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Linewidth for fluxon oscillators

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Soliton perturbation theory is used to calculate the fluxon oscillator linewidth arising from fluxon interaction with background radiation. Detailed calculations in the case of an oscillator that is long compared with the Josephson length and for which the radiation field is thermal establish lower bounds for the linewidth of a real oscillator. These lower bounds are not in disagreement with recent, instrument-limited measurements of X-band linewidths less than 5 kHz.

I. INTRODUCTION

In 1973 Fulton and Dynes pointed out that the "zero-field steps" observed in the voltage-current characteristics of long Josephson junctions could be ascribed to oscillatory behavior of internal fluxons (or magnetic solitons).[1] Subsequent observations of microwave radiation[2] led to the hope that such structures could play a technically useful role as oscillators into the millimeter wave range.[3] Recently, some long Josephson junctions of high quality were fabricated and tested at the University of Salerno[4] and sent to the Physikalische Technische Bundesanstalt in Berlin[5] and the Technical University of Denmark[6,7] for detailed measurements in the microwave range. Comparison of these microwave measurements with numerical and analog computations of fluxon dynamics (based on a structurally perturbed version of the sine-Gordon equation)[8] confirms that the original idea of Fulton and Dynes is correct.

Among other results emerging from these experimental studies has been the observation of a suprisingly narrow oscillator linewidth: less than 5 kHz (the instrument limit) at a fundamental oscillator frequency of 10 GHz.[7] Our aim in this paper is to present a theory of fluxon oscillator dynamics which allows us to predict the linewidth of a long Josephson junction oscillator.

Our approach is based upon the description of a Josephson transmission line as the sine-Gordon equation with structural perturbations that represent dissipation and input of energy.[9,10] We extend a recently developed soliton perturbation theory[11] to second order in a small parameter proportional to the structural perturbations in order to calculate the effect of background radiation on soliton dynamics.[12] This calculation allows us to define an "instantaneous frequency" which leads directly to an explicit formula for oscillator linewidth as a function of the background radiation in the junction. Such background radiation may be generated in several ways: (i) electrical noise conducted to the oscillator through bias and output leads, (ii) radiation generated by spatial inhomogeneities of the junction, (iii) radiation generated during reflection of a fluxon from the end of the junction, and (iv) thermal noise in the cavity modes of the junction. To obtain a lower bound on oscillator linewidth, we assume the radiation field to be entirely thermal noise. Under this assumption, and with some simplifications, we calculate suitably normalized values for linewidth as a function of temperature and average fluxon velocity. The worst (i.e., largest) value of linewidth that we calculate under these assumptions is less than the instrument-limited value of 5 kHz.[7]

Although the work reported here is related to recent studies of chaotic behavior in the sinusoidally driven nonlinear pendulum and sine-Gordon equation,[13] we emphasize that our results depend upon the assumption that the trajectory of the fluxon oscillation is not trapped in a region of phase space that contains a "strange attractor."[14] The abovementioned numerical studies[8] support this assumption.

II. DESCRIPTION OF OSCILLATORS

Our analysis of fluxon oscillators is based upon a previously developed theoretical model for the Josephson transmission line[9,10] (JTL), which is briefly recapitulated here for the convenience of the reader. Figure 1 shows a transmission line equivalent circuit[15] for JTL in which L is series inductance per unit length (pul) related to superconducting surface currents, R is series resistance pul

