PHYSICAL REVIEW B VOLUME 26, NUMBER 1 1 JULY 1982

Path-integral approach to the statistical mechanics of solitons

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(Received 16 February 1982)

Universal features for the statistical mechanics of a general class of systems with a one-component field in one dimension have been obtained using path-integral techniques in a physically revealing and more direct way than was possible before from the transfer-operator method. Results are also extended to systems that can support more than one type of soliton. Spin-wave and soliton contributions are calculated simultaneously, thus enabling us to investigate the relative importance of spin-wave and soliton contributions in a given physical quantity. These general results are then applied to the double-sine-Gordon model. We discuss the statistical-mechanical properties of the model as the parameters are varied. Assessments of the validity of our results are also made.

I. INTRODUCTION

An anisotropic Heisenberg ferromagnetic chain in the presence of an applied magnetic field has been mapped approximately onto the double-sine-Gordon (DSG) model within the classical limit.[1] Treating the DSG system as our model we have investigated various mechanical properties for the solitons in great detail. Here, we are concerned instead with their thermal behavior, and in particular the precise role they play in determining various thermodynamic quantities and static correlation functions. Their dynamical behavior, although equally fascinating, will not be considered here.

Two very fruitful approaches have been employed in the past to study the statistical mechanics of soliton-bearing systems, namely the transfer-operator method (TOM) and the phenomenology of an ideal gas of solitons (PIGS). With these methods striking universal features have been discovered and discussed at length for a number of nonlinear models in one dimension (1D).[2–5]

The TOM has the obvious advantage that it provides formally exact procedures for obtaining any static quantity for any 1D classical system.[6] It is applicable at all temperatures, and for discrete as well as continuous systems. It also has the advantage that contributions from all kinds of linear and nonlinear excitations in the system are automatically accounted for. However, the price to pay for this latter feature is that one often finds it difficult to isolate the effects of a specific kind of excitation, such as the soliton, and the interpretation of the physical origins of the results is in general not very easy. Moreover, for rather general Hamiltonians, the TOM often requires numerical solution, and interesting nonlinear behavior may sometimes be overlooked in such computations.

The PIGS is a physically appealing method for obtaining the statistical mechanics of solitons[7,3,2,8] and other nonlinear excitations.[9] One works directly with the nonlinear mode in which one is interested. It has the disadvantage that it is applicable only at low temperatures (at least at the present stage of development), where the solitons form a dilute gas of noninteracting particles. In addition, because of its phenomenological nature, its validity within a given class of systems must be gauged by other formally more exact calculations. But once the PIGS is set up, it is extremely important as a means for interpreting the results obtained from other methods. It was, in fact, precisely this method which made possible the first proper identification of the soliton density by Krumhansl and Schrieffer.[7]

Here we employ, instead, path-integral techniques (PIT). The approach, in its general form, is a rather old one, but it has regained popularity in recent years as the need to study intrinsically nonlinear problems arises in many branches of physics. We are interested here specifically in its use for studying the statistical mechanics of solitons. The ease of implementation of the method has been improving over the last decade, as important ideas and special techniques for handling functional integrals involving solitons (or instantons) have been rapidly accumulating.[10,11] Like the PIGS, the PIT has very physically appealing features, and quite often the physical origins of the results are immediately transparent from the way these quantities are derived. The method is well founded, and one works directly with the excitations of interest, so that their effects can be studied separately from those of other modes. The method can be extended to higher dimensions, to include multicomponent fields,[11] and to the dynamics.[12] There are, however, two serious drawbacks of the method at least in its present form. First, it is applicable only to the low-temperature region where the solitons form a dilute noninteracting gas. Second, one is confined to work within the continuum limit, and thus effects associated with the discreteness of realistic systems cannot be studied with this method.

The major contributions of the present paper are as follows. Universal features for the statistical mechanics for a general class of systems with a one-component field in 1D (the so-called class A systems[2,3]) have been obtained using the PIT in a physically revealing and more direct way than was possible before using the TOM.[1–5] We have also extended the earlier results to include systems that can support more than one type of soliton.[13] We can calculate simultaneously spin-wave and soliton contributions, thus enabling us to investigate the relative importance of each of these contributions to a given physical quantity. This capability is especially important, as the relative importance of spin-wave and soliton contributions to the central peak in the neutron inelastic scattering experiment of $CsNiF_3$ (Ref. 14) is still somewhat unsettled.[15] We have also applied our results here to the thermodynamics for various cases of the DSG model.[1] The results for case (i) of the DSG model agree with those obtained previously using the TOM and PIGS.[13] Results for the other cases have not been studied, and can also be obtained with little extra work by adopting a proper potential. Moreover, we have studied the interconnections of the results for various cases by examining the statistical mechanical properties of the system as the model parameters are varied. This enables us, in addition, to make an assessment of the degree of validity of our results.

Previous studies of the statistical mechanics of solitons using the PIT focus on specific models, namely the ϕ^4 model,[16] which belong to the class A systems treated here, and a two-component uniaxial magnetic chain, which belongs to a different class of systems.[17] No attempt to arrive at general results was made. For the ϕ^4 model,[16] because of an incorrect treatment of the translational mode of the soliton, the expression for the soliton density is different from ours.

Employing the PIT, we present here a detailed study of the thermodynamics and static correlation functions for the class A systems. In Sec. II, the functional integral for the partition function is evaluated at low temperatures by the method of steepest descent.[18] Besides the trivial solution to the classical equation of motion, the full nonlinear soliton solutions are used. Gaussian fluctuations about both of these paths are considered by studying the normal modes of their corresponding stability operators. The functional integral is then replaced by ordinary integrals over the normal-mode amplitudes. Divergences occur when the presence of solitons destroys certain continuous symmetries of the system, and these divergent integrals have to be handled separately. Techniques to do this are by now well known.[11] In evaluating the infinite product of eigenvalues for the Gaussian fluctuation equation for the soliton, certain tricks of Coleman[11] are utilized so that the results do not depend on the specific form of the model, but instead are valid for our entire class of systems. This is the subject of Appendix A.

In Sec. III, the static correlation functions are calculated specifically for various cases of the DSG model, following the method of Polyakov.[19] In the study of correlation functions it is important to make the distinction between soliton-sensitive and soliton-insensitive quantities.[20] The former are those quantities which take on different values on opposite sides of a soliton, and therefore are characteristic of a true domain wall. The corresponding correlation length is then determined by the mean distance between solitons (i.e., the average domain size), and thus is extremely large at temperatures much lower than the soliton creation energies.[21,22] In contrast, for the latter case these quantities do not vary (apart from the small-amplitude spin-wave-like fluctuations) on opposite sides of a soliton. Thus these solitons do not play the role of true domain walls for these physical quantities, and the corresponding correlation length goes to a constant value at low temperatures.[21,22] In the present approach using the PIT, we see that for soliton-sensitive quantities, the contribution from a single soliton to the corresponding two-point correlation function decreases linearly with distance, as was found in the displacement-displacement correlation function for the ϕ^4 model by Polyakov.[19] This implies that solitons are crucial in determining the long-distance behavior of these quantities. On the contrary, for soliton-insensitive quantities we find that the soliton contributions to the corresponding correlation functions decrease exponentially with distance, thus suggesting that solitons have little effect on the long-distance behavior of these quantities.[22]

All the results derived here can be expressed in terms of two types of simple integrals which involve the explicit form of the potential. One is needed for the functional forms for each type of soliton, and the other for the soliton creation energies. Thus there are two nonuniversal numbers for each type of soliton. Universal temperature dependences are found for the soliton-related quantities, as was obtained previously using the TOM and the PIGS.[2–5] Some of these results are extended here to systems which can support any number of types of solitons.

The results of this extension can be found in Sec. IV together with a brief discussion on the topological constraints which must be imposed on the sequencing of solitons in real space. With these general results here, explicit expressions for the free energies for various cases of the DSG model are summarized also in Sec. IV.

Section V gives interpretations of our findings for the DSG model. We pay special attention to the specific roles of the solitons in determining these finite-temperature results. For a given physical quantity, the relative importance between soliton and spin-wave contributions is analyzed in detail, with the help of the results of Appendix B which are based on harmonic spin-wave calculations. We investigate the interconnections among the results for various cases of the DSG model as the parameters of the model are varied. The failure of our results in a certain region in parameter space and the physical reasons behind such failure are discussed.

Section VI is a qualitative discussion on the statistical mechanical behavior of the DSG model for general parameter values.

II. FREE ENERGY

We are interested in the statistical mechanics of solitons for a rather general class of 1D classical systems with a single scalar field $\phi(x)$. The Hamiltonian we adopt here has the form

$$H=2J\int_{-\infty}^{+\infty}\left[\frac{\dot{\phi}^2}{2c^2}+\frac{\phi_x^2}{2}+V(\phi)\right]. \tag{2.1}$$

The energy scale for H is set by $2J$, which, for spin systems, is just the exchange coupling parameter. The constant c, which is the intrinsic velocity for the linear modes of the system, also plays the role of a limiting velocity for the solitons. There are three general requirements on the form of the potential $V(\phi)$. (1) $V(\phi)$ should contain no derivative interaction, nor can it be an explicit function of x or t, so that it depends only on ϕ. (2) The system must have more than one degenerate ground state. These states should form a discrete (but possibly an infinite) set. (3) We also require the system to possess a discrete internal symmetry which connects all these degenerate ground states; thus the system has well-characterized linear modes. [The curvatures near the absolute minima of $V(\phi)$ must therefore be the same.] Other than these requirements, the detailed form of $V(\phi)$ can be quite arbitrary. Since the energy of the system must be unbounded below, we can always add a constant to $V(\phi)$ so that it is non-negative, and takes on the absolute minimum value of zero only for the degenerate ground states.

