PHYSICAL REVIEW B VOLUME 26, NUMBER 1 1 JULY 1982

Critical behavior of m-component magnets with correlated impurities

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(Received 16 February 1982)

We study the critical behavior of an m-component classical spin system with quenched impurities correlated along an ϵ_d-dimensional "line" and randomly distributed in $d-\epsilon_d$ dimensions ($d=4-\epsilon$). The presence of this line of impurities makes the system anisotropic and the interactions highly nonlocal. The renormalization group (RG) is used to approach the critical region and the quantities of interest are calculated in a double ϵ,ϵ_d expansion. A two-loop calculation is needed to expose fully the divergent structure, and the theory is proved to be renormalizable up to this order. A consequence of the double ϵ,ϵ_d expansion is the fact that the RG functions and consequently the critical exponents depend on the ratio $\epsilon_d/(\epsilon+\epsilon_d)$. The solution of the RG equations leads to the existence of two correlation lengths: parallel to the "line" and perpendicular to it, with critical exponents $\nu_{\parallel}$ and $\nu_{\perp}$, respectively, with the relation $\nu_{\parallel}=z\nu_{\perp}$. The exponent z results from the presence of anisotropy in the system. New scaling laws are found for the critical exponents: $\gamma=\nu_{\perp}(2-\eta)$ and $\alpha=2-(d-\epsilon_d)\,\nu_{\perp}-\epsilon_d\nu_{\parallel}$. We establish a relation between our model and a quantum model in one less dimension with random pointlike impurities. For this system we predict a quantum-to-classical crossover at finite temperature with crossover exponent $\nu_{\parallel}^{-1}$.

I. INTRODUCTION

The effect of randomly distributed pointlike impurities on the critical behavior of magnets has been studied by several authors.[1–3] Lubensky[4] has considered a model with random pointlike impurities, and using renormalization-group (RG) techniques he found fixed points in an ϵ expansion ($\epsilon=4-d$, d is the dimensionality of the system) and calculated the critical exponents. More recently Dorogovstev[5] has studied the case of line defects using the RG approach and ϵ expansion. He proposes a model in which the impurities are correlated in ϵ_d dimensions and distributed at random in $d-\epsilon_d$ dimensions.

In his work an expansion in terms of both ϵ and ϵ_d is suggested and the RG functions are calculated to order ϵ,ϵ_d. The case $\epsilon=2$, $\epsilon_d=1$ corresponds to the model studied by McCoy and Wu.[6]

However, in this paper we point out that the full structure of the theory has not been taken into account properly. Actually the system is no longer isotropic, and as will be shown in the next section, the interactions are highly nonlocal. As a result, new divergent quantities appear and the infrared behavior of the theory is drastically changed. The idea of two different correlation lengths naturally arises since we expect the system to behave differently along directions "parallel" to the impurity "line" and in the "perpendicular" directions.

We will use the RG and the ϵ expansion[7–9] to study the critical region. Even though we will be ultimately interested in the case where ϵ_d is an integer, it will be argued in the following sections that an expansion in terms of ϵ and ϵ_d must be used. The anisotropy is taken into account by defining a parameter α_0 that differentiates between the parallel and perpendicular directions. We show that this parameter is indeed nontrivially renormalized already in lowest order in perturbation theory, and that in order to expose the full structure of the RG a calculation up to second order must be performed.

A novel feature of the calculation is that the RG functions and critical exponents depend on the ratio $\epsilon_d/(\epsilon+\epsilon_d)$. The solution to the RG equations brings a new set of scaling laws between exponents and we learn that $\gamma=\nu_{\perp}(2-\eta)$ and $\alpha=2-(d-\epsilon_d)\nu_{\perp}-\epsilon_d\nu_{\parallel}$, where γ and α are the usual susceptibility and specific-heat exponents and $\nu_{\parallel}$ and $\nu_{\perp}$ are the critical exponents for the parallel and perpendicular correlation lengths, respectively.