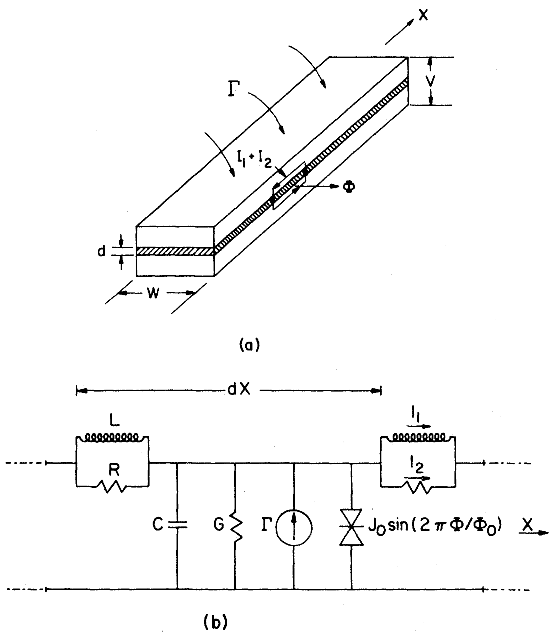

FIG. 1. (a) Physical model of the JTL (not to scale). (b) Transmission line equivalent circuit of the JTL.

related to normal surface currents, C is shunt capacitance pul related to electric field in the junction, G is shunt conductance pul related to normal electron conduction across the junction, Γ is an externally imposed bias current pul, and, finally, $J_0 \sin(2\pi\Phi/\Phi_0)$ is the Josephson current pul crossing the junction. Kirchhoff's equations for this JTL model lead to the following partial differential equation for transverse voltage (V):

$$\frac{L}{R}\Phi_{XXT}+\Phi_{XX}-LC\Phi_{TT}-GL\Phi_T = J_0 L \sin(2\pi\Phi/\Phi_0)+\Gamma L \,, \quad (1)$$

where X and T are laboratory space and time, $\Phi_0 = h/2e$ is the flux quantum, and

$$\Phi \equiv \int V dT \,. \quad (2)$$

Series inductance (L) and shunt capacitance (C) are related to junction geometry by

$$L=\mu_0 \frac{2\lambda_L + d}{W} \quad (3)$$

and

$$C=\epsilon \frac{W}{d} \,, \quad (4)$$

where λ_L is "London" penetration depth for surface currents, W is junction width, d is thickness of the barrier region, ϵ is dielectric permittivity for the barrier, and μ_0 ($=4\pi \times 10^{-7}$ H/m) is the magnetic susceptibility of free space.

For analysis it is convenient to normalize these variables as follows:

$$\phi = 2\pi\Phi/\Phi_0 \,, \quad (5)$$

$$x = X/\lambda_J \,, \quad (6)$$

$$t = T/\tau_J \,, \quad (7)$$

where λ_J is the "Josephson" penetration length and

$$\tau_J \equiv \lambda_J \sqrt{LC} \,. \quad (8)$$

With these normalizations, velocity is measured in units of

$$u_0 = \lambda_J/\tau_J = 1/\sqrt{LC} \,, \quad (9)$$

and (1) becomes

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - \beta\phi_{xxt} + \gamma \,, \quad (10)$$

where

$$\alpha \equiv GL/\tau_J \,, \quad (11a)$$

$$\beta \equiv L/R\tau_J \,, \quad (11b)$$

$$\gamma \equiv 2\pi L \Gamma / \Phi_0 \lambda_J^2 \,. \quad (11c)$$

With α, β, and $\gamma=0$, (10) is recognized as the sine-Gordon equation with the exact soliton solution[16]

$$\phi = 4\tan^{-1}\left[\exp\left[\pm\frac{x-ut}{(1-u^2)^{1/2}}\right]\right] , \quad (12)$$

which represents the propagation of a magnetic flux quantum or "fluxon" along the junction. To make a fluxon oscillator, one must design a physical path over which the fluxon can execute periodic motion. Two examples are shown in Fig. 2. In the "line oscillator" [Fig. 2(a)], a fluxon approaches one end, is reflected as an anitfluxon [change of sign in (12)], propagates to the other end, and is reflected as a fluxon, etc. In the "ring oscillator" [Fig. 2(b)] the fluxon proceeds at constant velocity around the ring. In our calculations, an important parameter is the total path, l, over which the fluxon travels to complete a cycle of oscillation normalized to λ_J. Thus for the *line oscillator* [Fig. 2(a)]

