag{126}$$

The leading approximation for F given in Sec. IV gives

$$F_0=\tfrac{1}{2}\ln\frac{E}{2\pi}-\frac{\sigma-1}{4}\ln[1-(\sigma+1)\varphi]+\frac{\sigma+1}{4}\ln(1-\sigma\varphi) \tag{127}$$

from which we find for $n\to0$,

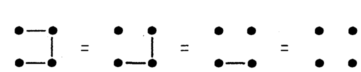

FIG. 6. Evaluation of Eq. (119) for Γ_1 (a square).

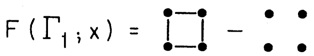

FIG. 7. Contraction of a term in Eq. (119) according to Eq. (58).

$$c_n^0=\frac{\sigma^n(2n-1)!}{2}\sum_{m=0}^{n}\frac{2m+1}{(n+m+1)!(n-m)!}\sigma^{-m}\,. \tag{128}$$

The analog of Eq. (67a) gives

$$x_c^{-1}=\lim_{n\to\infty}(c_n)^{1/n}=4\sigma\,. \tag{129}$$

As before the expansion for F is

$$F(x)=F_0(x)+\sum_{\Gamma}w(\Gamma)F(\Gamma;x) \tag{130a}$$

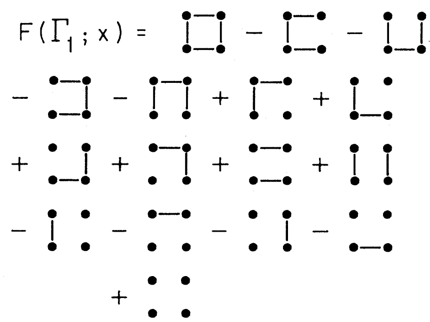

FIG. 8. Result for $F(\Gamma_1 x)$ when the procedure of Fig. 7 is repeatedly performed.

$$=F_0(x)+\sum_{\Gamma}\delta F(\Gamma;x)\,, \tag{130b}$$

where $F(\Gamma;x)$ is the contribution to F from Γ.

$$F(\Gamma;x)=-\lim_{n\to 0}\frac{1}{n}\int\prod_{i\alpha}d\psi_{i\alpha}e^{-E\psi_{i\alpha}^2/2}\prod_i h_i^z\prod_{\langle ij\rangle}\left[\frac{\cosh(t\vec{\psi}_i\cdot\vec{\psi}_j)}{h_ih_j}-1\right], \tag{131}$$

where $\vec{A}\cdot\vec{B}=\sum_\alpha A_\alpha B_\alpha$ and below $A^2=\vec{A}\cdot\vec{A}$. We now analyze F diagrammatically. We represent the factor $\exp(-\frac{1}{2}E\psi_i^2)h_i^z$ by a dot at vertex i and the factor $\cosh(t\vec{\psi}_i\cdot\vec{\psi}_j)/h_ih_j$ by a bond between sites i and j. We expand the product of $n_b(\Gamma)$ factors in the product over $\langle i,j\rangle$ in Eq. (131) into its $2^{n_b(r)}$ terms. Schematically we obtain for the square the result shown in Fig. 6. In all these diagrams we can contract free ends using Eq. (58). Figure 7 shows a typical result of this procedure. Altogether the result for the square is shown in Fig. 8. The generalization of this result is as follows. For diagrams Γ containing at least one bond, we define $G(\Gamma;x)$ by

$$G(\Gamma;x)=-\lim_{n\to 0}\frac{1}{n}\left[\int\prod_{i\alpha}d\psi_{i\alpha}e^{-E\psi_{i\alpha}^2/2}\prod_i h_i^z\prod_{\langle ij\rangle}\frac{\cosh(t\vec{\psi}_i\cdot\vec{\psi}_j)}{h_ih_j}-\int\prod_{i\alpha}d\psi_{i\alpha}e^{-E\psi_{i\alpha}^2/2}\prod_i h_i^z\right]. \tag{132}$$

Then

$$F(\Gamma;x)=G^c(\Gamma;x)\,, \tag{133}$$

where the cumulant $G^c(\Gamma;x)$ is defined recursively in terms of $G(\Gamma;x)$ as in Eq. (3). The second integral in Eq. (132) is found to be

$$\left[\frac{2\pi}{E-z\Delta}\right]^{nn_s(\Gamma)/2}C^{zn_s(\Gamma)}\,, \tag{134}$$

so that

$$G(\Gamma;x)=-\lim_{n\to 0}\frac{\partial}{\partial n}\left[C^{-2n_b(\Gamma)}(E-z\Delta)^{nn_s(\Gamma)/2}\int\prod_i\left(e^{-[E-z\Delta+n_i(\Gamma)\Delta]\psi_i^2/2}\right)\prod_{\langle ij\rangle}\cosh(t\vec{\psi}_i\cdot\vec{\psi}_j)\prod_i D\psi_i\right], \tag{135}$$

where $D\psi_i=\prod_\alpha[d\psi_{i\alpha}/(2\pi)^{1/2}]$ and $n_i(\Gamma)$ is the number of bonds which intersect at site i in the diagram Γ.

As in the animals problems we now perform a bond renormalization. We integrate over all $\vec{\psi}$ pertaining to sites connected to only two bonds. In this way we obtain an indirect interaction $V_{ij}^{(k)}$ for sites i and j which are connected by a chain of $k+1$ bonds. Construction of $V_{ij}^{(k)}$ requires performing the Gaussian integration over variables of k sites of the form

$$V_{ij}^{(k)}=\int \left[\prod_{r=1}^{k} D\psi_r\right]\exp\left[-\tfrac{1}{2}\sum_{rs}\vec{\psi}_r\cdot\vec{\psi}_s M_{rs}\right]e^{-\gamma\vec{\psi}_i\cdot\vec{\psi}_1-\gamma'\vec{\psi}_j\cdot\vec{\psi}_k} . \tag{136}$$

The tridiagonal matrix M is easily inverted. We find

$$G(\Gamma;x)=-\frac{n_s(\Gamma)}{2}\ln(E-z\Delta)+\frac{n_b(\Gamma)}{2}\ln\frac{E-z\Delta}{E-\sigma\Delta}-\lim_{n\to 0}\frac{\partial}{\partial n}\left[\int\prod_i{}' D\psi_i e^{-\psi_i^2[E-z\Delta+n_i(\Gamma)\Delta]/2}\prod_{ij}{}' V_{ij}^{(k)}\right] , \tag{137}$$

where

$$V_{ij}^{(k)}=\exp\left[\frac{\Delta}{2}(\psi_i^2+\psi_i^2)\frac{1-(\Delta/t)^{2k}}{1-(\Delta/t)^{2k+2}}\right]\left[\left[\frac{\Delta}{t^2}\right]^k\frac{1-(\Delta/t)^2}{1-(\Delta/t)^{2k+2}}\right]^{n/2}\cosh\left[t\vec{\psi}_i\cdot\vec{\psi}_j\frac{1-(\Delta/t)^2}{1-(\Delta/t)^{2k+2}}\right] \tag{138}$$

is the effective interaction between sites connected by a chain of length $k+1$ bonds. When $k=0$, $V_{ij}^{(k)}$ reduces to its bare value. In obtaining these results we used Eq. (125) in the form