The paper is organized as follows. In Sec. II we define the model and extract some exact results about the relevance of randomness in the system. In Sec. III the technical details of the renormalization program are discussed and the calculation of the different functions through second order is performed. Section IV is devoted to the solution of the RG equations obeyed by the two-point vertex function and the specific-heat vertex in the scaling regime and the scaling laws are extracted. In Sec. V we discuss the connection of our model with a quantum spin system at finite temperature with point impurities distributed at random. We learn that our model is equivalent to a quantum impurity system for $\epsilon_d = 1$, and the crossover from quantum-to-classical behavior is discussed. An appendix is devoted to some technical details in the calculation of Feynman integrals.

II. THE MODEL

We start with a model described by a Hamiltonian of the Landau-Ginzburg type for a magnet with $O(m)$ symmetry:

$$\mathscr{H}[\varphi,\delta r] = \int d^d x \left[\frac{1}{2} \sum_{i=1}^{m} [\nabla \varphi_i(x)]^2 + \sum_{i=1}^{m} [r + \delta r(x)] \varphi_i^2(x) + \frac{u}{4!} \left[\sum_{i=1}^{m} \varphi_i^2(x) \right]^2 \right], \tag{2.1}$$

where $\delta r(x)$ is the impurity field and r is the mass of the field $\varphi(x)$.

The partition function of the system for a given (quenched) configuration of impurities is

$$Z[\delta r] = \int \mathscr{D}\varphi(x) e^{-\mathscr{H}[\varphi,\delta r]}, \tag{2.2}$$

and the free energy is

$$\mathscr{F}[\delta r] = -\ln Z[\delta r]. \tag{2.3}$$

The quenched free energy is given by the average over the probability[10] distribution $P[\delta r]$:

$$\mathscr{F} = \langle\langle \mathscr{F}[\delta r] \rangle\rangle. \tag{2.4}$$

(We denote $\langle\langle A \rangle\rangle = \mathrm{Tr}_{\{\delta r(x)\}}\{P[\delta r(x)] A\}$.) Using the identity $\ln Z = \lim_{n\to 0}(Z^n - 1)/n$ the free energy is given by

$$\mathscr{F} = -\lim_{n\to 0} \left[\frac{\langle\langle Z^n \rangle\rangle}{n} - \frac{1}{n} \right]. \tag{2.5}$$

Write Z^n as the product of n replicas of the same model. Then

$$\langle\langle Z^n \rangle\rangle = \int \mathscr{D}\varphi_1(x) \ldots \mathscr{D}\varphi_n(x) \mathrm{Tr}_{[\delta r(x)]} P[\delta r(x)] \times \exp\left[- \left\{ \int d^d x \sum_{\alpha=1}^{n} \left[\frac{1}{2} \sum_{i=1}^{m} [\nabla \varphi_{\alpha i}(x)]^2 + \sum_{i=1}^{m} [r + \delta r(x)] \varphi_{\alpha i}^2(x) + \frac{u}{4!} \left[\sum_{i=1}^{m} \varphi_{\alpha i}^2(x) \right]^2 \right] \right\} \right]. \tag{2.6}$$

Now we define the impurity-probability distribution to yield

$$\langle\langle \delta r(x) \rangle\rangle = 0, \tag{2.7a}$$

and

$$\langle\langle \delta r(x) \delta r(x') \rangle\rangle = \Delta \delta^{d-\epsilon_d}(x - x') \tag{2.7b}$$

(with $\Delta > 0$). We expand (2.6) in cumulants:

$$\left\langle\left\langle \exp\left[-\int d^dx \sum_{\alpha=1}^{n}\sum_{i=1}^{m}\delta r(x)\varphi_{\alpha i}^2(x)\right]\right\rangle\right\rangle = \exp\left[\tfrac{1}{2}\int d^dx\, d^dx' \langle\langle \delta r(x)\delta r(x')\rangle\rangle \times \sum_{\alpha i}\sum_{\beta j}\varphi_{\alpha i}^2(x)\varphi_{\beta j}^2(x') + \cdots\right], \tag{2.8}$$

where the ellipsis represents higher-order cumulants of the form: $\cdots \varphi_{\alpha i}^2(x')\varphi_{\beta j}^2(x')\varphi_{\gamma k}^2(x'')\cdots$. These operators may be dropped because they are irrelevant by power counting, near four dimensions.[11]