Examples of systems belonging to this class of Hamiltonians (the so-called class A systems[2,3]) include the ϕ^4 model, the double-quadratic system, the π- and 2π-SG solitons, the DSG systems, and the double-quadratic chain. In the first two cases $V(\phi)$ is unbounded for $|\phi|\to\infty$, and the number of degenerate ground states is finite (namely two). The remaining ones are either singly or doubly periodic systems, and therefore have an infinite number of degenerate ground states.

In this section, the PIT will be used to calculate the free energy for this class of systems at low temperatures in the dilute soliton density limit. Special attention will be given to the contributions from the solitons. Despite the substantial arbitrariness of $V(\phi)$, in fact rather general expressions can be obtained for the static quantities of these class A systems.

The classical partition function here can be separated into a kinetic and a configurational contribution. To obtain the kinetic contribution, note that the momentum conjugate to the field $\phi(x)$ is

$$P_\phi(x)=\frac{2J}{c^2}\dot{\phi}(x) \tag{2.2}$$

[for planar spin systems P_ϕ corresponds precisely to the small out-of-plane angle.[1]] The kinetic part of the partition function Z_K is then easily calculated by performing Gaussian integrals over P_ϕ in the form

$$h^{-1}\int_{-\infty}^{+\infty} dP_\phi \exp\left[-\frac{\beta c^2}{4J}P_\phi^2\right], \tag{2.3}$$

where h is Planck's constant, to get

$$Z_K=\left[\frac{4\pi J}{\beta h^2 c^2}\right]^{L/2}. \tag{2.4}$$

The kinetic contribution to the free energy per particle is then given by

$$f_K=-\frac{1}{2\beta}\ln\left[\frac{4\pi J}{\beta h^2 c^2}\right]. \tag{2.5}$$

We must later add this to the configurational part to obtain the total free energy per particle. With the kinetic part taken care of, it is understood, hereafter, that we are always referring to the configurational contribution, which therefore will not carry any labels.

The (configurational) partition function Z, can be expressed in terms of a functional integral as

$$Z=\int D\phi(x)\exp\left[-2\beta J\int_{-L/2}^{L/2}dx\left[\frac{\phi_x^2}{2}+V(\phi)\right]\right]. \tag{2.6}$$

We work with a finite but very large L, and eventually we take L to infinity. The (configurational) Hamiltonian,

$$H[\phi(x)]\equiv\int_{-L/2}^{L/2}dx\left[\frac{\phi_x^2}{2}+V(\phi)\right], \tag{2.7}$$

plays the role of an action. At low temperatures ($\beta J\gg 1$) the functional integral can be evaluated by the method of steepest descent.[11] In this temperature region Z is dominated by paths that make $\delta H=0$. These paths $[\phi^c(x)]$ are just the classical solutions of the Euler-Lagrange equation, which from Eq. (2.7) is

$$\phi_{xx}^c=V'(\phi^c). \tag{2.8}$$

The prime denotes the derivative with respect to the argument. With the boundary conditions $\phi_x^c(|x|\to\infty)=0$, the first integral of Eq. (2.8) is

$$\phi_x^c=\pm[2V(\phi^c)]^{1/2}. \tag{2.9}$$

Interpreting x as "time" and ϕ as "coordinate," this equation describes the motion of a classical particle of unit mass moving in a potential of $-V(\phi)$. The energy of this particle is easily seen to be zero. This interpretation is extremely helpful in visualizing the forms of the classical solutions $\phi^c(x)$ without having to solve Eq. (2.9) explicitly. (See Fig. 1.)

Among all the classical solutions there are always the uniform (x-independent) solutions ϕ^0, which are given by the minima of $V(\phi)$. These paths correspond to a particle sitting forever on top of one of the peaks of $-V(\phi)$. The classical action E^c is given by

$$E^c\equiv H[\phi^c]=2J\int_{-L/2}^{L/2}dx\,(\phi_x^c)^2 \tag{2.10a}$$

$$=2J\int_{\phi^c(-L/2)}^{\phi^c(L/2)}d\phi[2V(\phi)]^{1/2}, \tag{2.10b}$$

where use has been made of Eq. (2.9). It follows that the uniform solutions have no action, since $\phi_x^0=0$ and $V(\phi^0)=0$. Besides these trivial solutions there are of course also the soliton solutions ϕ^s. These large-amplitude solutions correspond to a particle which at time $-\infty$ sits on top of one of the peaks of $-V(\phi)$, rolls down the hill, and at time $+\infty$ reaches the top of the adjacent peak. The fraction of time the particle spends away from the peaks is therefore very small, and thus the solitons are very localized objects in space. For a given $V(\phi)$ there may be more than one $\phi^s(x)$; their explicit forms can be found by solving Eq. (2.9). Their actions E^s, which correspond also to the creation energies of the solitons, must be finite, and can be obtained from Eq. (2.10). Multisoliton paths are important, and will be treated later within the dilute soliton gas approximation.

Next we consider the Gaussian fluctuations about the classical path. For this purpose, a func-

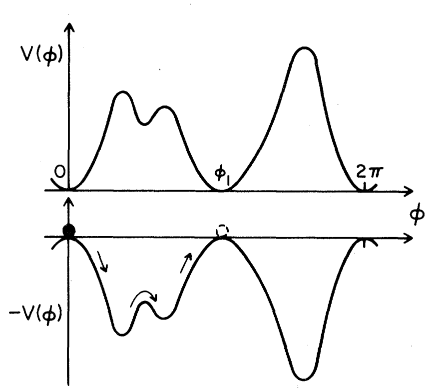

FIG. 1. For the particular potential $V(\phi)$ as shown there are two types of solitons, one type taking $\phi(x)$ from 0 to ϕ_1, and another taking $\phi(x)$ from ϕ_1 to 2π. We can interpret, e.g., the first type of soliton as describing the motion of a classical particle, which at time $x=-\infty$ sits at $\phi=0$, then rolls down the hill of $-V(\phi)$, and at time $x=+\infty$ reaches $\phi=\phi_1$.

tion $\eta^c(x)$ is defined for each path such that

$$\eta^c(x)=\phi(x)-\phi^c(x) ,$$
$$\eta^c(-L/2)=\eta^c(L/2)=0 . \tag{2.11}$$

The action is then expanded up to second order in $\eta(x)$ yielding

$$H[\phi(x)]\cong H[\phi^c(x)]+\tfrac{1}{2}\delta^2 H[\eta^c(x)] . \tag{2.12}$$

The zeroth-order term is just the classical action. The eigenvalues and eigenfunctions of $\delta^2 H$ obey the equations

$$-\frac{1}{2}\frac{d^2}{dx^2}\eta_n^c(x)+\tfrac{1}{2}V''(\phi^c)\eta_n^c(x)=\lambda_n^c\eta_n^c(x) ,$$
$$\eta_n^c(-L/2)=\eta_n^c(L/2)=0 . \tag{2.13}$$

Now $\delta^2 H$ can be expressed as

$$\tfrac{1}{2}\delta^2 H[\eta^c(x)]=\sum_n \lambda_n^c(a_n^c)^2 , \tag{2.14}$$

where a_n^c are the expansion coefficients of $\eta^c(x)$ in terms of the normalized eigenfunctions $\eta_n^c(x)$; i.e.,

$$\eta^c(x)=\sum_n a_n^c\eta_n^c(x) . \tag{2.15}$$

The functional integral in Eq. (2.6) is then replaced by ordinary integrals over these coefficients (or normal-mode amplitudes), thus

$$D\phi(x)\to\prod_n\int\frac{da_n^c}{\pi^{1/2}} . \tag{2.16}$$

Assuming for the moment that all the eigenvalues λ_n^c were positive definite, then each Gaussian integral over a_n^c would yield a factor $(\lambda_n^c)^{-1/2}$. Thus the partition function can be written

$$Z\cong Z^0+Z^s , \tag{2.17}$$

where

$$Z^0=N\prod_n(\lambda_n^0)^{-1/2} \tag{2.18}$$

is the contribution from ϕ^0 (the zero-soliton sector), and

$$Z^s=Ne^{-\beta E^s}\prod_n(\lambda_n^s)^{-1/2} \tag{2.19}$$

is the contribution from a single soliton. N is an unimportant normalization constant. The above calculation neglects effects due to anharmonic spin waves and mutual soliton interactions. Contributions from multisoliton configurations will be taken into account later.

However, not all λ_n^s are positive definite; there must be at least one eigenfunction $\eta_0^s(x)$ with the corresponding eigenvalue $\lambda_0^s\equiv 0$. The presence of this zero-frequency mode is a direct consequence of the translational invariance of the Hamiltonian. The soliton can therefore be centered anywhere along x, thus instead of just one soliton solution there is a whole family of soliton solutions $\phi^s(x-x_0)$ labeled by the center of the soliton, x_0. Shifting x_0 by a small amount implies

$$\phi^s(x-x_0-\delta x_0)-\phi^s(x-x_0)=\phi_x^s(x-x_0)\delta x_0 . \tag{2.20}$$

But ϕ_x^s must be proportional to η_0^s, since by differentiating both sides of Eq. (2.8), it is easily seen that ϕ_x^s satisfies Eq. (2.13) with $\lambda^s=0$. The proportionality constant follows directly from the normalization condition for $\eta_0^s(x)$ and Eq. (2.10a), giving the result

$$\eta_0^s(x)=\left[\frac{2J}{E^s}\right]^{1/2}\phi_x^s(x) . \tag{2.21}$$

Thus within the steepest descent approach, the integral of the functional integral for Z remains constant in the direction of $\eta_0^s(x)$. The integration over a_0^s is therefore divergent and must be handled separately. It is clear that there is such a zero-frequency mode corresponding to each continuous symmetry of the system that is broken by the soliton. The remedy is by now well known.[11] For our problem here, instead of integrating over a_0^s we will integrate over the centers of the solitons, x_0. The required transformation Jacobian can be obtained by considering a path that lies entirely in the $\eta_0^s(x)$ direction, then one has

$$\eta_0^s(x)da_0^s=\phi^s(x-x_0-dx_0)-\phi^s(x-x_0)$$
$$=\phi_x^s(x)dx_0 . \tag{2.22}$$

This gives, using Eq. (2.21),

$$da_0^s=\left[\frac{E^s}{2J}\right]^{1/2}dx_0 . \tag{2.23}$$

Thus it follows that

$$\int\frac{da_0^s}{\pi^{1/2}}=\frac{2L}{\pi^{1/2}B}\left[\frac{E^s}{2J}\right]^{1/2} . \tag{2.24}$$

The factor of 2 in Eq. (2.24) is to take into account contributions from both solitons and antisolitons. A constant B has also been inserted in order to avoid overcounting multisoliton states. Its value depends only on the topology of the soliton under consideration, as will be discussed in more detail later in Sec. IV.