$$l = \frac{2a}{\lambda_J} \quad (13)$$

while for the *ring oscillator* [Fig. 2(b)]

$$l = \frac{2\pi R}{\lambda_J} \,. \quad (14)$$

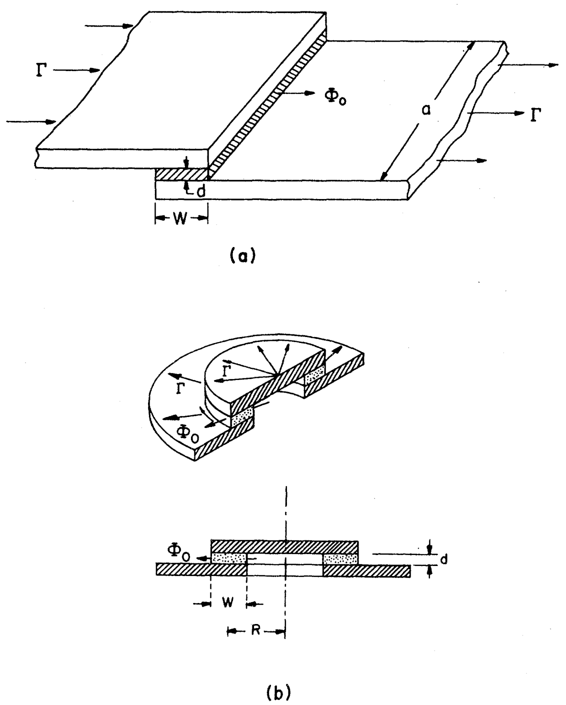

FIG. 2. (a) Line oscillator. (b) Ring oscillator.

The effect of the term γ in (10) is to pump energy into the fluxon motion while the α and β terms dissipate energy. In the following section we use soliton perturbation theory to calculate effects of these terms on the motion.

III. OUTLINE OF PERTURBATION APPROACH

The approach to sine-Gordon soliton perturbation analysis in Ref. 11 begins with a nonlinear equation

$$N\bar{\phi}=\epsilon\bar{f}(\bar{\phi}) , \tag{15}$$

where $N\bar{\phi}=0$ is a completely integrable (i.e., "soliton") equation $\bar{\phi}=\mathrm{col}(\phi,\phi_t)$, where

$$\mathrm{col}(x,y)\equiv \begin{bmatrix} x \\ y \end{bmatrix} ,$$

and ϵ is a small parameter. Expanding

$$\bar{\phi}=\bar{\phi}_0+\epsilon\bar{\phi}_1+\epsilon^2\bar{\phi}_2+\cdots \tag{16}$$

one finds that

$$N\bar{\phi}_0=0 , \tag{17}$$

so $\bar{\phi}_0$ is an exact multisoliton solution which depends upon certain constant parameters p_j (e.g., the speeds and positions of the solitons). Thus

$$\bar{\phi}_0=\bar{\phi}_0(x,t,\{p_j\}) . \tag{18}$$

If $\bar{\phi}_1$ and $\bar{\phi}_2$ are secular (i.e., grow linearly in time) the second and third terms on the right-hand side (RHS) of (16) are useful only for times of order ϵ^{-1} and ϵ^{-2}, respectively. To overcome this objection one can allow order ϵ time variations in the p_j's of ϕ_0 so $\bar{\phi}_1$ and $\bar{\phi}_2$ satisfy

$$L\bar{\phi}_1=\bar{F}_1(\bar{\phi}_0) , \tag{19}$$

$$L\bar{\phi}_2=\bar{F}_2(\bar{\phi}_0,\bar{\phi}_1) , \tag{20}$$

where $\bar{F}_1$ and $\bar{F}_2$ acquire extra terms because of the modulations of the p_j's, and L is a linearization of N around $\bar{\phi}_0$. Now secular growth of $\bar{\phi}_1$ and $\bar{\phi}_2$ can be avoided by requiring that

$$\bar{F}_1\perp\mathcal{N}_d(L^\dagger) , \tag{21}$$

$$\bar{F}_2\perp\mathcal{N}_d(L^\dagger) , \tag{22}$$

where $\mathcal{N}_d(L^\dagger)$ is the discrete null space of the adjoint of L. From (21) and (22) one obtains ordinary differential equations (ODE) for the order ϵ and order ϵ^2 variations in the p_j's.