$$E-z\Delta=(E-\sigma\Delta)-\Delta=(t^2/\Delta)-\Delta . \tag{139}$$

It is convenient to have G in terms of the dimensionless variable y:

$$y=(\Delta/t)^2=\varphi^2/x=\varphi/(1-\sigma\varphi) , \tag{140}$$

whence

$$G(\Gamma;x)=-\tfrac{1}{2}[n_s(\Gamma)-n_b(\Gamma)]\ln(1-y)-\lim_{n\to 0}\frac{\partial}{\partial n}\left[\int\prod_i{}' D\psi_i e^{-\psi_i^2[1-y+n_i(\Gamma)y]/2}\prod_{ij}{}' V_{ij}^{(k)}\right] \tag{141}$$

with

$$V_{ij}^{(k)}=e^{(\psi_i^2+\psi_j^2)(1-Y)/2}Y^{n/2}\cosh(y^{(k+1)/2}Y\vec{\psi}_i\cdot\vec{\psi}_j) , \tag{142}$$

where

$$Y=\frac{1-y}{1-y^{k+1}} . \tag{143}$$

For a polygon of s sides we eliminate all but one site by using $V_{ij}^{(k)}$ with $k=s-1$ and $i=j$. Then

$$G(\Gamma;x)=-\lim_{n\to 0}\frac{\partial}{\partial n}\int D\psi\, e^{-\psi^2(1+y)/2}\left[\frac{1-y}{1-y^s}\right]^{n/2}\exp\left[\psi^2\left[1-\frac{1-y}{1-y^s}\right]\right]\cosh\left[y^{s/2}\frac{1-y}{1-y^s}\psi^2\right] \tag{144a}$$

$$=\tfrac{1}{2}\ln(1-y^s)\equiv G^c(\Gamma;x) . \tag{144b}$$

The second equality in Eq. (144b) results from the fact that there are no nonzero subtractions involved in forming the cumulant $G^c(\Gamma;x)$ when Γ is a polygon. To verify this one has only to observe that $G^c(\Gamma;x)$ vanishes if Γ has any free ends, as is the case for all subdiagrams of a polygon.

We now study the diagrams having the topology of Γ_5. Thus we consider a diagram denoted $\Gamma(k_1,k_2,k_3)$ having two sites connected via three chains of length k_1, k_2, and k_3 bonds. We use the effective interaction for each of the three chains and find thereby

$$G^c(\Gamma;x)=\tfrac{1}{2}\ln(1-y)$$

$$-\lim_{n\to 0}\frac{\partial}{\partial n}\int D\psi_1 D\psi_2\left\{e^{-(\psi_1^2+\psi_2^2)(1+2y)/2}\prod_{i=1}^{3}\left[\left[\frac{1-y}{1-y^{k_i}}\right]^{n/2}\exp\left[\frac{\frac{1}{2}(\psi_i^2+\psi_2^2)(y-y^{k_i})}{1-y^{k_i}}\right]\right.\right.$$

$$\left.\left.\times\cosh\left[y^{k_i/2}\frac{1-y}{1-y^{k_i}}\vec{\psi}_1\cdot\vec{\psi}_2\right]\right]\right\}$$

$$-\tfrac{1}{2}\ln(1-y^{k_1+k_2})-\tfrac{1}{2}\ln(1-y^{k_1+k_3})-\tfrac{1}{2}\ln(1-y^{k_2+k_3})\,, \tag{145}$$

where the last three terms are the lower-order cumulants for the three diagrams found by joining the chains to form polygons. These cumulants are given in Eq. (144b). We rewrite Eq. (145) as

$$G^c(\Gamma;x)=-\ln(1-y)-\tfrac{1}{2}\sum_i \ln\left[\frac{1-y^{k_1+k_2+k_3-k_i}}{1-y^{k_i}}\right]$$

$$+\tfrac{1}{8}\lim_{n\to 0}\sum_{\eta_i}\frac{\partial}{\partial n}\int D\psi_1 D\psi_2$$

$$\times\exp\left[-\tfrac{1}{2}(\psi_1^2+\psi_2^2)\left[2y-2+\sum_i\frac{1-y}{1-y^{k_i}}\right]+\vec{\psi}_1\cdot\vec{\psi}_2\sum_i\eta_i y^{k_i/2}\frac{1-y}{1-y^{k_i}}\right] \tag{146a}$$

$$=\tfrac{1}{16}\sum_{\eta_i=\pm 1}\ln\left[\frac{1-2A+B_1+2B_2-4C}{1-B_1+D-C^2}\right], \tag{146b}$$

where the sum over each $\eta_i=\pm 1$ reproduces the cosh factor in Eq. (145) and

$$C=y^{k_1+k_2+k_3}\,, \tag{147a}$$

$$A=\eta_1\eta_2\eta_3 C^{1/2}\sum_i(\eta_i/y^{k_i/2})\,, \tag{147b}$$

$$B_1=C\sum_i y^{-k_i}\,, \tag{147c}$$

$$D=C\sum_i y^{k_i}\,, \tag{147d}$$

and $B_2=(A^2-B_1)/2$. To low order in y we obtain

$$G^c(\Gamma;x)=-y^{k_1+k_2+k_3}\left[4+\tfrac{3}{2}\sum_i y^{k_i}\right]. \tag{148}$$

Finally, we consider two polygons having perimeters k and l connected by a chain of length m bonds. We write

$$G^c(\Gamma;x)=\tfrac{1}{2}\ln(1-y)-\left\{\lim_{n\to 0}\frac{\partial}{\partial n}\int D\psi_1 D\psi_2 e^{-(\psi_1^2+\psi_2^2)(1+2y)/2}\right.$$
$$\times\left[\frac{(1-y)(1-y)(1-y)}{(1-y^k)(1-y^l)(1-y^m)}\right]^{n/2}\exp\left[\tfrac{1}{2}\psi_1^2\left[3-2\frac{1-y}{1-y^k}-\frac{1-y}{1-y^m}\right]\right]$$
$$\times\exp\left[\tfrac{1}{2}\psi_2^2\left[3-2\frac{1-y}{1-y^l}-\frac{1-y}{1-y^m}\right]\right]\cosh\left[y^{k/2}\frac{1-y}{1-y^k}\psi_1^2\right]$$
$$\left.\times\cosh\left[y^{l/2}\frac{1-y}{1-y^l}\psi_2^2\right]\cosh\left[y^{m/2}\frac{1-y}{1-y^m}\vec{\psi}_1\cdot\vec{\psi}_2\right]\right\}$$
$$-\tfrac{1}{2}\ln(1-y^k)-\tfrac{1}{2}\ln(1-y^l)\,, \tag{149}$$

where we have again used Eq. (144b) for the lower-order cumulants subtracted off in the last two terms of Eq. (149). After some rearrangement we obtain

$$G^c(\Gamma;x)=\tfrac{1}{8}\sum_{\eta_1\eta_2}\ln\left[1-\frac{4y^m\eta_1\eta_2 y^{(k+l)/2}}{(1-\eta_1 y^{k/2})(1-\eta_2 y^{l/2})}\right]. \tag{150}$$