Note from Eq. (2.9) that $\phi_x^c(x)$ cannot be zero for any finite x. Thus $\eta_0^c(x)$ has no nodes and must therefore be the eigenfunction of Eq. (2.13a) with the lowest eigenvalue, i.e., $\eta_0^c(x)$ is the Jacobi field. All other modes must then have $\lambda_n^0>0$, and thus the remaining integrals over a_n^s with $n>0$ are finite. The soliton path is therefore stable in all directions except along $\eta_0^s(x)$, where it is only marginally stable.

Next we must take into account multisoliton states. In the dilute soliton gas limit the procedure amounts to writing[11]

$$Z\cong Z^0\left[1+\frac{Z^s}{Z^0}\right]$$

$$\cong Z^s e^{Z^0/Z^s} . \tag{2.25}$$

Thus the free energy per spin is

$$f\equiv-\lim_{L\to\infty}\frac{1}{\beta L}\ln Z=f^0+f^s , \tag{2.26}$$

where

$$f^0\equiv-\lim_{L\to\infty}\frac{1}{\beta L}\ln Z^0 \tag{2.27}$$

is the contribution from the spin wave, and f^s is the soliton contribution. Using Eqs. (2.18), (2.19), and (2.24) we have

$$f^s=-\lim_{L\to\infty}\frac{2}{\beta\pi^{1/2}B}\left[\frac{E^s}{2J}\right]^{1/2}\frac{\prod_n(\lambda_n^0)^{1/2}}{\prod_n'(\lambda_n^s)^{1/2}} . \tag{2.28}$$

The prime in the infinite product of eigenvalues means that λ_0^s is not to be included. The problem now is to evaluate the ratio

$$R\equiv\frac{\prod_n\lambda_n^0}{\prod_n'\lambda_n^s} . \tag{2.29}$$

One could obtain R by explicitly solving for Eq. (2.13). However, except for very simple cases this is in general an exceedingly formidable analytical task. Moreover, it is difficult to see how to write down the results for a general $V(\phi)$. Following the work of Coleman,[11] one can in fact get around this problem and derive a general expression for R. Details can be found in Appendix A.

From Eqs. (2.28), (2.29), and (A18) the soliton part of the free energy per spin is

$$f^s=-\frac{2}{\beta B}\left[\frac{2\beta J}{\pi}\right]^{1/2}\kappa^{1/2}N^s e^{-\beta E^s} . \tag{2.30}$$

For systems that can support more than one type of soliton we must sum over the contributions from each type of soliton separately. An important contribution from the PIGS is the proper identification of the soliton (solitons and antisolitons of all types) density $n_{\rm tot}^s$, which for our class of systems can be written

$$n_{\rm tot}^s=\frac{2}{B}\left[\frac{2\beta J}{\pi}\right]^{1/2}\kappa^{1/2}\sum_i N_i^s e^{-\beta E_i^s} , \tag{2.31}$$

where i is to be summed over the types of solitons. Thus we have

$$f^s=-\frac{n_{\rm tot}^s}{\beta} . \tag{2.32}$$

There are three points that deserve comment here. First, although we are working at low temperatures where $(\beta J)^{-1}\ll 1$, the above results for the solitons cannot be obtained through any finite-order perturbation calculation. The solitons are intrinsically nonperturbative in nature. Second, the temperature dependences of $n_{\rm tot}^s$ are universal for our entire class of systems. An important advantage for the present approach using the PIT is that the origin of this universal form for $n_{\rm tot}^s$ is very clear. The activation form of $e^{-\beta E^s}$ comes from the fact that topological solitons have finite actions. The $\beta^{1/2}$ temperature dependence for the prefactor of $n_{\rm tot}^s$ is clearly due to the zero-frequency mode. In general one expects that for a system with n continuous symmetries broken by the solitons, the prefactor should be proportional to $\beta^{n/2}$. A system with two broken zero-frequency modes has recently been studied by both the PIT and the TOM,[17] and indeed a β-dependent prefactor is found. Third, in Eq. (2.31) the temperature-independent quantities, κ, N_i^s, and E_i^s are nonuniversal, but can be calculated for a given $V(\phi)$. (See, e.g., the DSG model in Sec. IV.)

Putting the kinetic part [Eq. (2.5)], the spin-wave part [Eq. (A4)], and the soliton part [Eq. (2.32)] together, the total free energy per spin is then given by

$$f_{\rm tot}=-\frac{1}{2\beta}\ln\left[\frac{4\pi J}{\beta h^2c^2}\right]+\frac{\kappa}{2\beta}-\frac{n_{\rm tot}^s}{\beta}\ . \quad (2.33)$$

Clearly, not only for the soliton part, but for the entire expression for $f_{\rm tot}$ the temperature dependence is of universal form.

III. STATIC CORRELATION FUNCTIONS

From Eqs. (2.33) and (2.31) we see that the soliton contribution to the free energy per spin is extremely small in comparison with the spin-wave contribution at k_BT much lower than the soliton creation energy. Therefore, the solitons have very little effect on thermodynamic quantities, like the internal energy and the specific heat, which are derivable from the free energy.[21,22] For spin systems, where $V(\phi)$ depends on the external magnetic field, small soliton contributions to the longitudinal components[23] of the magnetization and the spin susceptibility are also expected. We want to know if there are static physical quantities which are signficantly affected by the solitons. From previous works[21,22] we know that the solitons play a vital role in a given physical quantity when they separate the system into domains where this quantity takes on different values in adjacent domains. In other words, it is the presence of the solitons that prevents this quantity from ordering. This order-parameter correlation length is therefore given by the average size of the domains and thus varies as $(n_{\rm tot}^s)^{-1}$. As $T\to 0$, the soliton density decreases exponentially with β to zero, meaning that the domain sizes grow rapidly to infinity, therefore quantities like the order-parameter correlation length and static susceptibility must likewise increase anomalously. At $T=0$, there can be no soliton in the system; the domain size and the correlation length are therefore infinite. Thus $T=0$ can be considered as a critical point of the system.

These true domains can exist in spin systems which support solitons which are not 2π-like, i.e., $|Q|\neq 1$.[24] To find properties which are dominated by solitons one must consider quantities which are associated with the order parameter, but are not derivable from the free energy. This is the case for the transverse components of the static two-spin correlation function and the static susceptibility for case (i) of the DSG system[1] and for the π-SG system, as will be shown in more detail below. The vital role played by these solitons will be evident.

In contrast, we will see that the corresponding quantities for the longitudinal components of the spins are not much affected by the solitons.[21,22] This comes about because these components have practically the same value on opposite sides of a soliton. The corresponding correlation length therefore remains finite as $T\to 0$. Similar behaviors are found for either spin component for the other DSG solitons with $|Q|=1$.

Here we will focus our attention on the static correlation functions for various cases of the DSG system. The calculation of the static two-spin correlation functions follows closely the approach of Polyakov.[19] In terms of path integrals we can write the cosine correlation function as

$$\langle\cos\phi(0)\cos\phi(r)\rangle=\frac{\int D\phi(x)e^{-\beta H[\phi(x)]}\cos\phi(0)\cos\phi(r)}{\int D\phi(x)e^{-\beta H[\phi(x)]}}\ , \quad (3.1)$$

with a similar expression for the sine components. We have in mind here case (i) of the DSG model where the magnetic field is applied along $\hat{y}$ (see Sec. II A of paper I) and ϕ is measured from $\hat{x}$. Thus the cosine (sine) components are along $\hat{x}$ ($\hat{y}$) and transverse (parallel) to the field. The "action" in Eq. (3.1) is the same as in Eq. (2.7), and without loss of generality we have taken r to be non-negative. The evaluations of the functional integrals in Eq. (3.1) follow practically the same procedures as before, giving

$$\langle\cos\phi(0)\cos\phi(r)\rangle\cong\frac{\cos^2\phi^0\prod_n(\lambda_n^0)^{-1/2}+e^{-\beta E^s}\prod_n{}'(\lambda_n^s)^{-1/2}\frac{2}{B}\left[\frac{\beta E^s}{\pi}\right]^{1/2}\int dx_0\cos\phi^s(x_0)\cos\phi^s(x_0-r)}{\prod_n(\lambda_n^0)^{-1/2}+e^{-\beta E^s}\prod_n{}'(\lambda_n^s)^{-1/2}\frac{2}{B}\left[\frac{\beta E^s}{\pi}\right]^{1/2}\int dx_0}\ . \quad (3.2)$$

In both the numerator and the denominator of this expression, the first term comes from the trivial solution ϕ^0 and the second term comes from the soliton solution ϕ^s. Although not written out explicitly for the soliton part, a sum over the types of soliton is to be carried out when the system can support more than one type of soliton. Thus in case (i) of the DSG model,[1] we should sum over the type (*a*) and (*b*) solitons. In the dilute soliton density limit we can write from Eq. (3.2)

$$\langle \cos\phi(0)\cos\phi(r)\rangle \cong \cos^2\phi^0 + \frac{2}{B}\left[\frac{\beta E^s}{\pi}\right]^{1/2} R^{1/2} e^{-\beta E^s} \int dx_0[\cos\phi^s(x_0)\cos\phi^s(r-x_0)-\cos^2\phi^0] \; . \tag{3.3}$$

In the second term on the right-hand side of Eq. (3.3), the factor in front of the integral is just the soliton density n^s (for a given type of soliton), and it carries all the temperature dependences for the correlation function. The integral over the centers of the solitons will be denoted by $F_c(r)$,

$$F_c(r) \equiv \int dx_0[\cos\phi^s(x_0)\cos\phi^s(r-x_0)-\cos^2\phi^0] \; . \tag{3.4}$$

$F_c(r)$ does not vary with the temperature, but depends in general on the detail shape of the soliton, and is the only r-dependent factor in the correlation function. Similarly for the sine component we define $F_s(r)$ by

$$F_s(r) \equiv \int dx_0[\sin\phi^s(x_0)\sin\phi^s(r-x_0)-\sin^2\phi^2] \; . \tag{3.5}$$

As shown below, the behavior of $F_c(r)$ and $F_s(r)$ as functions of r will in fact tell us the specific roles of the solitons in the corresponding correlation functions.