The strategy of our calculation is as follows. Order ϵ corrections obtained from (21), are used to calculate the effects of α, β, and γ terms in (10) on the steady motion of a JTL fluxon. The radiation field $\bar{\phi}_1$ is then determined from (19). This permits us to evaluate the orthogonality condition (22) which gives ODE's that determine the effects of $\bar{\phi}_1$ (radiation field) on the fluxon motion. In our picture it is this interaction of the fluxon motion with the radiation field that leads to an instantaneous frequency and therefore to a nonzero oscillator linewidth.

Our analysis proceeds as follows (see Ref. 11 for details). The exact single fluxon solution (12) of the unperturbed sine-Gordon equation is modified to

$$\phi_0=4\tan^{-1}[\exp(\zeta)] , \tag{23}$$

where

$$\zeta\equiv\gamma(t)[x-X(t)] . \tag{24}$$

Thus $X(t)$ specifies the trajectory of the fluxon and $\gamma(t)$ its relativistic contraction. Two elements of $\mathcal{N}_d(L^\dagger)$ are

$$\bar{b}_1\equiv\begin{bmatrix}\phi_{0,tt}\\ -\phi_{0,t}\end{bmatrix} \tag{25}$$

and

$$\bar{b}_2\equiv\begin{bmatrix}\phi_{0,xt}\\ -\phi_{0,x}\end{bmatrix} . \tag{26}$$

The conditions $\bar{F}_1 \perp \bar{b}_1$ and $\bar{F}_1 \perp \bar{b}_2$ imply

$$\gamma = \frac{1}{(1-\dot{X}^2)^{1/2}} . \tag{27}$$

Thus, for typographical convenience, we define

$$\dot{X} \equiv u . \tag{28}$$

The time dependence of $\dot{X}$ is divided into order ϵ contributions, calculated from (21), and order ϵ^2 contributions, calculated from (22). Thus

$$\dot{X} = \dot{X}_1 + \dot{X}_2 , \tag{29}$$

where

$$\ddot{X}_1 = -\frac{\epsilon}{4}(1-u^2)^{3/2} \times \int_{-\infty}^{\infty} f(\phi_0(\zeta)) \operatorname{sech}\zeta \, d\zeta , \tag{30}$$

$$\ddot{X}_2 = \frac{\epsilon^2}{4}(1-u^2)^{3/2} \times \int_{-\infty}^{\infty} [f(\phi_1) - \tfrac{1}{2}\phi_1^2 \sin\phi_0] \operatorname{sech}\zeta \, d\zeta . \tag{31}$$

IV. GENERAL CALCULATION OF LINEWIDTH

From the results of the previous section we see that, under steady-state oscillator conditions, the fluxon speed is

$$\dot{X} = u_c + u_v(t) , \tag{32}$$

where u_c is a constant (power balance) velocity. The time-varying component u_v arises from interaction of the fluxon with the radiation field and, from (30) and (31) obeys the ODE

$$\dot{u}_v = \ddot{X}_2 - \langle \ddot{X}_2 \rangle_{av} , \tag{33}$$

where $\langle \ \rangle_{av}$ indicates a time average.