The partition function reads:

$$Z = \int \mathscr{D}\varphi_\alpha(x)\exp\left[-\left\{\int d^dx\left[\sum_{\alpha,i}\frac{[\nabla\varphi_{\alpha i}(x)]^2}{2} + r\sum_{\alpha i}\varphi_{\alpha i}^2(x) + \frac{u}{4!}\sum_{\alpha}\left[\sum_i[\varphi_{\alpha i}^2(x)]\right]^2\right] - \frac{\Delta}{2}\int d^dx\int d^dx'\delta^{d-\epsilon_d}(x-x')\sum_{\alpha i}\varphi_{\alpha i}^2(x)\sum_{\beta j}\varphi_{\beta j}^2(x')\right\}\right]. \tag{2.9}$$

In this expression $\alpha,\beta=1,2,\ldots,n$ are the replica labels and $i,j=1,2,\ldots,m$ are $O(m)$ symmetry indices. Expression (2.9) clearly indicates that the system is anisotropic in space: The impurities are correlated along ϵ_d dimensions, whereas in the $d-\epsilon_d$ remaining directions the impurity-correlation function (2.7b) is zero for impurities at different points.

Since we will use the renormalization-group[9,11] approach to understand the critical behavior, this leads us to the search for fixed points in the parameter space. We will calculate the functions of interest in perturbation theory, so that in order to have consistent and meaningful results we have to compute the contributions from perturbation theory in an $\epsilon(d=4-\epsilon)$ expansion. However, our system is characterized by two dimensionalities, namely ϵ and ϵ_d, and the Feynman diagrams of the perturbative expansion will involve, in general, both dimensionalities.

In the next section it will be shown that the framework of a double expansion in terms of ϵ and ϵ_d is needed. The perturbation expansion is defined in terms of the coupling constants u and Δ represented by the vertices in Fig. 1.

Hereafter we set the (renormalized) mass r_R to zero and work with a massless theory. Since the theory is anisotropic, there is the need to distinguish between the parallel and the perpendicular components of the momenta. We then generalize the (bare) propagator to be

$$G^0(p^{\|},p^{\perp}) = \frac{1}{\alpha_0 p^{\|2} + p^{\perp 2}}, \tag{2.10}$$

where it will emerge that the anisotropy constant α_0 is renormalized.

In order to define dimensionless couplings let us analyze the canonical dimensionalities[11] of the various operators and couplings. Throughout this analysis we will regard the components of the momenta to have different units along ϵ_d directions than those along $d-\epsilon_d$ dimensions.

The propagator (2.10) suggests

$$[\alpha] = \frac{[\kappa^{\perp}]^2}{[\kappa^{\|}]^2}, \tag{211.a}$$

where [ ] stands for units in terms of momentum

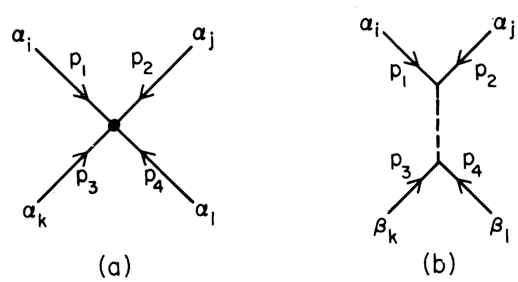

FIG. 1. Interaction vertices:

$$(a) = -\frac{u}{4!}S_{ijkl}\delta^{\|}\Big(\sum p_i^{\|}\Big)\delta^{\perp}\Big(\sum p_i^{\perp}\Big),$$

$$S_{ijkl} = \tfrac{1}{3}\times(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}).$$

$$(b) = \frac{\Delta}{2}T_{ijkl}\delta^{\|}(p_1+p_2)\delta^{\|}(p_3+p_4)\delta^{\perp}\Big(\sum p_i^{\perp}\Big),$$

$$T_{ijkl} = \delta_{ij}\delta_{kl}.$$

The superscripts $\|$ and $\perp$ mean along ϵ_d directions and along $d-\epsilon_d$ directions, respectively.