We first discuss the case where the system can only support solitons which are 2π-like, i.e., $|Q|=1$. Examples can be found in cases (ii) and (iii) of I. Case (iv) also belongs here unless $\nu\equiv 0$. In all these cases both x and y components of the spins assume practically the same values on opposite sides of the soliton. Small deviations come only from spin-wave-like fluctuations. By inspecting the integrands of Eqs. (3.4) and (3.5), we see that there are basically two contributions to $F_{c,s}(r)$. Since the solitons are rather localized objects, for r much larger than the width of the soliton the integrands are nonappreciable only in the immediate vicinity of $x_0=0$ and $x_0=r$. An integral over these two neighborhoods should yield a term which is independent of r, but rather depends on the detailed shape of the soliton. After multiplying by the factor n^s this term contributes only to the r-independent part of the correlation function, thus representing a very small decrease in the corresponding components of the magnetization as a result of the spins residing within the solitons. The second contribution comes from the overlap of the two solitons centered at $x_0=0$ and $x_0=r$, and is therefore r dependent. Since the solitons are rather localized, for $r \gg \kappa^{-1}$ the overlap is very small (even before multiplying by n^s). Moreover, from Eq. (I3.5) [Eq. (3.5) of I)] this contribution must decrease exponentially to zero as $r\rightarrow\infty$, meaning that these $|Q|=1$ solitons are irrelevant in determining the long-distance behavior for either of the two spin components.[25] We have in fact exactly evaluated some of these integrals for the $|Q|=1$ DSG solitons for all r, and in the limit $r\gg\kappa^{-1}$, we find precise agreement between the above simple intuition and the calculated results. Thus in general 2π solitons have little contribution to the long-distance behavior of these $|Q|=1$ soliton systems. As will be seen in Sec. V, the major contributions to the statics come in fact from the spin waves.[25]

Next we consider the static correlation functions for systems which support solitons with $|Q|\neq 1$. For concreteness, and for use later, we focus our attention on case (i) of the DSG system. Although the solitons here have $|Q|\neq 1$, because they carry the field $\phi(x)$ from $\phi_0\equiv\sin^{-1}(\mu/\lambda)$ to $\pi-\phi_0$, the sine component of spins nevertheless assumes practically the same value on opposite sides of a soliton. Thus this component is noncritical, and the soliton contribution to the corresponding static correlation function at long distances is expected to be very small, as is also the case for other static quantities. As we will see in Sec. V, the major contributions to these quantities in fact come from spin waves which are somehow modified by the presence of the two degenerate ground states.

The cosine components of spins however behave drastically different in that on opposite sides of a soliton, they assume the same magnitude of $[1-(\mu/\lambda)^2]^{1/2}$ but with opposite signs. Thus solitons will be vital in determining this component of the static correlation function, susceptibility, and correlation length. To have a clearer understanding of these soliton effects, we have calculated $F_c(r)$ exactly using the soliton solutions in Eq. (I2.12). The resulting expressions are somewhat complicated and will not be written out here, but what is important here is their behavior for

$r \gg \kappa^{-1}$. We find for both type (*a*) and (*b*) solitons that in this limit

$$F_c(r) \cong -2r\left[1-\left[\frac{\mu}{\lambda}\right]^2\right] + O(e^{-\kappa r}) \ . \qquad (3.6)$$

Putting this back into Eq. (3.3) we see that although the solitons have a finite creation energy, and so their densities are very low at $k_B T \ll E_a^s$ and E_b^s, their effects on the order-parameter correlation function in fact decrease linearly with distance. This indicates that the solitons dominate the long-distance behavior of this correlation function. On the other hand, the spin-wave contribution is comparatively much less important. This behavior is just the opposite of what we found for the sine (longitudinal) component, and for both components in systems with only 2π solitons.

The result of Eq. (3.6) could in fact be obtained easily without detailed calculations. For $r \gg \kappa^{-1}$ the detailed structure of the soliton is then insignificant, the only importance of the soliton is that they behave as true domain walls for the cosine component of the spins. Thus using Eq. (I2.12) and letting $\kappa \to \infty$, we have the expected result

$$\cos\phi_{a,b}^s(x) \cong \pm\left[1-\left[\frac{\mu}{\lambda}\right]^2\right]^{1/2} \mathrm{sgn}(x) \ . \qquad (3.7)$$

This component of the spins therefore behaves as an Ising system, with the solitons corresponding to the locations where spin flips take place. Using Eq. (3.7) in Eq. (3.4) gives

$$F_c(r) \cong \left[1-\left[\frac{\mu}{\lambda}\right]^2\right] \int dx_0[\mathrm{sgn}(x_0)\mathrm{sgn}(x_0-r)-1]$$

$$= -\left[1-\left[\frac{\mu}{\lambda}\right]^2\right] 2r \ , \qquad (3.8)$$

which is precisely the correct answer.[19]

Equations (3.3), (3.4), and (3.6) imply that for $r \gg \kappa^{-1}$ we can write, after summing over the two types of solitons for case (i) of the DSG system

$$\langle \cos\phi(0)\cos\phi(r)\rangle \cong \cos^2\phi^0[1-2(n_a^s+n_b^s)r] \ , \qquad (3.9)$$

where n_a^s and n_b^s are the total soliton densities for type (*a*) and type (*b*) solitons, respectively. The expression for the soliton densities can be found in Table I. Of course the correlation function cannot decrease without bound as r increases; this failure comes from the fact that we have only considered the contribution from a single soliton. For large r, multisoliton configurations must be taken into consideration. Following Polyakov,[19] Eq. (3.9) gives

$$\langle \cos\phi(0)\cos\phi(r)\rangle \cong \cos^2\phi^0 e^{-2n_{\mathrm{tot}}^s r} \qquad (3.10)$$

after taking into account multisoliton configurations within the dilute gas approximation. This is precisely the result one would expect from the TOM.[6] The exponential decay of correlations is of course expected for 1D systems at finite temperature. The correlation length which is given by

$$\xi_x = (2n_{\mathrm{tot}}^s)^{-1} \ , \qquad (3.11)$$

however, can be extremely long at temperatures much lower than E_a^s and E_b^s. The correlation function in wave-vector space is Lorentzian, with a very narrow width equal to $2n_{\mathrm{tot}}^s$:

$$\mathscr{S}_x(k) = \frac{4n_{\mathrm{tot}}^s}{(2n_{\mathrm{tot}}^s)^2+k^2} \ . \qquad (3.12)$$

The $k=0$ limit of $\mathscr{S}_x(k)$ gives the static susceptibility ($g\mu_B \equiv 1$)

$$\chi_x = \frac{\beta}{n_{\mathrm{tot}}^s} \ . \qquad (3.13)$$

Both ξ_x and χ_x have just the right forms appropriate to a system with an Ising symmetry, with E_a^s (the smaller of the soliton creation energies) playing the part of the Ising coupling constant.[26]

IV. TOPOLOGICAL CONSTRAINTS, EXTENSION TO POLYSOLITON SYSTEMS AND SUMMARY OF RESULTS FOR THE DSG MODEL

We must now consider the factor B which has been inserted in Eq. (2.24) and appears eventually in the expression for the soliton density [see Eq. (2.31)]. This factor is associated with the fact that when one tries to construct from single soliton solutions a multisoliton configuration that consists of solitons widely separated from each other, the topology of the single-soliton solution may impose very strict constraints on the sequencing of solitons in space. This observation was noted first for the ϕ^4 problem[27] and later discussed in more detail for other potentials.[11,2,3] Following the same line of reasoning, we see that for the 2π solitons there is no constraint on the sequencing of solitons, a soliton can be either followed by an antisoliton or another soliton. In constructing multisoliton configurations one must therefore treat the solitons and the antisolitons as different objects. Thus the

TABLE. I. Summary of the results for various cases of the DSG model. In the first column the upper [lower] sign is for the type (a) [(b)] solitons.