If $u_v = 0$, the fluxon executes a perfectly periodic motion over a path l with frequency

$$\nu_c = u_c / l . \tag{34}$$

In general we can define a (time-dependent) instantaneous frequency as

$$\nu(t) = \nu_c + u_v / l . \tag{35}$$

The rms derivation of $\nu(t)$ from its mean value ν_c is

$$\Delta\nu = \{ \langle [\nu(t) - \nu_c]^2 \rangle_{av} \}^{1/2} . \tag{36}$$

We take $\Delta\nu$ as a convenient measure of oscillator linewidth. Since the radiation field in (31) is not periodic we take the average in (36) over a long time as

$$\Delta\nu = \left[\lim_{T\to\infty} \frac{1}{T} \int_0^T dt [\nu(t) - \nu_c]^2 \right]^{1/2} . \tag{37}$$

Equation (37), together with (35) and (31), provides a straightforward procedure for calculating the linewidth of a fluxon oscillator. We do this for a particular example in the following section.

V. THERMAL LINEWIDTH OF A SINGLE FLUXON OSCILLATOR

We now turn to a practical question of fluxon oscillator design: calculation of linewidth when the radiation field is assumed to be in thermal equilibrium with its environment. This calculation neglects other sources of the radiation field (electrical noise, radiation emitted from the fluxon, etc.) thus it should give a lower bound for realizable linewidths and some idea about how the linewidth depends upon oscillator parameters and temperature. The analysis is restricted to a single fluxon oscillation to avoid analytical difficulties associated with the phenomena of "bunching."[8]

We employ (31) where, from (10),

$$\epsilon f(\phi_1) \equiv \alpha \left[\phi_{1,t} - \frac{\beta}{\alpha}\phi_{1,xxt} + \frac{\gamma}{\alpha} \right] , \tag{38}$$

thus α is our small parameter in the perturbation analysis. Since we are assuming that the radiation field arises because the linear modes of the oscillator are in thermal equilibrium, (31) takes the form

$$\ddot{X}_2 = \tfrac{1}{4}(1-u^2)^{3/2} \int_{-\infty}^{\infty} [\alpha f(\eta\phi_1) + (1-u^2)(\eta\phi_1)^2 \tanh\zeta \operatorname{sech}\zeta] \operatorname{sech}\zeta \, d\zeta , \tag{39}$$

where α is a small parameter that measures the structural perturbation, and η is a small parameter that measures the amplitude of the radiation field.

In the following analysis we make two simplifying assumptions:

$$\beta = 0 \tag{40}$$

and

$$l \gg 1 . \tag{41}$$

The first of these is not a serious restriction if one assumes a somewhat larger value of α to account for dissipation in the β term of (38).

We calculate the thermal radiation field $\eta\phi_1$ as a sum of individual photon modes of a cavity which contains a single fluxon moving with constant velocity u.[17] Thus

$$\eta\phi_1 = \sum_n \frac{A_n}{\sqrt{2\pi}} \frac{\sqrt{1-u^2}}{\omega_n - u k_n} \left[\frac{k_n - u\omega_n}{\sqrt{1-u^2}} \cos(k_n x - \omega_n t) - \sin(k_n x - \omega_n t) \tanh\left[\frac{x-ut}{\sqrt{1-u^2}} \right] \right] , \tag{42}$$

where

$$k_n^2 = \omega_n^2 - 1 \tag{43a}$$

and

$$k_n = \frac{2\pi n}{l} . \tag{43b}$$

Since the second term in the integral (39) is an odd function of ζ while the first term is even, the contribution of the second term is small except when $n \simeq l$. Then the ratio of the first to second term is of order l and, under assumption (41), we can neglect the second term. Thus (39) takes the form

$$\ddot{X}_2 = \frac{\alpha}{4}(1-u^2)^{3/2} \int_{-\infty}^{\infty} \left[\eta\phi_{1,t} + \frac{\gamma}{\alpha} \right] \operatorname{sech}\zeta \, d\zeta . \tag{44}$$

The term γ/α in (44) merely contributes a constant to $\ddot{X}_2$ which is absorbed in the power balance condition that determines u_c. Thus it does not enter into our calculation of $\Delta\nu$.