(inverse length). Since the argument of the exponential in (9) must be dimensionless, we then find

$$[\varphi]=[\kappa^{\perp}]^{d/2-1}\left[\frac{\kappa^{\parallel}}{\kappa^{\perp}}\right]^{\epsilon_d/2}, \tag{2.11b}$$

$$[u]=[\kappa^{\perp}]^{4-d}\left[\frac{\kappa^{\perp}}{\kappa^{\parallel}}\right]^{\epsilon_d}, \tag{2.11c}$$

$$[\Delta]=[\kappa^{\perp}]^{4-d+\epsilon_d}. \tag{2.11d}$$

This analysis leads us to define

$$g=u\alpha^{-\epsilon_d/2}\kappa^{-\epsilon},\quad \delta=\Delta\kappa^{-\tilde{\epsilon}}, \tag{2.12}$$

where g and δ are dimensionless and $\tilde{\epsilon}=\epsilon+\epsilon_d$. After the perturbative series is summed we are instructed to let the number of replicas go to zero [Eq. (2.5)]. However, each Feynman diagram is analytic in terms of n, so we can take the limit $n\to 0$ order by order in perturbation theory. As a result, those diagrams with an internal loop carrying a free-replica index will vanish in this limit. A loop carrying a free-replica index is connected to the rest of the diagram only by dashed lines (Fig. 2).

Before we begin the task of renormalizing the theory, some exact results can be inferred. If the term $\delta r(x)\varphi^2(x)$ in (2.6) is regarded as a perturbation, the free energy can be expanded around the "pure" term, and averaging over the quenched probability distribution we get

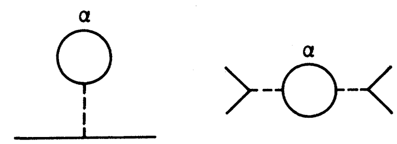

FIG. 2. Graph with an internal free-replica index α.

$$\langle\langle \mathscr{F}\rangle\rangle=\mathscr{F}_{\text{pure}}+\Delta\int d^d r\int d^d r'\delta^{d-\epsilon_d}(r-r')\times\langle\varphi^2(r)\varphi^2(r')\rangle_{\text{pure}}+\cdots, \tag{2.13}$$

where $\langle\,\rangle_{\text{pure}}$ stands for the average in the pure theory, and we have used (2.7a) and (2.7b).

The term $\int d^d r\langle\varphi^2(r)\varphi^2(r')\rangle_{\text{pure}}$ is proportional to the specific heat of the pure system and (2.13) can be written as

$$\langle\langle \mathscr{F}\rangle\rangle=\mathscr{F}_{\text{pure}}+\Delta\int d^{\epsilon_d}r'\,t^{-\alpha}f(r'/\xi)+O(\Delta^2), \tag{2.14}$$

where ξ is the correlation length in the pure system, and t is the reduced temperature. This relation can be cast as

$$\langle\langle \mathscr{F}\rangle\rangle=\mathscr{F}_{\text{pure}}+\text{const}\times\Delta t^{-(\alpha+\epsilon_d\nu)}+O(\Delta^2). \tag{2.15}$$

For $\alpha+\epsilon_d\nu>0$ randomness is relevant. We identify $\phi=\alpha+\epsilon_d\nu$ as the crossover exponent. This is the generalization of the Harris criterion.[12]

TABLE I. Contributions to $\partial\Gamma^{(2)}/\partial p^{\parallel 2}$ and $\partial\Gamma^{(2)}/\partial p^{\perp 2}$ from graphs in Fig. 3. $P(p)=\ln(\alpha p^{\parallel}/\kappa)$.