	(i)	(ii)	(iii)-(iv)
$V(\phi)$	$\lambda\left[\sin\phi-\frac{\mu}{\lambda}\right]^2$	$-\lambda\cos^2\phi+2\mu(1-\sin\phi)$	$\lambda\sin^2\phi+2\nu(1-\cos\phi)$
$\tilde{V}$	$-\frac{\mu^2}{\lambda}-\lambda$	-2μ	$-2\nu-\lambda$
ϕ^0	$\sin^{-1}\left[\frac{\mu}{\lambda}\right]$ and $\pi-\sin^{-1}\left[\frac{\mu}{\lambda}\right]$	$\frac{\pi}{2}$	0
$\kappa\equiv V''[(\phi_0)]^{1/2}$	$(2\lambda)^{1/2}\left[1-\left[\frac{\mu}{\lambda}\right]^2\right]^{1/2}$	$(2\lambda)^{1/2}\left[\frac{\mu}{\lambda}-1\right]^{1/2}$	$(2\lambda)^{1/2}\left[\frac{\nu}{\lambda}+1\right]^{1/2}$
$\phi^s(x)$	$2\tan^{-1}\left[\left[\frac{\lambda\mp\mu}{\lambda\pm\mu}\right]^{1/2}\tanh\frac{\kappa}{2}x\right]\pm\frac{\pi}{2}$	$2\sin^{-1}\left\{\left[\cosh^2\kappa x+\left[\frac{\lambda}{\mu-\lambda}\right]\sinh^2\kappa x\right]^{-1/2}\right\}+\frac{\pi}{2}$	$2\sin^{-1}\left\{\left[\cosh^2\kappa x-\left[\frac{\lambda}{\lambda+\nu}\right]\sinh^2\kappa x\right]^{-1/2}\right\}$
βE^s	$2(2\beta J)(2\lambda)^{1/2}\left\{\frac{\kappa}{(2\lambda)^{1/2}}+\frac{\mu}{\lambda}\left[\sin^{-1}\left[\frac{\mu}{\lambda}\right]\mp\frac{\pi}{2}\right]\right\}$	$4(2\beta J)(2\lambda)^{1/2}\left\{\frac{\kappa}{(2\lambda)^{1/2}}+\frac{\mu}{\lambda}\sin^{-1}\left[\left[\frac{\lambda}{\mu}\right]^{1/2}\right]\right\}$	$4(2\beta J)(2\lambda)^{1/2}\left\{\frac{\kappa}{(2\lambda)^{1/2}}+\frac{\nu}{\lambda}\sinh^{-1}\left[\left[\frac{\lambda}{\nu}\right]^{1/2}\right]\right\}$
$\eta_0^s(x)$	$\left[\frac{2\beta J}{\beta E^s_{a,b}}\right]^{1/2}\frac{\kappa\left[\frac{\lambda\mp\mu}{\lambda\pm\mu}\right]^{1/2}}{\left[\cosh^2\frac{\kappa x}{2}+\left[\frac{\lambda\mp\mu}{\lambda\pm\mu}\right]\sinh^2\frac{\kappa x}{2}\right]}$	$\left[\frac{2\beta J}{\beta E^s}\right]^{1/2}\frac{2(2\mu)^{1/2}\cosh\kappa x}{\left[\cosh^2\kappa x+\left[\frac{\lambda}{\mu-\lambda}\right]\sinh^2\kappa x\right]}$	$\left[\frac{2\beta J}{\beta E^s}\right]^{1/2}\frac{2(2\nu)^{1/2}\cosh\kappa x}{\left[\cosh^2\kappa x-\left[\frac{\lambda}{\lambda+\nu}\right]\sinh^2\kappa x\right]}$
N^s	$\frac{2\kappa^2}{(2\lambda)^{1/2}}$	$\frac{4\kappa^2}{(2\mu)^{1/2}}$	$\frac{4\kappa^2}{(2\nu)^{1/2}}$
f	$2J\tilde{V}+\frac{\kappa}{2\beta}-\frac{n^s_{\rm tot}}{\beta}$	$2J\tilde{V}+\frac{\kappa}{2\beta}-\frac{n^s_{\rm tot}}{\beta}$	$2J\tilde{V}+\frac{\kappa}{2\beta}-\frac{n^s_{\rm tot}}{\beta}$
$n^s_{\rm tot}$	$2\left[\frac{2\beta J}{\pi}\right]^{1/2}\frac{\kappa^{5/2}}{(2\lambda)^{1/2}}(e^{-\beta E^s_a}+e^{-\beta E^s_b})$	$8\left[\frac{2\beta J}{\pi}\right]^{1/2}\frac{\kappa^{5/2}}{(2\mu)^{1/2}}e^{-\beta E^s}$	$8\left[\frac{2\beta J}{\pi}\right]^{1/2}\frac{\kappa^{5/2}}{(2\nu)^{1/2}}e^{-\beta E^s}$

factor of 2 which has been inserted in Eq. (2.24) to account separately for contributions from the solitons and the antisolitons is appropriate. Thus $B=1$ for these cases. Belonging to this category are cases (ii)—(iv) of the DSG system.

The situation for case (i) is, however, a bit different. The topology of the system requires that in going from one soliton to the next either the type of soliton or the sign of Q (Ref. 24) must change, but not both. Thus if two adjacent particles are of the same type then one must be a soliton and the other an antisoliton. And if they are of different types then they must both be solitons or both antisolitons. Since contributions from the two types of solitons are being summed up separately, the counting of multisoliton configurations here is the same as for the ϕ^4 problem.[27] We must therefore consider the soliton and the antisoliton of a given type not as different particles, but as the same object in a different state. Thus the factor of 2 in Eq. (2.24) is unnecessary and we must divide the result by 2. As for case (i) of the DSG system, we must put B equal to 2.

It is interesting to note that there are two equivalent views on the topological constraints for the π-SG solitons. We can view the system as consisting of only one type of soliton, thus the counting of multisoliton configurations is the same as for the 2π solitons (i.e., the solitons and the antisolitons are to be treated as different objects), and thus $B=1$. We can also consider the π-SG system as a degenerate limit for case (i) of the DSG system, i.e., we view the solitons in regions (a) and (b) as belonging to different types (as was the view we adopted in I). We must then treat the solitons and the antisolitons of a given type as the same object (but in a different state), thus we put $B=2$. However, to obtain the total soliton contribution we must also sum up the contributions from the two types of solitons separately. The final result is therefore exactly the same as from the previous interpretation, as it must be.

The above considerations are very helpful in trying to extend our results here to include spin systems which can support more than two different types of soliton. Spin systems are necessarily 2π periodic, but within a 2π interval the potential $V(\phi)$ can have in general a number of degenerate ground states as a result of various anisotropies. Let this number be N_s so that there are N_s peaks separating these potential minima within the 2π interval. Let the peaks be labeled in increasing order in ϕ by an integer $j=1,2,\ldots,N_s$. To each peak there corresponds one type of soliton, which we denote by $S^j_\pm$ where j, defined modulo N_s, indicates the type of soliton, and $\pm$ gives the sign of the topological charge (i.e., $+$ for solitons and $-$ for antisolitons). It is easily seen that the topological constraint in constructing multisoliton configurations is that $S^j_\pm$ must be followed by either $S^{j\pm1}_\pm$. Note that requirement (3) on the potential implies that V'' at the minima ϕ^0 have the same value κ^2. Thus we can write

$$n^s_{\rm tot}=\frac{1}{B}\sum_{j=1}^{N_s} n^s_j\;, \tag{4.1}$$

where

$$n^s_j\equiv 2\left[\frac{2\beta J}{\pi}\right]^{1/2}\kappa^{1/2}N^s_j e^{-\beta E^s_j} \tag{4.2}$$

is the total density of type-j solitons plus antisolitons without imposing any topological constraints. The topological factor B can be obtained by demanding that in the limit that all types of solitons are identical, the result for $n^s_{\rm tot}$ should be the same as for a system with only one type of soliton. This gives $B=N_s$. Thus in general $n^s_{\rm tot}$ is just the average of the soliton densities over the number of types of solitons,

$$n^s_{\rm tot}=\frac{1}{N}\sum_{j=1}^{N_s} n^s_j\;. \tag{4.3}$$

Clearly, the N_s types of solitons need not be all different. For $N_s=1$ and 2 the previous results are recovered from Eq. (4.3). The extension of the calculation of static correlation functions (Sec. III) to systems with $N_s\geq 3$ is less trivial, and will not be reported here.

It is now straightforward to apply the results of Sec. II to obtain the free energy for each case of the DSG system. These results and other relevant quantities are summarized in Table I. The organization of the table and the notations used are as follows. Results for cases (i), (ii), and (iii)-(iv) are listed in columns 1, 2, and 3, respectively. The potential $V(\phi)$ is what we put into our Hamiltonian in Eq. (2.1). A constant term $\widetilde{V}$ has been separated out from the original potential as given by Eq. (I2.56) so that $V(\phi)$ in Eq. (2.1) is non-negative. The inverse trigonometric functions are understood to take on their principal values only. We have assumed, without loss of generality, that the parameters λ, μ, and ν, as defined in Eq. (I2.5c), are non-negative.[1] For case (i), (a) and (b) refer to the two types of solitons discussed in Sec. II A of I. No labels are put on quantities which are common to either type of soliton. In the expressions for the total soliton densities, which include for each case solitons and antisolitons of all possible types, the correct topological factors B have already been accounted for. Note that with the exception of $\widetilde{V}$ the results in column 3 can be obtained from those in column 2 by formally letting $\lambda \rightarrow -\lambda$, $\mu \rightarrow \nu$, and $\phi \rightarrow \phi - \pi/2$. The free energy per spin as given in the table represents only the configurational contribution; the kinetic part from Eq. (2.5) must be added to obtain the total free energy per spin. For case (i) the free energy and the order-parameter static correlation function of Eq. (3.10) have been derived before from the TOM and the PIGS.[13] The rest are new results.

V. INTERPRETATIONS OF RESULTS FOR THE DSG SYSTEM

With the results for various cases of the DSG system explicitly written out in Table I, we are now in a position to interpret these findings. The thermodynamic quantities discussed in I are only valid at $T=0$; we will see here now these results are modified by finite temperature fluctuations. For a given static quantity, the relative importance of solitons and spin-wave contributions is discussed in some detail. To facilitate our analysis, the predictions which are based on a purely harmonic theory can be found in Appendix B.

The total free energy per spin consists of a kinetic part [Eq. (2.5)] and a configurational part which is given for various cases in Table I. The physical origins of these contributions are quite clear from the way they are derived. This is precisely one of the advantages of the PIT. From Eq. (2.2) we see that the kinetic part is due to the small out-of-plane motions of the spins. For the configurational free energy, the first term, which is temperature independent, must be the ground-state contribution. The second term comes from Gaussian fluctuations about the ground state, and therefore represents the spin-wave contribution. The last term arises from the soliton solution and the fluctuations about this solution, and therefore represents the contribution from the "dressed" solitons. It must be emphasized that our results neglect spin-wave interactions and soliton-soliton interactions, and are applicable only at low temperatures and at temperatures $k_B T \ll E^s$ where the solitons form a dilute noninteracting gas.