The component of $\dot{X}_2$ that depends on the radiation field is

$$\dot{X}_{2R} = \sum_n \alpha C_n \cos[(k_n u - \omega_n)t + \theta_n] , \tag{45}$$

where

$$C_n = \frac{u^2(1-u^2)^{3/2}}{4\sqrt{2\pi}} \frac{k_n A_n}{(k_n u - \omega_n)} \times \operatorname{sech}\left[\frac{\pi k_n}{2}(1-u^2)^{1/2} \times \exp[-\pi k_n (1-u^2)^{1/2}] \right] . \tag{46}$$

To calculate the mode amplitudes $\{A_n\}$, we assume a mode at frequency ω_n to have the energy

$$E_n = \frac{\hbar\omega_n}{\exp\left[\dfrac{\hbar\omega_n}{k_B T} - 1 \right]} . \tag{47}$$

Strictly speaking, the relation between E_n and A_n should be calculated for a cavity containing a fluxon; however, from inequality (41) this relation is the same as that for an empty cavity. Thus in normalized units

$$A_n = \left[\frac{8\pi\hbar u_0 k_n \omega_n / n\Phi_0^2}{\left[\exp\left[\dfrac{\hbar\omega_n}{k_B T \tau_J} \right] - 1 \right] \left[\dfrac{k_n^2}{L} + C u_0^2 \omega_n^2 + \dfrac{2eJ_0\lambda_J^2}{\hbar} \right]} \right]^{1/2} . \tag{48}$$

From (48), (45), and (37) we obtain

$$\frac{\Delta\nu}{\alpha} = \frac{u^2(1-u^2)^{3/2}}{2\sqrt{2}\Phi_0 l} \sqrt{\hbar u_0} \times \left[\sum_n \frac{1}{n} \frac{\omega_n k_n^4 \left[\operatorname{sech}\left[\dfrac{\pi}{2} k_n (1-u^2)^{1/2} \right] \exp[-k_n \pi (1-u^2)^{1/2}] \right]^2}{\left[\exp\left[\dfrac{\hbar\omega_n}{k_B T \tau_J} \right] - 1 \right] \left[\dfrac{k_n^2}{L} + C u_0^2 \omega_n^2 + \dfrac{2eJ_0 \lambda_J{}^2}{\hbar} (k_n u - \omega_n)^2 \right]} \right]^{1/2} . \tag{49}$$

Equation (49) is the main result of this paper. To appreciate the dependence of linewidth $\Delta\nu$ upon oscillator parameters and temperature that it implies, we turn to two examples of JTL that have been thoroughly studied.[10] Important parameters are recorded in Table I. From these parameters we have plotted in Figs. 3 and 4 $l\Delta\nu/\alpha$ as a function of u for several values of temperature and l. Since these calculations are rather insensitive to l (see Fig. 5) we can assume $\Delta\nu \propto l^{-1}$. We see that $\Delta\nu(u)$ rises to a maximum value at

$$u \simeq 0.8 \tag{50}$$

and, as we expect, falls to zero in limits $u \to 0$ and 1. The main difference between N25L and N53C is in the value for λ_J, but this has a relatively small effect upon $\Delta\nu$. We find, of course, that $\Delta\nu$ falls with decreasing temperature, but it is interesting to observe that the curves $\Delta\nu(u)$ show little change in shape.

Dueholm *et al.*[7] have reported an instrument-limited measurement that

$$\Delta < 5 , \tag{51}$$

where Δ is the linewidth (in units of kHz) for a line oscillator with

$$\alpha = 0.01 ,$$

$$l = 12 .$$

From our calculation the thermal linewidth in laboratory units is given by $\Delta\nu/\tau_J$. Assuming $l=12$ [i.e., $a/\lambda_J=6$ in Fig. 2(a)] we find for N25L that the maximum linewidth is equal to 260 Hz and for N53C, the maximum linewidth is equal to 550 Hz. These results are not inconsistent with (51).