The next step is to obtain the $\delta c_n(\Gamma)$ via Eq. (126). We see that one power of y (or φ) is equivalent to one power of $(1/\sigma)$. Thus where convenient, as in Eq. (148), we have truncated the $G^c(\Gamma)$ into the form given in Table VI. There we also give results pertaining to $\delta c_n(\Gamma)$. Collecting these we have

$$\frac{\delta c_n}{c_n^0}=\sigma^{-2}(n-15)+\sigma^{-3}(n-89)+\sigma^{-4}\left[\frac{n^2}{2}+\frac{51n}{2}\right]+\sigma^{-5}(n^2+159n)\,, \tag{151}$$

from which

$$\lim_{n\to\infty}\frac{1}{n}\ln\frac{c_n}{c_n^0}=\sigma^{-2}+\sigma^{-3}+\frac{81}{2}\sigma^{-4}+263\sigma^{-5}\,, \tag{152}$$

which gives the mobility energy E_c as

$$E_c^2=4\sigma t^2\left[1+\frac{1}{\sigma^2}+\frac{1}{\sigma^3}+\frac{41}{\sigma^4}+\frac{264}{\sigma^5}+\cdots\right]. \tag{153}$$

Values of E_c are given in Table VII as a function of dimension for various orders of Eq. (153). Also shown, for comparison, is the exact band edge for this model at $E=zt$. From the ratio of the last two terms one might guess that this series would diverge at $\sigma\sim 6$ which gives $d=2.5$. It is believed[18] that $d=2$ is the marginal dimensionality for localization. Normally, one would expect the series for the critical value of the coupling constant to diverge at low dimension, but here the series for (t^2/E^2), the reciprocal of that in Eq. (153), shows no hint of growing unbounded positively as d decreases. Perhaps at some $d=d^*$ the mobility edge merges with the band edge and the transition changes its character at d^*. Harris and Lubensky[10] find special effects at $d=4$, and Lubensky and McKane[19] predict qualitative changes in the nature of the localization transition at $d=4\frac{2}{3}$, but our series is too irregular to give conclusive enough results for us to comment on this. One measure of the uncertainty in our result is to compare the result to order σ^{-5} with that obtained by adding a term in σ^{-6} such that $F_c^2=4\sigma t^2$ for $\sigma=1$, as it should because no closed loops can be formed in one dimension. This *ad hoc* construction gives

$$E_c^2=4\sigma t^2\left[1+\frac{1}{\sigma^2}+\frac{1}{\sigma^3}+\frac{41}{\sigma^4}+\frac{264}{\sigma^5}-\frac{308}{\sigma^6}\right]. \tag{154}$$

As can be seen from Table VII, E_c seems to be accurately given for d larger than, say, 4, and the difference between E_c and the exact band edge is striking for large d.

TABLE VI. Diagrammatic results for localization to order σ^{-5}.

Γ	$-G^c(\Gamma)$	$\dfrac{\delta c_n(\Gamma)^a}{\sigma^n}$	$\dfrac{\delta c_n(\Gamma)}{c_n^0}$
1	$\frac{1}{2}w(\Gamma_1)y^4$	$\left[\dfrac{1}{2\sigma^2}-\dfrac{1}{\sigma^4}\right]f_4$	$\dfrac{(n-15)}{\sigma^2}-\dfrac{(3n-51)}{\sigma^3}+\dfrac{3n}{\sigma^4}-\dfrac{n}{\sigma^5}$
2	$\frac{1}{2}w(\Gamma_2)y^6$	$\left[\dfrac{2}{\sigma^3}-\dfrac{9}{2\sigma^4}-\dfrac{2}{\sigma^5}\right]f_6$	$\dfrac{4(n-35)}{\sigma^3}-\dfrac{21n}{\sigma^4}+\dfrac{39n}{\sigma^5}$
3	$\frac{1}{2}w(\Gamma_3)y^8$	$\left[\dfrac{27}{2\sigma^4}-\dfrac{77}{\sigma^5}\right]f_8$	$\dfrac{27n}{\sigma^4}-\dfrac{235n}{\sigma^5}$
4	$\frac{1}{2}w(\Gamma_4)y^{10}$	$\dfrac{124}{\sigma^5}f_{10}$	$\dfrac{248n}{\sigma^5}$
5	$w(\Gamma_5)(4y^7+\frac{3}{2}y^8)$	$\left[\dfrac{2}{\sigma^4}-\dfrac{4}{\sigma^5}\right]\left[7f_7+\dfrac{3}{\sigma}f_8\right]$	$\dfrac{28n}{\sigma^4}-\dfrac{128n}{\sigma^5}$
6	$4w(\Gamma_6)y^8$	$\dfrac{32}{\sigma^5}f_8$	$\dfrac{64n}{\sigma^5}$
7	$4w(\Gamma_7)y^9$	$\dfrac{144}{\sigma^5}f_9$	$\dfrac{288n}{\sigma^5}$
8	$4w(\Gamma_8)y^9$	$\dfrac{6}{\sigma^5}f_9$	$\dfrac{12n}{\sigma^5}$
9[b]	$\left[1-\dfrac{3}{\sigma}\right]\dfrac{\sigma^4y^8}{4}$	$\dfrac{n+40}{4\sigma^4}\left[1-\dfrac{3}{\sigma}-\dfrac{48}{n\sigma}\right]f_8$	$\dfrac{n}{2\sigma^4}(n-23)$
	$+\left[\dfrac{3-2\sigma y}{1-\sigma y}+2\sigma y^2-\dfrac{3}{\sigma}\right]$	$+\dfrac{10}{\sigma^5}f_{10}$	$+\dfrac{n}{\sigma^5}(-3n+68)$
10[b]	$\dfrac{2\sigma^5y^{10}}{1-\sigma y}(3-2\sigma y)$	$\dfrac{2}{\sigma^5}(n+50)f_{10}$	$\dfrac{4n}{\sigma^5}(n-49)$

[a]Here $f_k=[(2n-1)!]/[(n-k)!][(n+k)!]$.
[b]The results for Γ_9 and Γ_{10} are for the sum over $m\geq 0$ of all diagrams in the respective family.

VII. CONCLUSIONS

We state here the major results of our work.

(1) We have presented a powerful method of constructing a site potential h which eliminates diagrams with free ends from the diagrammatic expansion of thermodynamic functions.

(2) In the cases considered here, we cannot only construct h, but we can also subsequently perform a bond renormalization whereby the sites in a chain connecting sites i and j are eliminated and are replaced by an effective interaction V_{ij} between sites i and j.

(3) For the Cayley tree, on which all diagrams have free ends, the effective potential h leads to the exact solution of the problem. One new result given here is the exact solution for the dilute polymer problem on the Cayley tree for general values of the fugacities for f-functional units (i.e., for vertices for f bonds come together). As expected, our result, Eq. (43), is quite similar to the mean-field approximation of Lubensky and Isaacson.[7]

TABLE VII. Numerical results for E_c^2/t^2 to nth order in $1/\sigma$, denoted E_{cn}^2/t^2.