From the free energy we can obtain by differentiations various thermodynamic quantities of interest, e.g., the specific heat and the longitudinal components of the magnetization and the static susceptibility. (For the DSG system, longitudinal and transverse refer, respectively, to directions within the easy plane which are parallel and perpendicular to the applied magnetic field. Note that depending on the case under consideration the direction of the applied field may be different.) The specific heats obtained for various cases have rather similar behavior; they always go to a constant value k_B, as $T \rightarrow 0$, as expected for our classical treatment of the problem. Other results, like the magnetization and the susceptibilities do, however, vary from case to case, and have rather interesting properties, as will be discussed below under different subheadings.

A. Case (i)

Since the magnetic field here is along $\hat{y}$, one can easily obtain from the free energy, using Table I, the longitudinal components of the magnetization

$$M_y = M_{\parallel} = \frac{\mu}{\lambda}\left[1 + \frac{1}{4\beta J \kappa_i} + \cdots \right], \tag{5.1}$$

and the static susceptibility

$$\chi_y = \chi_{\parallel} = \frac{1}{4J\lambda} + \frac{1}{8\beta J^2 \kappa_i^3} + \cdots . \tag{5.2}$$

The soliton contributions, which of course can be easily obtained, have not been written out explicitly in the above equations, since they are extremely small in the temperature region of interest. Comparing Eq. (5.2) with $\chi_{\parallel}$ calculated in the spin-wave approximation [Eq. (B11)] we see that although the y components of the spins do not form true domains (as opposed to the x components), the spin-wave contribution to χ_y is nevertheless modified by the presence of these solitons with $|Q| \neq 1$. This may be due to the fact that there are two equivalent ground states from which spin waves can be excited. The $T=0$ limit of Eqs. (5.1) and (5.2) of course agrees with the earlier results mentioned in I.

Next we consider physical quantities associated with the x components of the spins. Because the soliton here switches these components between the two distinct ground states (i.e., $\cos\phi^0 = \pm[1$

$-(\mu/\lambda)^2]^{1/2})$, M_x is identically zero at all finite temperatures. However, at $T=0$ there are no solitons, the degeneracy of the ground states is thereby broken given a nonzero value to M_x. This indicates again that $T=0$ can be viewed as a critical point, with the x component of the spin S_x as the order parameter. It follows that as $T\to 0$, χ_x must diverge. As we have discussed before, the long-distance behavior of S_x is dominated by the solitons; we can write from Eq. (3.13), neglecting the small and nonsingular contributions from the spin waves,

$$\chi_x = \frac{\beta}{n^s_{\rm tot}} , \tag{5.3}$$

which indeed diverges as $T\to 0$. The correlation length ξ_x which is given by $(2n^s_{\rm tot})^{-1}$ also has the same sort of essential singularity at $T=0$. Because of this dependence of ξ_x on $n^s_{\rm tot}$ we can view the "phase transition" at $T=0$ as caused by the *disappearance of solitons.* The behavior here is reminiscent of the Kosterlitz-Thouless transition of 2D *XY* systems,[28,29] which is driven by the disappearance of free vortices as $T\to T^+_{\rm KT}$. The solitons in 1D here and the free vortices in Kosterlitz-Thouless systems (2D) both are topological excitations which drive the phase transitions in these systems, and cause the correlation lengths to be nonanalytic at their respective transition temperatures. The usual strong scaling laws are obeyed provided that they are expressed in terms of their respective correlation lengths.[29] For our problem here we find $\tilde{\alpha}\equiv\alpha/\nu=-d=-1$, $\tilde{\gamma}\equiv\gamma/\nu=1$, $\delta=\infty$, and $\eta=1$.

Next we consider the behavior of the static quantities as μ (the magnetic field along $\hat{y}$) is varied. From the discussions in I, we do not expect any anomaly as $\mu\to 0$ at any temperature. At $\mu=0$ the forms of the potential in regions (a) and (b) are identical, but as μ is increased above zero, E^s_a decreases while E^s_b increases. The total soliton density in fact becomes higher, implying that χ_x and ξ_x will both decrease, as expected since the magnetic field suppresses spin fluctuations. However, as μ is increased to the bifurcation value λ, $\kappa\to 0$, and our results become nonsensical (e.g., $M_y\to\infty$ and $n_s\to 0$). The failure of our results near the bifurcation point will be discussed in more detail after considering case (ii).

B. Case (ii)

From the second column in Table I we easily find the results

$$M_y = M_{\|} = 1-(8\beta J\kappa_{ii})^{-1} + \cdots , \tag{5.4}$$

$$\chi_y = \chi_{\|} = (32\beta J^2\kappa_{ii}^3)^{-1} + \cdots , \tag{5.5}$$

omitting the negligible soliton contributions. Comparing Eqs. (5.4) and (5.5) with Eqs. (A8) and (A11) we see that M_y and χ_y are given to extremely high accuracy by spin-wave theory. Thus we expect that $\xi_y=\xi_{\|}=(2\kappa_{ii})^{-1}$, as given by Eq. (B9). For $\mu>\lambda$, M_x must vanish at all temperatures. Section III tells us that solitons can be ignored for the x components of the static correlation function; thus we can write from Eqs. (B10) and (B9)

$$\chi_x = \chi_\perp = (2J\kappa_{ii}^2)^{-1} , \tag{5.6}$$

$$\xi_x = \xi_\perp = \kappa_{ii}^{-1} . \tag{5.7}$$

Again our results are not expected to work near the bifurcation point.

Combining the results for M_y and χ_y for cases (i) and (ii), their behavior as a function of the magnetic field along $\hat{y}$ are shown in Figs. 2(a) and 2(b), respectively. The dot-dashed lines represent the re-

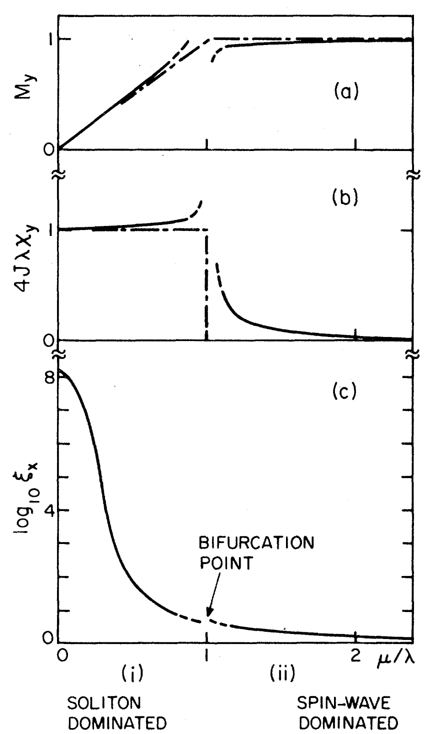

FIG. 2. Longitudinal magnetization (a), the longitudinal susceptibility (b), and the transverse correlation length (c) as a function of the magnetic field along $\hat{y}$. The parameters correspond roughly to $\beta J=10$ and $\lambda=0.2$. The dot-dashed lines represent the results at $T=0$.

sults at $T=0$. The curves have not been drawn close to the bifurcation point where our results are not expected to work.

Note that both ξ_x and χ_x for $\mu<\lambda$ are proportional to $(n^s_{\rm tot})^{-1}$, and therefore are extremely sensitive to the temperature and are very large. But when $\mu\to\lambda$, they become comparatively very small and are T independent [see Eqs. (5.6) and (5.7)]. Thus as μ is increased from zero to values above the bifurcation point, ξ_x and χ_x both decrease dramatically to T-independent values. The magnitudes for these decreases become even larger as T is reduced, because eventually at $T=0$ both ξ_x and χ_x are infinite for $\mu<\lambda$. The behavior for ξ_x is depicted in Fig. 2(c). The bifurcation region thus marks the boundary between a soliton-dominated regime and a spin-wave-dominated one. It should be very interesting to see how a theory which remains valid across the bifurcation region will connect these two drastically different regimes.

Although the mechanical properties of the solitons derived in I remain valid even at $\mu=\lambda$, their thermal properties discussed here certainly do not. Our results are based on the dilute soliton gas approximation, where soliton-soliton interactions are neglected. This is a quite reasonable assumption under the usual situation where the topological solitons are very localized in space (see Sec. III A of I), and when one works at temperatures much lower than the creation energies of any type of solitons in the system. Thus near the bifurcation point there are two major reasons to suspect the validity of our statistical mechanical results. First, for μ close to λ, κ_i and κ_{ii} are both very small; thus the solitons whose sizes are given under usual circumstances by κ^{-1} are certainly not localized. In fact at $\mu=\lambda$ they become the "algebraic solitons" as discussed in I. Second, the creation energy of the type (a) soliton approaches zero as $\mu\to\lambda^-$, thus for $\mu\lesssim\lambda$ a large number of these solitons are excited. All these features near the bifurcation point suggest that the solitons there are strongly interacting and therefore the noninteracting soliton gas picture is no longer valid.