Graph	$\frac{\partial}{\partial p^{\parallel 2}}$	$\frac{\partial}{\partial p^{\perp 2}}$
(1)	$-4\frac{\alpha_0}{\tilde\epsilon}\left[1-\frac{\tilde\epsilon}{2}-\frac{\tilde\epsilon}{2}P(p)\right]$	
(2)	$-16\frac{\alpha_R}{\tilde\epsilon^2}[1-\frac{3}{2}\tilde\epsilon-\tilde\epsilon P(p)]$	
(3)	$-24\frac{\alpha_R}{\tilde\epsilon^2}\left[1-\frac{\tilde\epsilon}{2}-\tilde\epsilon P(p)\right]$	$-\frac{2}{\tilde\epsilon}[1-\tilde\epsilon P(p)]$
(4)	$4\alpha_R\frac{m+2}{3}\frac{1-\frac{\tilde\epsilon}{2}-\frac{\epsilon+\tilde\epsilon}{2}P(p)}{\epsilon(\epsilon+\tilde\epsilon)}$	$4\frac{m+2}{3}\frac{1-\frac{\epsilon+\tilde\epsilon}{2}P(p)}{2(\epsilon+\tilde\epsilon)}$
(5)	$-\frac{m+2}{18}\frac{\alpha_R[1-\epsilon P(p)]}{8\epsilon}$	$-\frac{m+2}{18}\frac{1-\epsilon P(p)}{8\epsilon}$

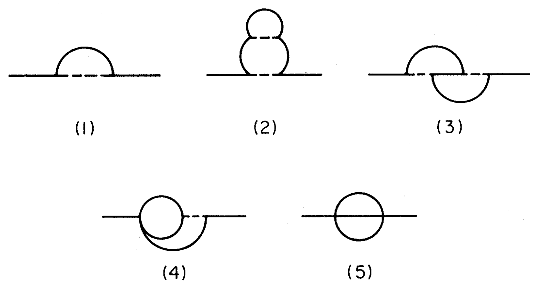

FIG. 3. Diagrams contributing to $\Gamma^{(2)}$. The expressions for the derivatives with respect to $p^{\|2}$ and $p^{\perp 2}$ are given in Table I.

III. RENORMALIZATION

In this section we develop the renormalization program in terms of the ($1PI$) two- and four-point vertices $\Gamma^{(2)}$ and $\Gamma_u^{(4)},\Gamma_\Delta^{(4)}$, respectively.[11] The subscripts u and Δ in the $\Gamma^{(4)}$ differentiate between the tensorial and momentum-conservation structure associated with each of these vertices. Dimensional analysis[9,11] shows that these are the only relevant (in the renormalization group sense) quantities near four dimensions. The integrals related to the Feynman diagrams have singularities as $\epsilon,\epsilon_d \to 0$ and we will compute them using the dimensional regularization scheme.[11,13,14] We have carried the renormalization program through two loops. A calculation to at least this level is necessary to reveal the full structure of the renormalization group. The loop integrals can be expanded in a Laurent series in terms of ϵ,ϵ_d . In general the regular (as $\epsilon,\epsilon_d \to 0$) part of the integrals depend upon the external momenta in a very complicated way. However, if we use the minimal subtraction[11,13] method only the poles in terms of ϵ,ϵ_d are needed. We do not use the subtraction point[11] scheme because the integrals are too difficult to evaluate. However, both methods differ only by a finite renormalization. The graphs contributing to $\Gamma^{(2)}$ $(p^{\|},p^{\perp})$ are shown in Fig. 3 (see also Table I). There are two types of divergences associated with these graphs, given by

$$\frac{\partial\Gamma^{(2)}(p^{\|},p^{\perp})}{\partial p^{\|2}} \quad \text{and} \quad \frac{\partial\Gamma^{(2)}(p^{\|},p^{\perp})}{\partial p^{\perp 2}} \; .$$

We recognize the first as the corrections to α_0, and the second as the wave-function renormalization.