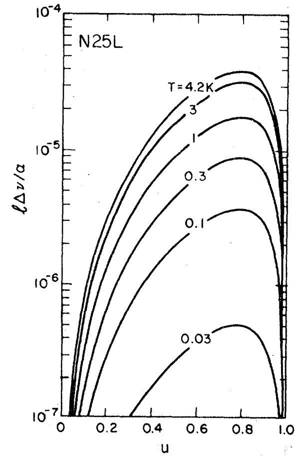

FIG. 3. Normalized thermal linewidth as a function of average fluxon velocity and absolute temperature for JTL No. N25L.

VI. CONCLUDING DISCUSSION

The main result of this paper is (49) which gives $\Delta\nu/\alpha$ as a function of the oscillator parameters where l is the total fluxon path length for a cycle of oscillation, measured in units of λ, u is the average fluxon speed normalized to u_0 $(=\lambda_J/\tau_J)$, and T is the absolute temperature, in addition to the JTL parameters λ_J, τ_J, C, L, and J_0 defined in Sec. II.

From (49) the rms deviation of the oscillator linewidth is equal to $\Delta\nu/\tau_J$ Hz, where α measures the shunt oscillator losses (including loading). In deriving (49) the following assumptions have been made:

(1) only a single fluxon is present in the cavity,

(2) $l \gg 1$,

(3) the background radiation field is entirely thermal, and

(4) surface losses [$\beta\phi_{xxt}$ in (10)] are neglected.

Thus our calculations give a lower bound for the linewidth to be found in a real oscillator. Additional contributions to oscillator linewidth may arise

TABLE I. Josephson transmission lines.

Parameter	N25L	N53C	Unit
α	0.0052	0.00555	
u_0	2.3×10^{7}	1.76×10^{7}	m/s
L	2.1×10^{-9}	2.5×10^{-9}	H/m
C	0.9×10^{-6}	1.3×10^{-6}	F/m
λ_J	1.27×10^{-3}	2.63×10^{-4}	m
J_0	9.7×10^{-2}	1.9	A/m
τ_J	0.55×10^{-10}	1.5×10^{-11}	s

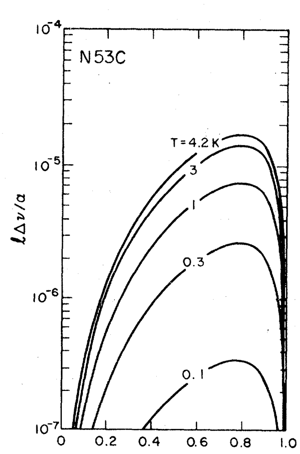

FIG. 4. Same as Fig. 3 for JTL no. N53C.

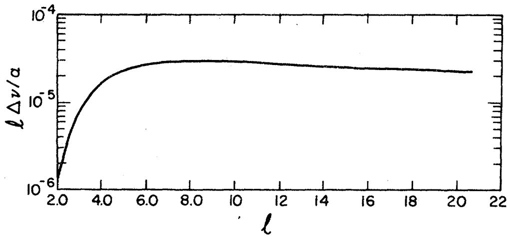

FIG. 5. $l\Delta\nu/\alpha$ vs l for $T=3$ K, and $u=0.8$.

from excess electrical noise and radiation from the fluxon itself.

Equation (44) shows an exact mechanism for influence of electrical noise on the fluxon motion through stochastic behavior of the bias current γ. A line oscillator may have larger linewidth than a corresponding ring oscillator because the kink-antikink reflection that take place in a line oscillator generate an additional component of radiation[18] that is not present in a ring oscillator. Since we see no special difficulties in making ring oscillators, we suggest that they be considered experimentally.

Finally, Figs. 3 and 4 show $\Delta\nu$ rising to a maximum value around $u=0.8$ Although this result is obtained for thermal linewidth, we feel that this behavior should be found when a more general radiation field is present. An experimental check of this suggestion should be possible with instrumental resolution of linewidth only an order of magnitude better than that reported in Ref. 7.

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