The fact that κ_i and κ_{ii} both go to zero at the bifurcation point[1] implies that the quadratic terms in the Taylor expansion of $V(\phi)$ about the ground states also vanish there. Thus for μ sufficiently close to λ it becomes important that one should incorporate into the path-integral formalism quartic as well as quadratic fluctuations about the classical paths. A rough estimate of where this will occur may be obtained by studying the equation

$$\tfrac{1}{2}\kappa^2\langle\psi^2\rangle \gg \frac{1}{4!}V_0^{(4)}\langle\psi^4\rangle\ , \tag{5.8}$$

where $\psi\equiv\phi-\phi^0$, and $V_0^{(4)}=6\lambda$. We use the spin-wave results for $\langle\psi^2\rangle$ and $\langle\psi^4\rangle$ from Eq. (B6) by setting $r=0$. Equation (5.8) then becomes

$$\kappa^3 \gg \frac{\lambda}{4\beta J}\ . \tag{5.9}$$

From Eqs. (I3.10) and (I3.11) we can write, for μ near λ,

$$\kappa_i^3=(4\lambda\Delta)^{3/2}\ , \tag{5.10a}$$

$$\kappa_{ii}^3=(-2\lambda\Delta)^{3/2}\ , \tag{5.10b}$$

where $\Delta\equiv1-\mu/\lambda$ measures the distance away from the bifurcation point. Thus we expect that quartic fluctuations are not important when

$$\Delta \gg \Delta_L \equiv \frac{1}{4\lambda}\left[\frac{\lambda}{4\beta J}\right]^{2/3} \quad (\mu<\lambda) \tag{5.11a}$$

or

$$|\Delta| \gg \Delta_R\equiv 2\Delta_L \quad (\mu>\lambda)\ . \tag{5.11b}$$

Note that Δ_R is exactly twice Δ_L, and both are temperature dependent. They shrink in size as the temperature is reduced, since spin fluctuations are then suppressed. The above considerations suggest that our results may be trusted when μ/λ is far from the region given by $1-\Delta_L<\mu/\lambda<1+\Delta_R$. However, there is an additional consideration that has to be made. Recall from I that for $\mu\to\lambda^-$, besides the flattening of the two potential minima, the height of the smaller peak separating them also goes to zero, as they approach each other at $\pi/2$. This means that for $\mu\lesssim\lambda$ it is very easy for the spins to fluctuate across these two minima, since E_a^s becomes very small. Thus the density of the type (a) solitons cannot be dilute and the soliton gas picture is then no longer valid. This picture is further hampered since the size of both type (a) and (b) solitons also increases as $\mu\to\lambda^-$. This breakdown is just the second condition we have mentioned above.

We consider the restrictions which are imposed by the second condition on the parameters of the theory. Since the width of a soliton is essentially given by κ^{-1},[1] for the soliton density to be low we need to have

$$\kappa_i \gg n^s_{\rm tot}\ . \tag{5.12}$$

Note that the restrictions as given by Eqs. (5.11) and (5.12) depend on the temperature and the parameters of the theory in quite a different way. One must have both of these restrictions checked individually before any of the results here may be used.

C. Cases (iii) and (iv)

For these cases the magnetic field whose magnitude is parametrized by $\mu \equiv g\mu_B H_x/4J$, is applied along $\hat{x}$. From the free energy in the last column of Table I, one can easily obtain

$$M_x = 1-(8\beta J\kappa_{iv})^{-1} + \cdots \,, \tag{5.13}$$

$$\chi_x = (32\beta J^2\kappa_{iv}^3)^{-1} + \cdots \,. \tag{5.14}$$

Again the small soliton contributions have been omitted in writing these equations. From Appendix B we see that these results, with the soliton contributions omitted, agree *exactly* with the spin-wave predictions. Although for ν very close to zero n_{tot}^s becomes very large (see Table I) and our results can no longer be valid, they should be applicable for ν not too small. Moreover, we expect that all static quantities to be given essentially by spin waves; the contributions from these 2π solitons are exponentially small. This conclusion is also suppported by a detailed calculation using the TOM for the 2π-SG system.[22] Thus we can write from Appendix B,

$$\xi_x = (2\kappa_{iv})^{-1} = \tfrac{1}{2}\xi_y \,, \tag{5.15}$$

$$\chi_y = (2J\kappa_{iv}^2)^{-1} \,. \tag{5.16}$$

Symmetry implies that $M_y \equiv 0$ at all temperatures. Note that ξ_x saturates in the temperature range of interest. However, at $\nu=0$ the DSG system becomes the π-SG system, and we know, either by putting $\mu=0$ in case (i) or from a previous work,[21,22] that this system has at low T an extremely long and temperature-sensitive order-parameter correlation length:

$$\xi_x(\nu=0) = (2n_{\text{tot}}^{\pi})^{-1} \,. \tag{5.17}$$

The density for the π-SG solitons n_{tot}^{π} can be obtained here either from the first or the last column of Table I by setting the fields equal to zero. We see from Eq. (5.15) that as ν is decreased, ξ_x increases gradually (in a temperature-independent way), as expected, since spin fluctuations are enhanced. This trend continues across $\nu=\lambda$ to small ν values, until ν is so close to zero that Eq. (5.15) no longer applies. But at $\nu=0$, ξ_x must have somehow increased to an extremely large and T-dependent value given by $(2n_{\text{tot}}^{\pi})^{-1}$. The lower the temperature the more this anomalous increase has to be. The same behavior, of course, also applies to χ_x, since at $\nu=0$ it is given by $\beta/n_{\text{tot}}^{\pi}$. It should be very interesting to see how a theory which remains valid even for $\nu \simeq 0$ would give results which would connect these two very different behaviors. We do not expect any anomaly, however, across the point $\nu=\lambda$. Figure 3 illustrates M_x and ξ_x as a function of ν.

As $\nu \to 0$ our calculation implies a diverging soliton energy, which is definitely an absurd result and signifies, in fact, a breakdown of the present formalism. Since the lifetime of the metastable state is approaching infinity in this limit, its effect must now be accounted for. Furthermore, the occurrence of reactions where a 2π-DSG soliton dissociates into two π-SG solitons at $\nu=0$ [see Eq. (I3.15)] implies that at the moment these π solitons are formed they must repel each other very strongly. We can also consider the 2π-DSG solitons just before their dissociation. The groups of spins with ϕ between 0 and π, and between π and 2π, must interact very strongly with each other. The repulsion between these two groups of spins is counteracted by the binding force of the soliton, thus resulting in a wobbling motion about their equilibrium separation. This separation increases with the size of the soliton as $|\ln\nu|$ (see Fig. I5). At $\nu=0$ their repulsion finally wins, and they separate to form two π-SG solitons. This sort of wobbler for small ν has been observed for example in self-induced transparency experiments in sodium vapor.[30] These internal structures, as well as the in-

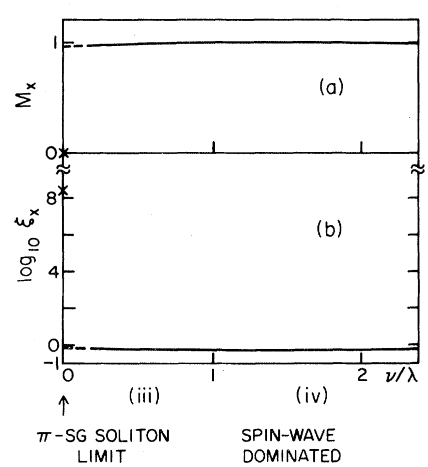

FIG. 3. Longitudinal components of the magnetization and the correlation length as a function of the magnetic field along $\hat{x}$. The parameters correspond roughly to $\beta J=10$ and $\lambda=0.2$.

crease in size of the 2π-SG solitons as $\nu\rightarrow 0$, $\nu \lesssim 0$, cause serious problems in the present formalism which is based on a noninteracting soliton-gas picture.

VI. CONCLUDING REMARKS

Although we have studied the statistical mechanics of the DSG model only along certain symmetry directions in the entire (λ,μ,ν) parameter space [see Fig. I1(a)], the qualitative behavior for general parameter values is quite clear based on what we have learned here. In the $\nu=0$ plane and for μ less than but some distance away from λ, the bifurcation value, the long-distance behavior of the order-parameter of the system is dominated by the (*a*) and the (*b*) DSG solitons at low temperatures. The dilute soliton gas picture is valid for $k_B T \ll E_a^s$ and E_b^s. The order-parameter correlation length and the susceptibility are both very large. Slightly away from this plane, i.e., $|\nu| \gtrsim 0$, but still within the shaded region of Fig. I6, either the type (*a*) or the type (*b*) soliton becomes unstable. The low-temperature statistical mechanics are then strongly influenced by the presence of metastable solitons and solitons that are strongly interacting. Our approach here, which is based on a dilute noninteracting soliton gas, is therefore invalid. Nevertheless, we can still expect the correlation length and the susceptibility to be rather large, although somewhat smaller than for the above $\nu=0$ case. Away from the shaded region, these quantities should decrease rapidly to values that are determined basically by spin-wave fluctuations. The behavior across the bifurcation point $\nu=0$, $\mu=\lambda$ has been discussed in Sec. V for a constant λ. The behavior of the system as the bifurcation point (line) is crossed from general directions should be even more intriguing, but again unfortunately cannot be obtained from the present work.

ACKNOWLEDGMENTS

The author wishes to thank Dan Hone for useful suggestions in the preparation of the manuscript. The finanacial support by NSF Grant No. DMR80-08004 is gratefully acknowledged.