We then define

$$\Gamma_R^{(2)}(p^{\|},p^{\perp},\alpha_R,g_R,\delta_R) = Z_\phi \Gamma^{(2)}(p^{\|},p^{\perp},\alpha_0,g,\delta) \; . \quad (3.1)$$

Now we write:

$$Z_\phi = (1 - z_1\delta_R^2 - z_2 g_R \delta_R - z_3 g_R^2 - \cdots) \; , \quad (3.2a)$$

$$\alpha_0 = \alpha_R(1 + \alpha_2\delta_R + \alpha_2\delta_R^2 + \alpha_3 g_R \delta_R + \alpha_4 g_R^2 + \cdots) \; . \quad (3.2b)$$

Following the minimal subtraction procedure, we will demand that the coefficients z_n cancel the poles in ϵ,ϵ_d of $\partial\Gamma_R^{(2)}/\partial p^{\perp 2}$, and that the coefficients α_n cancel the poles of $\partial\Gamma_R^{(2)}/\partial p^{\|2}$. This is equivalent to the conditions:

$$\frac{\partial\Gamma_R^{(2)}}{\partial p^{\perp 2}} = 1 + \cdots \quad \text{as} \quad \epsilon_d,\epsilon \to 0 \; , \quad (3.3a)$$

$$\frac{\partial\Gamma_R^{(2)}}{\partial p^{\|2}} = \alpha_R(1 + \cdots) \quad \text{as} \quad \epsilon,\epsilon_d \to 0) \; , \quad (3.3b)$$

where the ellipsis stand for regular terms.

Since we want to express the renormalization functions as a power series in the dimensionless couplings, we must extract a dimensionful factor from the Feynman integrals. This is achieved by rescaling all the momenta in the integral by a certain (arbitrary) scale factor κ. After this process, a typical integral, which depends upon a set of momenta, is written as

$$I(p_i^{\|},p_j^{\perp}) = \kappa^D F\left[\frac{p_i^{\|}}{\kappa},\frac{p_j^{\perp}}{\kappa}\right] , \quad (3.4)$$

where D is the canonical dimension of the integral (from power counting).

Another feature is the following: The loops associated with a vertex u have an integral over a parallel momentum, and this component is always multiplied by α. So if we define

$$\alpha^{1/2}q^{\|} = Q^{\|} \; , \quad d^{\epsilon_d}q^{\|} = \alpha^{-\epsilon_d/2} d^{\epsilon_d}Q^{\|} \; , \quad (3.5)$$

this picks out a term $\alpha^{-\epsilon d/2}$ which is absorbed in the definition of the dimensionless coupling [Eq. (2.12)]. As an example, use Eqs. (3.4) and (3.5) to compute the u^2 one-loop contribution to $\Gamma_u^{(4)}$. The Feynman integral is

$$J(\alpha k^{\|},k^{\perp}) = \int \frac{d^{\|}q\, d^{\perp}q}{(\alpha q^{\|2} + q^{\perp 2})[\alpha(q^{\|} - k^{\|})^2 + (q^{\perp} - k^{\perp})^2]} \; . \quad (3.6)$$

If we define

$$\alpha^{1/2}\frac{q^{\|}}{\kappa} = Q^{\|}, \quad \frac{q^{\perp}}{\kappa} = Q^{\perp}, \quad k = (\alpha^{1/2}k^{\|},k^{\perp}) \; ,$$

$$d^{\|}q\, d^{\perp}q = \kappa^{\epsilon_d}\alpha^{-\epsilon_d/2} d^{\|}Q \kappa^{d-\epsilon_d} d^{\perp}Q = \kappa^d \alpha^{-\epsilon_d/2} d^d Q \; ,$$

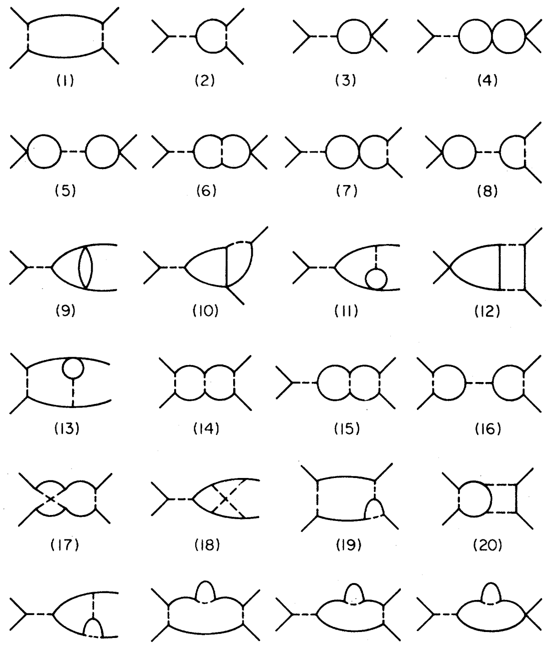

FIG. 4. Diagrams contributing to $\Gamma_\Delta^{(4)}$. The corresponding expressions are given in Table II.