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
E_{c0}^2/t^2	12.000	20.000	28.000	36.000	44.000	52.000	60.000
E_{c2}^2/t^2	13.333	20.800	28.571	36.444	44.364	52.308	60.267
E_{c3}^2/t^2	13.778	20.960	28.653	36.494	44.397	52.331	60.284
E_{c4}^2/t^2	19.852	22.272	29.131	36.719	44.520	52.406	60.333
E_{c5}^2/t^2	32.889	23.962	29.571	36.880	44.592	52.443	60.354
$E_{c6}^2/t^{2\,a}$	27.819	23.567	29.498	36.859	44.584	52.440	60.352
Band Edge[b]	16.000	36.000	64.000	100.000	144.000	196.000	256.000

[a]See Eq. (153).
[b]$E_c^2/t^2=4d^2$.

(4) Using the above renormalizations we are able to construct terms in the series for the critical coupling constant to order $(1/\sigma)^5$ in terms of diagrams involving at most two sites. We have studied three problems: dilute polymers (animals) at constant H, dilute polymers at constant Q, and localization, where H is the fugacity for free ends and Q is the density of free ends. For animals we obtain series for K_c, the critical monomer fugacity and for localization we obtain the series for t/E_c where E_c is the mobility energy and t is the magnitude of the hopping matrix element whose sign is random. We give results to order $(1/\sigma)^5$, except for the constant Q case, where our results go to order σ^{-4}. The only previous work on these expansions[5] was for K_c at constant H, where an effort comparable to ours produced an approximation to the leading correction of order $(1/\sigma)^2$.

(5) Since for animals we treat the critical point both when H is constant and when q is constant, we are able to obtain an expansion for $q_c(H=1)$, the value of q at the critical point for $H=1$.

(6) Since we can construct the effective interaction obtained by renormalizing chains, we are studying its use in the renormalization-group calculation at low spatial dimension based on an approach such as the Migdal-Kadanoff bond-moving scheme.[15] This program would be very interesting, since an important open question is whether or not localization and animals are in the same universality class for all dimensions.

APPENDIX A: CUMULANTS

The purpose of this appendix is to clarify the relation in terms of cumulants between the animals problem and percolation. We define the percolation susceptibility for the system Γ as

$$\chi_p(\Gamma)=\sum_i\sum_j(1-\delta_{ij})\chi_p(\Gamma;i,j)\ , \qquad \text{(A1)}$$

where $\chi_p(\Gamma;i,j)$ is the probability that sites i and j are connected when each bond of the system Γ is randomly present with probability p and absent with probability $1-p$. Also the animals susceptibility for the system Γ, $\chi_A(\Gamma)$ will be defined here as

$$\chi_A(\Gamma)=\sum_{n_b}\sum_{n_s}A(n_b,n_s;\Gamma)K^{n_b}n_s(n_s-1)\ , \qquad \text{(A2)}$$

where $A(n_b,n_s;\Gamma)$ is the number of ways a cluster of n_b bonds and n_s sites can be formed on the set Γ. For our purposes it is convenient not to include the $i=j$ term in Eq. (A1) and to use $n_s(n_s-1)$ rather than n^2 in Eq. (A2). It is helpful to give some concrete results for simple diagrams, such as those in Fig. 9, for which the various susceptibilities are listed in Table VIII. According to Eq. (2) the values of χ_p or χ_A for the infinite lattice can

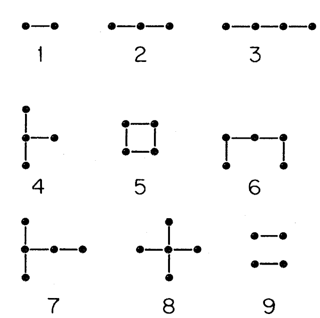

FIG. 9. A few low-order diagrams.

TABLE VIII. Various susceptibilities for small diagrams.

Γ[a]	$\chi_A(\Gamma)$	$\chi_A^c(\Gamma)$	$\chi_p(\Gamma)$	$\chi_p^c(\Gamma)$
1	$2K$	$2K$	$2p$	$2p$
2	$6K^2+4K$	$6K^2$	$6p^2+4p(1-p)$	$2p^2$
3	$12K^3+12K^2+6K$	$12K^3$	$12p^3+16p^2(1-p)$ $+6p(1-p)^2$	$2p^3$
4	$12K^3+18K^2+6K$	$12K^3$	$12p^3+18p^2(1-p)$ $+6p(1-p)^2$	0
5	$12K^4+48K^3$ $+24K^2+8K$	$12K^4$	$12p^4+48p^3(1-p)$ $+32p^2(1-p)^2+8p(1-p)^3$	$-12p^4$
6	$20K^4+24K^3$ $+18K^2+8K$	$20K^4$	$20p^4+40p^3(1-p)$ $+30p^2(1-p)^2+8p(1-p)^3$	$2p^4$
7	$20K^4+36K^3$ $+24K^2+8K$	$20K^4$	$20p^4+44p^3(1-p)$ $+32p^2(1-p)^2+8p(1-p)^3$	0
8	$20K^4+48K^3$ $+36K^2+8K$	$20K^4$	$20p^4+48p^3(1-p)$ $+36p^2(1-p)^2+8p(1-p)^3$	0
9	$4K$	0	$4p(1-p)+4p^2$	0

[a] See Fig. 9.

be obtained in terms of the cumulants of the susceptibilities which we also give in Table VIII.

To see what $\chi_A^c(\Gamma)$ is we temporarily replace K by an independent variable K_i for each bond i. The property of cumulants ensures that $\chi_A^c(\Gamma)$ vanishes if any K_i is zero. Thus $\chi_A^c(\Gamma)$ cannot contain terms in K of lower order than $K^{n_b(\Gamma)}$. Nor are higher-order terms permitted. Hence we see that for a connected cluster Γ we have

$$\chi_A^c(\Gamma)=n_s(\Gamma)[n_s(\Gamma)-1]K^{n_b(\Gamma)} . \tag{A3}$$

As we shall see $\chi_p^c(\Gamma)$ is not obtained from Eq. (A3) simply by replacing K by p. Reasoning similar to that leading to Eq. (A3) is needed to derive Eq. (74a) or (119a) in case one does not regard them as being obvious.

Now consider $\chi_p^c(\Gamma)$. If we temporarily replace p by independent variables p_i for each bond i, then the cumulant property ensures that $\chi_p^c(\Gamma)$ vanishes if any p_i is zero. As before, this argument tells us that $\chi_p^c(\Gamma)$ is proportional to $p^{n_b(\Gamma)}$, so that

$$\chi_p^c(\Gamma)=e(\Gamma)p^{n_b(\Gamma)} , \tag{A4}$$

where we need to determine $e(\Gamma)$. The coefficient of $p^{n_b(\Gamma)}$ is of course zero for subdiagrams of Γ. So $e(\Gamma)$ is simply the coefficient of $p^{n_b(\Gamma)}$ in $\chi_p(\Gamma)$. Considering the factors p and $(1-p)$ it is obvious that

$$e(\Gamma)=\sum_{\gamma\in\Gamma}(-1)^{n_b(\Gamma)-n_b(\gamma)}\chi_p(\gamma,p=1) . \tag{A5}$$

This relation indicates that

$$e(\Gamma)=\chi_p^c(\Gamma;p=1) , \tag{A6}$$

and in particular, $e(\Gamma)$ vanishes if Γ is disconnected. When Γ is connected, then

$$\chi_p(\Gamma;p=1)=n_s(\Gamma)[n_s(\Gamma)-1]$$

and $\chi_p^c(\Gamma,p=1)$ is the cumulant $\{n_s(\Gamma)[n_s(\Gamma)-1]\}^c$. Thus we have established that

$$\chi_A^c(C)=K^{n_b(C)}n_s(C)[n_s(C)-1] , \tag{A7a}$$

$$\chi_p^c(\Gamma)=p^{n_b(\Gamma)}\{n_s(\Gamma)[n_s(\Gamma)-1]\}^c , \tag{A7b}$$

where C is a connected cluster and Γ is arbitrary. Thus the coefficient of p^n in the percolation susceptibility is the cumulant, derived from the coefficient of K^n in the animals susceptibility.