APPENDIX A: EVALUATION OF R FOR GENERAL POTENTIALS

Here we will evaluate R of Eq. (2.29) for a general $V(\phi)$ following closely the work of Coleman.[11] First, use is made of the fact that the problem of finding the infinite product of eigenvalues of a Sturm-Liouville operator can be reduced to that of solving a corresponding initial value problem.[31] This allows one to write

$$R=\frac{g^0(L/2)\lambda_0^s(L)}{g^s(L/2)}\ , \qquad \text{(A1)}$$

where the function $g^c(x)$ satisfies the equation

$$-\frac{1}{2}\frac{d^2}{dx^2}g^c(x)+\tfrac{1}{2}V''[\phi^c(x)]g^c(x)=0\ , \qquad \text{(A2a)}$$

with the initial conditions

$$g^c(x)\Big|_{-L/2}=0,\quad \frac{d}{dx}g^c(x)\Big|_{-L/2}=1\ . \qquad \text{(A2b)}$$

For the trivial solution $\phi^c(x)=\phi^0$, since $V''(\phi^0)\equiv\kappa^2$ is independent of x, it is easy to calculate $g^0(x)$. For large L one finds

$$g^0(L/2)=\frac{e^{\kappa L}}{2\kappa}\ . \qquad \text{(A3)}$$

This gives, from Eqs. (2.27) and (2.18) the spin-wave contribution to the free energy per spin as

$$f^0=\frac{\kappa}{2\beta}\ . \qquad \text{(A4)}$$

To obtain $g^s(L/2)$, note that $\eta_0^s(x)$ obeys Eq. (A2a) but fails to satisfy the boundary conditions of Eq. (A2b) to be the desired function $g^s(x)$. However the second-order differential equation [Eq. (A2a)] must have a second linearly independent solution $\xi_0^s(x)$. The trick, which is due to Coleman,[11] is to choose the correct linear combination of η_0^s and ξ_0^s so as to fit the required initial conditions for $g^s(x)$. Therefore, we write

$$g^s(x)=a\eta_0^s(x)+b\xi_0^s(x)\ . \qquad \text{(A5)}$$

To properly normalize $\xi_0^s(x)$ we require the Wronskian to be unity:

$$\eta_0^s(x)\frac{d}{dx}\xi_0^s(x)-\xi_0^s(x)\frac{d}{dx}\eta_0^s(x)=1\ . \qquad \text{(A6)}$$

Demanding that $g^s(x)$ satisfies the initial conditions of Eq. (A2b), and utilizing the Wronskian condition gives $a=-\xi_0^s(-L/2)$ and $b=\eta_0^s(-L/2)$. Thus we have

$$g^s(L/2)=\eta_0^s(-L/2)\xi_0^s(L/2)-\xi_0^s(-L/2)\eta_0^s(L/2)\ . \qquad \text{(A7)}$$

To find the asymptotic forms of η_0^s and ξ_0^s, note from Eq. (I3.5) that η_0^s must behave as $e^{-\kappa|x|}$

asymptotically. Therefore we write, for $|x|\to\infty$,

$$\eta_0^s(x)\cong\left[\frac{2J}{E^s}\right]^{1/2}N^s e^{-\kappa|x|}\ , \tag{A8}$$

where the prefactor N^s, which is of course temperature independent, can be obtained from the explicit form for $\phi^s(x)$. This equation together with Eq. (A6) gives the asymptotic form for $\xi_0^s(x)$:

$$\xi_0^s(x)\cong\left[\frac{E^s}{2J}\right]^{1/2}(2\kappa N^s)^{-1}\mathrm{sgn}\left[x-\frac{L}{2}\right]e^{\kappa|x|}\ . \tag{A9}$$

Using Eqs. (A8) and (A9) in Eq. (A7) gives

$$g^s(L/2)=\kappa^{-1}\ . \tag{A10}$$

The final task in obtaining R is to compute $\lambda_0^s(L)$, which is of course identically zero in the $L\to\infty$ limit. However, for finite L, it is finite, because in that case $\eta_0^s(x)$, as given by Eq. (2.21), does not vanish at $\pm L/2$. The function violates the boundary conditions of Eq. (2.13b) by an exponentially small amount, as is indicated by Eq. (A8). The actual η_0^s is in fact a bit more localized than that of Eq. (2.21). The actual eigenvalue $\lambda_0^s(L)$ must therefore be slightly larger than zero. We expect it to be a positive but exponentially small quantity. The smallness of $\lambda_0^s(L)$ suggests that it may be calculated perturbatively from Eq. (2.13a).[11]

Consider the Schrödinger equation

$$(H^s+\lambda_0^s)\psi=0\ , \tag{A11a}$$

where

$$H^s\equiv-\frac{1}{2}\frac{d^2}{dx^2}+\tfrac{1}{2}V''[\phi^s(x)]\ , \tag{A11b}$$

and the wave function ψ satisfies exactly the boundary conditions

$$\psi(-L/2)=\psi(L/2)=0\ . \tag{A11c}$$

Let us define the unperturbed wave function $\psi_0(x)$ such that

$$H^s\psi_0=0$$

and

$$\psi_0(-L/2)=\psi_0(L/2)=0\ . \tag{A12}$$

The strategy for calculating λ_0^s is to identify $\lambda_0^s\psi(x)$ of Eq. (A11a) as a perturbation and calculate the effects on $\psi_0(x)$. This computation is facilitated by noting that since $H^s\eta_s^s=H^s\xi_0^s=0$, one can therefore construct the unperturbed Green's function $G(x,x')$ from η_0^s and ξ_0^s. The result is

$$G(x,x')=-2\theta(x-x')[\eta_0^s(x)\xi_0^s(x')-\xi_0^s(x)\eta_0^s(x')]\ . \tag{A13}$$

The exact wave function ψ is then given by the Lippmann-Schwinger equation as

$$\psi(x)=\psi_0(x)-\lambda_0^s\int_{-L/2}^{x}dx'G(x,x')\psi(x')\ . \tag{A14}$$

Born's approximation gives, to lowest order in λ_0^s,

$$\psi(x)=\psi_0(x)-\lambda_0^s\int_{-L/2}^{x}dx'G(x,x')\psi_0(x')\ . \tag{A15}$$

Since we are only interested in getting λ_0^s, but not $\psi(x)$, there is no need to properly normalize $\psi_0(x)$, which therefore can be constructed from η_0^s and ξ_0^s as

$$\psi_0(x)=\eta_0^s(x)+p\xi_0^s(x)\ . \tag{A16}$$

The condition $\psi_0(-L/2)=0$ implies that

$$p=-\eta_0^s(-L/2)/\xi_0^s(-L/2)\ .$$

Demanding that $\psi(L/2)=0$ in Eq. (A15), and using Eqs. (A13) and (A16) gives, in the limit $L\to\infty$,

$$\lambda_0^s(L)=\left[\frac{2J}{E^s}\right]2\kappa(N^s)^2e^{-\kappa L}\ . \tag{A17}$$

As anticipated, λ_0^s is indeed an exponentially small quantity.

Substituting Eqs. (A3), (A10), and (A17) into Eq. (A1) yields finally the result

$$R=\left[\frac{2J}{E^s}\right]\kappa(N^s)^2\ . \tag{A18}$$

It is reassuring to see that R is indeed finite, since the divergent factor λ_0^s has already been extracted.

APPENDIX B: SPIN-WAVE CALCULATIONS FOR PLANAR SYSTEMS

When the system has a single physically distinct ground state ϕ^0 we expect the solitons to give little contribution to thermodynamic quantities and static correlation functions. Thus for these properties it suffices to consider only the spin-wave (sw) excitations. In order to be consistent with the form of our Hamiltonian in Eq. (21), it is important that one should use a spin-wave theory which is appropriate for planar spin systems. Such a calculation has been given previously using Villain's theory[32]; however, for the establishment of our notations here and for references in Sec. V, the main results will be given below.

Defining ψ as $\phi-\phi^0$, the Hamiltonian of Eq. (2.1) can be written, up to quadratic terms in ψ, in the form

$$H_{\rm sw}=2J\int dx\left[\frac{\dot\psi^2}{2c^2}+\frac{\psi_x^2}{2}+\frac{\kappa^2}{2}\psi^2\right], \qquad \text{(B1)}$$

where $\kappa^2\equiv V''(\phi^0)$. For planar spin systems we have from Eq. (I2.4)

$$\dot\psi=\dot\phi\cong\frac{c^2}{2J}\theta\cong\frac{c^2}{2J}S^z . \qquad \text{(B2)}$$

Thus the harmonic Hamiltonian can be written, in terms of a pair of canonical operators S_k^z and ψ_k,[32] in the form

$$H_{\rm sw}=\tfrac{1}{2}\sum_k(a_kS_k^zS_{-k}^z+b_k\psi_k\psi_{-k}) , \qquad \text{(B3)}$$

where $a_k\equiv c^2(2J)^{-1}$ is in fact k independent, $b_k\equiv 2J\omega_k^2$, and $\omega_k\equiv(\kappa^2+k^2)^{1/2}$. From Eq. (B3) the spin-wave energy is easily found to be $E_k=\omega_kc$. This c is the intrinsic spin-wave velocity and κc is the spin-wave gap of the system. One also readily obtains the static correlation functions for spin components within the easy plane:

$$\mathscr{S}_\perp(r)\equiv\langle\sin\psi(0)\sin\psi(r)\rangle$$
$$\cong\langle\psi(0)\psi(r)\rangle$$
$$=(4\beta J\kappa)^{-1}e^{-\kappa r} , \qquad \text{(B4)}$$

$$\mathscr{S}_\perp(k)=(2\beta J)^{-1}(k^2+\kappa^2)^{-1} , \qquad \text{(B5)}$$

$$\mathscr{S}_{||}(r)=\langle\cos\psi(0)\psi(r)\rangle$$
$$\cong 1-\langle\psi^2\rangle+\tfrac{1}{4}\langle\psi^2(0)\psi^2(r)\rangle ,$$
$$=1-\frac{1}{4\beta J\kappa}+\frac{1}{2(4\beta J\kappa)^2}e^{-2\kappa r} , \qquad \text{(B6)}$$

$$\mathscr{S}_{||}(k)=\frac{1}{(4\beta J\kappa)^2}\frac{2\kappa}{(2\kappa)^2+k^2} . \qquad \text{(B7)}$$

The magnetization is given by

$$M_{||}=1-(8\beta J\kappa)^{-1} . \qquad \text{(B8)}$$

The correlation length for transverse spin fluctuations is therefore κ^{-1}, which coincides with the width of the soliton. It is also exactly twice as long as for the longitudinal components:

$$\xi_\perp=\kappa^{-1}=2\xi_{||} . \qquad \text{(B9)}$$

The static susceptibility can be obtained from the corresponding static correlation functions from Eqs. (B4) and (B7) by setting $k=0$ and multiplying by β to get, for $g\mu_B\equiv 1$,

$$\chi_\perp=(2J\kappa^2)^{-1} , \qquad \text{(B10)}$$

$$\chi_{||}=(32\beta J^2\kappa^3)^{-1} . \qquad \text{(B11)}$$

Obviously the results written here for the spin waves are universal for the class A systems.

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