then

$$J(\alpha k^{\|},k^{\perp})=\kappa^{-\epsilon}\alpha^{-\epsilon_d/2}\int\frac{d^dQ}{Q^2(Q-k/\kappa)^2}\,, \tag{3.7}$$

where the variable Q is dimensionless. The proof of this factorization to all orders can be given again in terms of a skeleton expansion. As a result, order by order in perturbation theory, the renormalization-group functions will depend on the renormalized dimensionless couplings but not on α.

The renormalizability[15] of the theory then implies

$$\Gamma_R^{(N)}(p_i^{\|},p_j^{\perp},\alpha_R,\delta_R,g_R,\kappa)=Z_\varphi^{N/2}\Gamma^{(N)}(p_i^{\|},p_j^{\perp},\alpha_0,\delta,g)\,, \tag{3.8}$$

where we relate the bare and renormalized couplings through a power series:

$$\begin{aligned}\delta&=\delta_R(1+D_1\delta_R+D_2g_R+D_3\delta_R^2\\&\quad+D_4g_R^2+D_5\delta_Rg_R+\cdots)\,,\\ g&=g_R(1+F_1g_R+F_2\delta_R+F_3g_R^2\\&\quad+F_4\delta_R^2+F_5\delta_Rg_R+\cdots)\,.\end{aligned} \tag{3.9}$$

Again, the coefficients D_i,F_i will be defined so as to cancel the divergences in the $\Gamma_\Delta^{(4)},\Gamma_u^{(4)}$. In Figs.

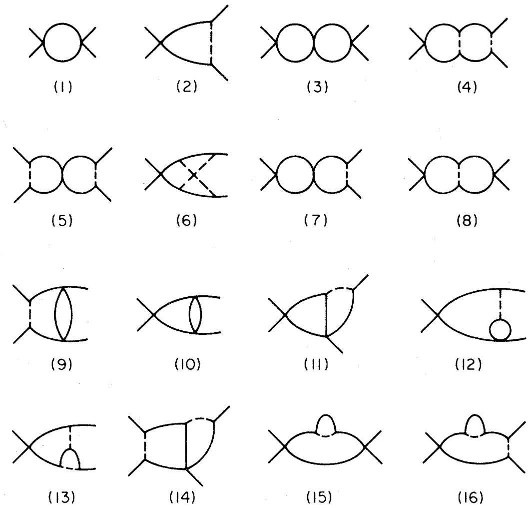

FIG. 5. Diagrams contributing to $\Gamma_u^{(4)}$. The result for the integrals is given in Table III.

4 and 5 the contributions to one and two loops for $\Gamma_\Delta^{(4)}$ and $\Gamma_u^{(4)}$, respectively, are given (see also Tables II and III). Define $\tilde{\epsilon}=\epsilon+\epsilon_d$; up to one loop we find

$$Z_\varphi=1,\quad \alpha_1=-\frac{4}{\tilde{\epsilon}},\quad \alpha_0=\alpha_R\left[1-\frac{4}{\tilde{\epsilon}}\delta_R\right], \tag{3.10}$$

where in the one-loop graphs for $\Gamma^{(2)}$ we set $\delta=\delta_R$, $g=g_R$. To this order,

$$\begin{aligned}\Gamma_u^{(4)}&=-g\alpha_0^{\epsilon_d/2}\kappa^{\epsilon}(1+b_1g+b_2\delta)\,,\\ \Gamma_\Delta^{(4)}&=12\delta\kappa^{\tilde{\epsilon}}(1+a_1\delta+a_2g)\,,\end{aligned} \tag{3.11}$$