APPENDIX B: $\delta c_n(\Gamma)$ FOR ANIMALS

In this appendix we will obtain the contributions to $\delta c_n(\Gamma)/c_n^0$ for the diagrams shown in Figs. 2–4. For diagrams $\Gamma_1,\Gamma_2,\ldots,\Gamma_8$ we express $\delta F(\Gamma)$ as a sum of terms, each of which is of the form

$$-\delta f(r,s)\equiv b^r(1+b)^{\sigma s} . \tag{B1}$$

We denote the corresponding contribution to $\delta c_n(\Gamma)$ as $\delta c_n(r,s)$. Use of Eq. (70) yields

$$\delta c_n(r,s)=\frac{\sigma(r+s)(\sigma n+\sigma s-1)!}{(n-r)!(\sigma n+\sigma s-n+r)!} . \tag{B2}$$

We are interested in the quantity $\delta c_n(r,s)/c_n^0$ which we write as

$$\frac{\delta c_n(r,s)}{c_n^0}=\frac{\sigma(r+s)\left[1+\dfrac{s}{n}\right]^{n-r-1}(n+1)!}{(\sigma+1)(\sigma n)^r\left[1+\dfrac{1}{n}\right]^{n-1}(n-r)!}\prod_{j=1}^{n-r-1}\left[1-\frac{j}{\sigma(n+s)}\right]\prod_{j=1}^{n-2}\left[1-\frac{j}{\sigma(n+1)}\right]^{-1} . \tag{B3}$$

So far we have not dropped any terms. We now carry out the expansion in powers of $1/\sigma$ and for each coefficient we arrange the result in descending powers of n. To obtain the final result for K_c/K_c^0 to order σ^{-5} we need keep only terms of the order kept in Eq. (86). Thus we write

$$\frac{\delta c_n(r,s)}{c_n^0}=\frac{n(r+s)e^{s-1}e^{-(s-1)\sigma[\ln(1-1/\sigma)+1/\sigma]}}{\sigma^r\left[1+\dfrac{1}{\sigma}\right]\left[-\dfrac{1}{\sigma}\right]^{r-1}}$$
$$\times\left[1-\frac{1}{2n}[(r+s)^2+2s-5]-\frac{1}{2n\sigma}(r+s+3)(r+s-2)-\cdots\right] , \tag{B4}$$

where the omitted terms in the last set of large parentheses of order $n^{-u}\sigma^{-v}$ with either $u\geq 2$ or $u=1$, $v\geq 2$ are not needed here. Note that $\delta c_n(r,s)/c_n^0$ is of order σ^{-r}. In view of Eq. (B1) this result indicates that we may consider b to be of order $1/\sigma$. Accordingly, we may expand $\delta F(\Gamma)$ in power of b so that each term is of the form considered in Eq. (B1). Thus we write

$$-\delta F(\Gamma_1)=b^4(1-4b+10b^2-20b^3+\cdots)[1-4(1+b)^\sigma] , \tag{B5}$$

so that

$$\frac{\delta c_n(\Gamma_1)}{c_n^0}=\frac{w(\Gamma_1)}{c_n^0}\{[\delta c_n(4,0)-4\delta c_n(5,0)+10\delta c_n(6,0)-20\delta c_n(7,0)]$$
$$-4[\delta c_n(4,1)-4\delta c_n(5,1)+10\delta c_n(6,1)-20\delta c_n(7,1)]\} . \tag{B6}$$

In this way we obtain the following results:

$$\frac{\delta c_n(\Gamma_1)}{c_n^0}=\frac{n}{2e\sigma^2}\left[1-\frac{7}{2\sigma}+\frac{103}{24\sigma^2}-\frac{95}{48\sigma^3}\right]+\frac{1}{e\sigma^2}\left[-\frac{7}{4}+\frac{101}{8\sigma}\right]$$
$$-\frac{n}{2\sigma^2}\left[5-\frac{14}{\sigma}+\frac{13}{\sigma^2}-\frac{4}{\sigma^3}\right]+\frac{1}{\sigma^2}\left[\frac{45}{2}-\frac{93}{\sigma}\right] , \tag{B7a}$$

$$\frac{\delta c_n(\Gamma_2)}{c_n^0}=\frac{2n}{e\sigma^3}\left[1-\frac{23}{4\sigma}+\frac{73}{6\sigma^2}\right]-\frac{25}{e\sigma^3}-\frac{2n}{\sigma^3}\left[7-\frac{143}{4\sigma}+\frac{64}{\sigma^2}\right]+\frac{280}{\sigma^3} , \tag{B7b}$$

$$\frac{\delta c_n(\Gamma_3)}{c_n^0}=\frac{n}{2e\sigma^4}\left[27-\frac{497}{2\sigma}\right]-\frac{n}{2\sigma^4}\left[243-\frac{2088}{\sigma}\right] , \tag{B7c}$$

$$\frac{\delta c_n(\Gamma_4)}{c_n^0}=\frac{124n}{e\sigma^5}-\frac{1364n}{\sigma^5} , \tag{B7d}$$

$$\frac{\delta c_n(\Gamma_5)}{c_n^0}=\frac{3n}{2e^2\sigma^4}\left[1-\frac{22}{3\sigma}\right]-\frac{35n}{2e\sigma^4}\left[1-\frac{63}{10\sigma}\right]+\frac{60n}{\sigma^4}\left[1-\frac{27}{5\sigma}\right], \tag{B7e}$$

$$\frac{\delta c_n(\Gamma_6)}{c_n^0}=\frac{n}{2\sigma^5}\left[\frac{7}{e^2}-\frac{88}{e}+333\right], \tag{B7f}$$

$$\frac{\delta c_n(\Gamma_7)}{c_n^0}=\frac{8n}{\sigma^5}\left[\frac{2}{e^2}-\frac{27}{e}+105\right], \tag{B7g}$$

$$\frac{\delta c_n(\Gamma_8)}{c_n^0}=\frac{n}{3\sigma^5}\left[\frac{2}{e^2}-\frac{27}{e}+115\right]. \tag{B7h}$$

For $\Gamma(9^{(m)})$ we write correctly to order σ^{-5}

$$\sum_{m=0}^{\infty}\delta c_n(\Gamma_9^{(m)})=\frac{1}{2\pi i}\int\frac{db}{b^{n+1}}(1+b)^{\sigma n-1}[1-(\sigma-1)b]$$
$$\times\sum_{m=0}^{\infty}\sigma^{m+8}\left[\frac{1}{8\sigma^4}\left[1-\frac{3}{\sigma}\right]-\frac{1}{8\sigma^5}\delta_{m0}\right]\left[\frac{b}{1+b}\right]^{m+8}\frac{1}{(1+b)^{\sigma+1}}Z^2, \tag{B8}$$

where

$$Z=[1+(5+b)(1+b)^{\sigma}]. \tag{B9}$$

The term in δ_{m0} gives a contribution, $\delta c_n^{(a)}(\Gamma_9^{(m)})$, which is evaluated using Eq. (B4) as

$$\frac{\delta c_n^{(a)}(\Gamma_9^{(m)})}{c_n^0}=-\frac{n}{\sigma^5}\left[\frac{7}{e^2}-\frac{80}{e}+225\right]. \tag{B10}$$

The remaining contributions, denoted $\delta c_n^{(b)}(\Gamma_9^{(m)})$ give a geometric series which, when summed, lead to a factor $[1-(\sigma-1)b]^{-1}$, which is essentially $\chi_0(b)$. This is to be expected, since the no-loop approximation for the two-point function $\chi_0(b)$ involves a sum over all chain diagrams. Thus we obtain

$$\delta c_n^{(b)}(\Gamma_9^{(m)})=\frac{1}{2\pi i}\int\frac{db}{b^{n-7}}(1+b)^{\sigma n-\sigma-10}\frac{\sigma^3}{8}Z^2(\sigma-3). \tag{B11}$$

Since the leading contribution from Γ_9 is of order σ^{-4}, we need only keep corrections of order $1/\sigma$ relative to the leading term. However, terms of order n^2 and of order n need to be retained. The calculations are straightforward but tedious. The result to order $(1/\sigma)^5$ is

$$\sum_{m=0}^{\infty}\frac{\delta c_n(\Gamma_9^{(m)})}{c_n^0}=\frac{n^2}{2}\left[\frac{5}{2\sigma^2}-\frac{6}{\sigma^3}-\frac{1}{2e\sigma^2}+\frac{7}{4e\sigma^3}\right]^2$$
$$+\frac{n}{16\sigma^4}\left[-1700+\frac{510}{e}-\frac{36}{e^2}+\frac{10840}{\sigma}-\frac{3801}{e\sigma}+\frac{314}{e^2\sigma}\right]. \tag{B12}$$

Similarly we find

$$\sum_{m=0}^{\infty}\frac{\delta c_n(\Gamma_{10}^{(m)})}{c_n^0}=\frac{n^2}{\sigma^5}\left[\frac{2}{e}-14\right]\left[\frac{1}{2e}-\frac{5}{2}\right]+\frac{n}{\sigma^5}\left[-\frac{33}{e^2}+\frac{510}{e}-1855\right]. \tag{B13}$$

APPENDIX C: EXPLICIT CALCULATIONS FOR CONSTANT q

The calculation of $f(b,K,\Gamma)$ are similar to, but more complicated than that for $H=1$. We start by constructing the effective interaction from Eq. (79). To eliminate H we write

$$\mathrm{Tr} S_i^\alpha h_i^{\sigma-1} = h + (1+b)^{\sigma-1} - 1 \tag{C1a}$$

$$= \frac{b}{K} - b(1+b)^{\sigma-1} , \tag{C1b}$$

where we used the trace rules to obtain the first line and Eq. (90) for the last line. Likewise we find

$$\mathrm{Tr}(S_i^\alpha)^k h_i^{\sigma-1} = (1+b)^{\sigma-1} , \quad k>1 . \tag{C1c}$$

We use these results to evaluate the right-hand side of Eq. (79), obtaining

$$V_{ij}^{(k)} = [K(1+b)^{\sigma-1}]^k \left[K \sum_\alpha S_i^\alpha S_j^\alpha - b \sum_\alpha (S_i^\alpha + S_j^\alpha) + \left[(k-1)b^2 - \frac{kb^2}{K(1+b)^{\sigma-1}} \right] \sum_{\alpha\beta} S_i^\alpha S_j^\beta + n \frac{b^2}{K} \delta_{ij} \right] , \tag{C2}$$

as the effective interaction between sites i and j via a chain of k sites. The term in δ_{ij} is needed for the treatment of polygons. As a check, we note that for $k=0$ the bare interaction is recovered, and for $H=1$ Eq (80) is reproduced.

For the polygon with s sides we use $V_{ij}^{(k)}$ with $k=s-1$ in Eq. (81) and in analogy with Eq. (82) we find

$$-F(\Gamma;b,K) = (KX)^{s-1}(K^2X + sb^2KX - sb^2)/K , \tag{C3}$$

where $X=(1+b)^{\sigma-1}$. As will become apparent, the present calculation is much more intricate than the one for $H=1$. Hence we only work to order σ^{-4}. In that case the two other diagrams whose evaluation are required are Γ_5 and $\{\Gamma_9^{(m)}\}$, for which we use Eq. (83), but with V replaced by the interaction of Eq. (C2). Since these diagrams only contribute to order σ^{-4}, we may drop correction terms involving powers of b. Thus we set

$$(1+b) \sim 1 , \tag{C4a}$$

$$\mathrm{Tr} S_i^\alpha h_i^{\sigma-1} \sim \frac{b}{K} , \tag{C4b}$$

$$\mathrm{Tr}(S_i^\alpha)^2 h_i^{\sigma-2} \sim X , \tag{C4c}$$

Then the analog of Eq. (83A) yields

$$-F(\Gamma_5;K,b) = \left\{ X^2 \left[\left[\frac{K}{b} \right]^3 - 6 \left[\frac{K}{b} \right]^2 + 6 \left[\frac{K}{b} \right] \right] + X \left[-4 \left[\frac{K}{b} \right] + 20 \right] + 4 \left[\frac{b}{K} \right] \right\} b^3 K^4 X^4 . \tag{C5}$$

The analog of Eq. (83b) yields

$$\sum_{m=0}^{\infty} -\delta F(\Gamma_9^{(m)};K,b) = \tfrac{1}{8} \chi_0 \sigma^3 K^5 X^7 \left[K - 2b - 3 \left[\frac{b^2}{KX} \right] \right]^2 . \tag{C6}$$

If we were to obtain a term on the right-hand side of Eq. (103b) of order χ_0, then the perturbative treatment we are using would not work. The offending term in Eq. (C6) drops out when we calculated $\delta\Omega_9$ defined as

$$\delta\Omega_9 = -\left[\sum_{m=0}^{\infty} \frac{K}{\sigma+1} \delta F(\Gamma_9^{(m)};K,b) \right] - \frac{K}{2(\sigma+1)^3} \chi_0(K,b) \left[\frac{\partial \delta F(\Gamma_1;K,b)}{\partial b} \right]^2 . \tag{C7}$$

We work to leading order in σ^{-1}. Then we have

$$-\frac{\partial \delta F(\Gamma_1;K,b)}{\partial b} = \tfrac{1}{2}\sigma^2(\sigma K^4 X^4 - 2bK^2X^3 - 3\sigma b^2 K^2 X^3) . \tag{C8}$$

Thus we finally get

$$\delta\Omega_9 = \sigma^2 K^5 X^5 [\sigma(K^2X^2 - 3b^2)^2 - 4b^2 KX]/8 . \tag{C9}$$

We now collect the contributions to Σ using the results of Eqs. (C3), (C5), and (C9). We then obtain Eq. (104) of Sec. V, where the coefficients are

$$\gamma_2=(KX\sigma)^3[2rKX\sigma-(1+6rq+\tfrac{9}{2}r^2q^2)]\ , \tag{C10a}$$

$$\begin{aligned}\gamma_3=&(KX\sigma)^3(1+12rq+21r^2q^2+9r^3q^3)+(KX\sigma)^4[-r(\tfrac{13}{2}+4rq)+(1+8rq+8r^2q^2)]\\&+(KX\sigma)^5[12rKX\sigma-(4+40rq+50r^2q^2)]\ ,\end{aligned} \tag{C10b}$$

$$\begin{aligned}\gamma_4=&(XK\sigma)^3(-6rq-\tfrac{57}{2}r^2q^2-36r^3q^3-\tfrac{27}{2}r^4q^4)+(XK\sigma)^4(-1-16rq-22r^2q^2+16r^3q^3+16r^4q^4)\\&+r(XK\sigma)^4(7+13rq+6r^2q^2)+r(XK\sigma)^5[-(2+20rq+25r^2q^2)]\\&+(XK\sigma)^5(13+200rq+476r^2q^2+270r^3q^3+\tfrac{225}{8}r^4q^4)\\&+(XK\sigma)^6[9r^2-r(\tfrac{169}{2}+96rq+27r^2q^2)+(6+72rq+108r^2q^2)]\\&+(XK\sigma)^7[\tfrac{49}{8}r^2-(27+378rq+\tfrac{1323}{2}r^2q^2)]+108r(XK\sigma)^8\ ,\end{aligned} \tag{C10c}$$

where $r=K\sigma$. For further reference we write

$$K_c\left.\frac{\partial\gamma_2}{\partial K_c}\right|_{K_{c0}}=r_0(10+8r_0q)-(3+27r_0q+\tfrac{81}{2}r_0^2q^2+\tfrac{27}{2}r_0^3q^3)\ , \tag{C11}$$

where we have noted that $XK\sigma=1$ when $K=K_{c0}$.

APPENDIX D: EXPANSION COEFFICIENTS FOR CONSTANT q

Here we record the explicit results for the coefficients introduced in Sec. V in Eqs. (108), (111), and (114). We substitute Eqs. (106) and (107) into Eq. (102). In view of Eq. (105) we may write

$$\Gamma_0(q,K_c)=K_{c0}\frac{\partial K_{c0}}{\partial K_c}\left[\frac{\beta_2}{\sigma^2}+\frac{\beta_3}{\sigma^3}+\frac{\beta_4}{\sigma^4}\right]+\frac{\beta_2^2K_{c0}^2}{2\sigma^4}\frac{\partial^2\Gamma_0}{\partial K_c^2}+\cdots\ , \tag{D1}$$

and from Eqs. (103a) and (51c) we have

$$K_c\left.\frac{\partial\Gamma_0}{\partial K_c}\right|_{K_{c0}}=-r_0(1+qK_{c0})^{\sigma-2}(1+\sigma K_{c0}q) \tag{D2a}$$

$$\sim-(1+r_0q)\left[1-\frac{r_0q}{\sigma}+\frac{r_0^2q^2}{\sigma^2}+\cdots\right] \tag{D2b}$$

and

$$K_{c0}^2\left.\frac{\partial^2\Gamma_0}{\partial K^2}\right|_{K_{c0}}=-r_0^2q(1+qK_{c0})^{\sigma-3}\left[2+\sigma K_{c0}q-\frac{2}{\sigma}-K_{c0}q\right] \tag{D3a}$$

$$\sim-r_0q(2+r_0q)\ . \tag{D3b}$$

Substituting Eqs. (D1) and (107) into Eq. (102) we find that

$$\beta_2=-\gamma_2D\ , \tag{D4a}$$

$$\beta_3=\beta_2r_0q-\gamma_3D\ , \tag{D4b}$$

$$\beta_4=\beta_3r_0q-\beta_2r_0^2q^2-\tfrac{1}{2}\beta_2^2r_0q(1+D)-\gamma_4D-\beta_2K_cD\frac{\partial\gamma_2}{\partial K_c}\ , \tag{D4c}$$

where $D=(1+r_0q)^{-1}$. Using the results of Appendix C for the γ_n's we then obtain Eq. (108).

Substituting Eq. (106) into Eq. (112) yields

$$\hat{\beta}_2=-2r_0^0D_0+\tfrac{9}{2}r_0^0q3+\tfrac{3}{2}-\tfrac{1}{2}D_0\ , \tag{D5a}$$

$$\hat{\beta}_3=r_0^0[2-\tfrac{17}{2}D_0+D_0^3]-\tfrac{9}{4}(r_0q)^2+\tfrac{135}{4}r_0^0q-\tfrac{25}{2}+\tfrac{29}{2}D_0-\tfrac{1}{4}D_0^2+\tfrac{1}{4}D_0^3\ , \tag{D5b}$$

$$\begin{aligned}\hat{\beta}_4=&(r_0^0)^2(2D_0^3+4D_0^2-\tfrac{9}{8}D_0)+\frac{r_0^0}{12}(-36r_0^0q+44-654D_0+12D_0^2+67D_0^3+6D_0^4-9D_0^5)\\&+\tfrac{1}{48}[348(r_0^0q)^3-2457(r_0^0q)^2-18955r_0^0q-11055+11625D_0+317D_0^2-317D_0^3+15D_0^4-9D_0^5]\ ,\end{aligned} \tag{D5c}$$

where $D_0=(1+r_0^0q)^{-1}$.

Substituting these results into Eq. (115) yields

$$\beta'_2=\hat{\beta}_2(q_0)\ , \tag{D6a}$$

$$\beta'_3=\hat{\beta}_3(q_0)+2r_0^0(q_0)q_1(D_{00}-D_{00}^3)+\tfrac{1}{2}q_1(9-9D_{00}+D_{00}^2-D_{00}^3)\ , \tag{D6b}$$

$$\begin{aligned}\beta'_4=&\hat{\beta}_4(q_0)+2q_2r_0^0(q_0)(D_{00}-D_{00}^3)+\tfrac{1}{2}q_2(9-9D_{00}+D_{00}^2-D_{00}^3)\\&+q_1r_0^0(q_0)(-2+\tfrac{21}{2}D_{00}-\tfrac{19}{2}D_{00}^3-2D_{00}^4+3D_{00}^5)\\&+q_1[-\tfrac{9}{2}r_0^0(q_0)q_0+\tfrac{153}{4}-\tfrac{153}{4}D_{00}-\tfrac{29}{2}D_{00}^2+15D_{00}^3-\tfrac{5}{4}D_{00}^4+\tfrac{3}{4}D_{00}^5]\\&-q_1^2r_0^0(q_0)(2D_{00}-3D_{00}^3-2D_{00}^4+3D_{00}^5)+\tfrac{1}{4}q_1^2(-9+9D_{00}+8D_{00}^2-10D_{00}^3+5D_{00}^4-3D_{00}^5)\ .\end{aligned} \tag{D6c}$$

Here $D_{00}=[1+r_0^0(q_0)q_0]^{-1}$.

APPENDIX E: THE ISING MODEL SUSCEPTIBILITY

In this appendix we apply the technique of eliminating free ends to the Ising model. For this model it is appropriate to study the susceptibility rather than the free energy, since the result for the free energy of the Cayley tree is not a good starting point for the method. Thus we consider the Ising model in a magnetic field, in which case Eq. (15) (in units where $k_BT=1$) is

$$h_i=\mathrm{Tr}_j e^{H\sigma_i}h_j^{z-1}e^{J\sigma_i\sigma_j}/\mathrm{Tr}_j e^{H\sigma_i}h_j^z\ , \tag{E1}$$

which has a solution of the form

$$h_i=Ae^{B\sigma_i}\ . \tag{E2}$$

By substituting this ansatz into Eq. (E1) one finds that A and B satisfy

$$A\cosh B=\frac{\cosh J\cosh(H+\sigma B)}{A\cosh(H+zB)}\ , \tag{E3a}$$

$$A\sinh B=\frac{\sinh J\sinh(H+\sigma B)}{A\cosh(H+zB)}\ , \tag{E3b}$$

where $\sigma=z+1$. We work to order H^2 and write

$$B=bH+O(H^3)\ , \tag{E4a}$$

$$A=(\cosh J)^{1/2}(1+\tfrac{1}{2}aH^2)+O(H^4)\ . \tag{E4b}$$

Use of Eq. (E3) leads to

$$a=-t(1+t)/(1-\sigma t)^2\equiv-t(1+t)\chi_0^2\ , \tag{E5a}$$

$$b=t\chi_0\ , \tag{E5b}$$

where t denotes $\tanh J$.

We now evaluate

$$\chi=-\frac{\partial^2F}{\partial H^2}=-\frac{\partial^2}{\partial H^2}\sum_\Gamma w(\Gamma)F^c(\Gamma) \tag{E6a}$$

$$=\sum_\Gamma w(\Gamma)\chi^c(\Gamma)\ , \tag{E6b}$$

where we set

$$F(\Gamma)=\mathrm{Tr}\prod_i(e^{H\sigma_i}h_i^z)\prod_{\langle ij\rangle}[e^{J\sigma_i\sigma_j}/(h_ih_j)]\ . \tag{E7}$$

The advantage of introducing h_i is that now only diagrams with no free ends enter in Eq. (E6). This statement holds even when $H\neq0$ as we take it to be here, providing Eq. (E1) is satisfied. Such a formulation is possible because the weak embedding constants for the diagrams with two free ends which usually are studied to obtain the susceptibility can be related to the weak embedding constants of diagrams with no free ends via a recursion relation. A direct diagrammatic approach to this problem can be formulated, but in essence it would be equivalent to Eq. (E1).

In Eq. (E6) we must include the diagram consisting of a single point. We denote this diagram Γ_0. Also note that it is necessary to form the cumulant $\chi^c(\Gamma)$ defined in terms of $\chi(\Gamma)$ via Eq. (3). Using Eqs. (E2) throught (E5) we write (E7) as

$$\chi(\Gamma)=[2n_b(\Gamma)-zn_s(\Gamma)]t(1+t)\chi_0^2 + \frac{\partial^2}{\partial H^2}\ln\mathrm{Tr}\prod_i e^{[1+n_i(\Gamma)b]H\sigma_i}\prod_{\langle ij\rangle} e^{J_{ij}\sigma_i\sigma_j}, \tag{E8}$$

where $n_i(\Gamma)$ is the number of sites in Γ neighboring to site i. For Γ_0 we find $\chi(\Gamma_0)=\chi_0(1+t)$. Again, for the diagrams we consider it is helpful to trace over sites internal to a chain connecting two sites a and b. If the chain contains k sites, we define $V_{ab}^{(k)}$ via

$$V_{ab}^{(k)}=\left[\prod_{i=1}^{k}\mathrm{Tr}_i e^{[1+(\sigma-1)b]H\sigma_i}\right] \times e^{J(\sigma_a\sigma_1+\sigma_k\sigma_b)}\prod_{i=1}^{k-1} e^{J\sigma_1\sigma_j}. \tag{E9}$$

Explicitly we obtain to order H^2:

$$V_{ab}^{(k)}=(\cosh J)^{k+1}\left[1+\frac{k}{2}H_2^2\right]\left\{1+\frac{tH_2^2}{1-t}\left[k-\frac{1-t^k}{1-t}\right] + (\sigma_a+\sigma_b)tH_2\frac{1-t^k}{1-t}+\sigma_a\sigma_b\left[t^{k+1}+H_2^2\left[\frac{t^2(1-t^k)}{(1-t)^2}-\frac{kt^{k+1}}{1-t}\right]\right]\right\}, \tag{E10}$$

where

$$H_2=H[1+(z-2)b]=H(1-t)\chi_0 .$$

To evaluate $\chi(\Gamma)$ for a polygon of s sides we use $V_{ab}^{(k)}$ with $k=s-1$ and $a=b$. For such a polygon we find

$$\chi(\Gamma)=s\chi_0(1+t)-2s\chi_0^2t^s\frac{1-t^2}{1+t^s}, \tag{E11}$$

which yields

$$\chi^c(\Gamma)=-2s\chi_0^2t^s\frac{1-t^2}{1+t^s}. \tag{E12}$$

For the other diagrams $\chi(\Gamma)$ can be expressed in terms of $V_{ab}^{(k)}$ as was done in Secs. IV and V. For instance

$$\chi(\Gamma_8)=(18-8z)t(1+t)\chi_0^2 + \frac{\partial^2}{\partial H^2}\ln\mathrm{Tr}e^{H[1+(z-3)b][\sigma_a+\sigma_b)}[V_{ab}^{(2)}]^3 . \tag{E13}$$

For the diagrams other than polygons, we find, keeping only orders of t which matter for our calculation

$$\chi^c(\Gamma_5)=8\chi_0^2t^7(1+4t), \tag{E14a}$$

$$\chi^c(\Gamma_6)=8\chi_0^2t^8, \tag{E14b}$$

$$\chi^c(\Gamma_7)=\chi^c(\Gamma_8)=8\chi_0^2t^9, \tag{E14c}$$

$$\chi^c(\Gamma_9^{(m)})=8\chi_0^2t^{m+8}(1+2t),\quad m\geq 0 \tag{E14d}$$

$$\chi^c(\Gamma_{10}^{(m)})=8\chi_0^2t^{m+10},\quad m\geq 0 . \tag{E14e}$$

Using these evaluations and the $w(\Gamma)$ from Table II in Eq. (E8) we reproduce Eq. (5.19) of FG.[3] This agreement checks that Table II correctly includes all diagrams needed for calculations to order $(1/\sigma)^5$.

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[4]For a review of localization, see D. J. Thouless, Phys. Rep. 13, 93 (1974).

[5]For the Cayley tree the exact results corresponding to Eqs. (7) and (9) are $N^{-1}\chi=[1+\tanh(\beta J)]/[1-\sigma\tanh(\beta J)]$ and $N^{-1}\chi=(1+p)/(1-\sigma p)$, respectively.

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