where in the one-loop diagrams $\alpha_0\simeq\alpha_R$. The replacement of α_0 by α_R in (3.11) will not contribute to one-loop (order $\epsilon,\tilde{\epsilon}$), so we set $\alpha_0=\alpha_R$ and replace (3.9) in (3.11) to find:

$$\begin{aligned}D_1&=-\frac{16}{\tilde{\epsilon}},\quad D_2=\frac{m+2}{3}\frac{1}{\epsilon},\\ F_1&=\frac{m+8}{6}\frac{1}{\epsilon},\quad F_2=-\frac{24}{\tilde{\epsilon}}\,.\end{aligned} \tag{3.12}$$

In order to study deviations from the critical massless theory we need to study correlation functions $\Gamma^{(E,L)}$, with an arbitrary number of insertions of the operator $\frac{1}{2}\varphi^2(x)$.[11] This introduces a new renormalization constant Z_{φ^2} defined by

$$\bar{Z}_{\varphi^2}=Z_{\varphi^2}Z_\varphi,\quad \Gamma_R^{E,L}=Z_{\varphi^2}^LZ_\varphi^{E/2}\Gamma^{(E,L)}\,, \tag{3.13}$$

where

TABLE II. Contributions to $\Gamma_{\Delta}^{(4)}$ from graphs in Fig. 4. Geometrical factors have been absorbed in the couplings.

(1) $=96I_7$	(2) $=96I_3$	(3) $=-12\frac{m+2}{3}I_1$
(4) $=\frac{12}{3}\left[\frac{m+2}{3}\right]^2 I_1^2$	(5) $=\frac{12}{4}\left[\frac{m+2}{3}\right]^2 I_1^2$	(6) $=-16(m+2)I_2$
(7) $=-48\frac{m+2}{3}I_3 I_1$	(8) $=-48\frac{m+2}{3}I_3 I_1$	(9) $=12\frac{m+2}{3}I_4$
(10 $=-192\frac{m+2}{3}I_5$	(11) $=-96\frac{m+2}{3}I_6$	(12) $=-96\frac{m+2}{3}I_5$
(13) $=-192\frac{m+2}{3}I_{10}$	(14) $=384I_7^2$	(15) $=384I_3^2$
(16) $=192I_3^2$	(17) $=768I_8$	(18) $=384I_{11}$
(19) $=1536I_8$	(20) $=768I_{11}$	(21) $=768I_{11}$
(22) $=384I_{13}$	(23) $=384I_{13}$	(24) $=2\frac{m+2}{3}I_{12}$

where

$$I_1=\frac{1}{\epsilon}\left[1-\frac{\epsilon}{2}-\frac{\epsilon}{2}L\right],\quad I_2=\frac{2}{\tilde\epsilon(\epsilon+\tilde\epsilon)}\left[1-\frac{\epsilon+\tilde\epsilon}{2}-\frac{\epsilon+\tilde\epsilon}{2}L\right],$$

$$I_3=\frac{1}{\tilde\epsilon}\left[1-\frac{\tilde\epsilon}{2}-\frac{\tilde\epsilon}{2}\tilde L_2\right],\quad I_4=\frac{1}{2\epsilon^2}\left[1-\frac{\epsilon}{2}-\epsilon L\right],$$

$$I_5=\frac{1}{\tilde\epsilon(\epsilon+\tilde\epsilon)}\left[1-\frac{\epsilon}{2}-\frac{\epsilon+\tilde\epsilon}{2}L\right],\quad I_6=\frac{1}{\epsilon(\epsilon+\tilde\epsilon)}\left[1-\frac{\tilde\epsilon}{2}-\frac{\epsilon+\tilde\epsilon}{2}\tilde L_2\right],$$

$$I_7=\frac{1}{\tilde\epsilon}\left[1-\frac{\tilde\epsilon}{2}-\frac{\tilde\epsilon}{2}\tilde L_1\right],\quad I_8=\frac{1}{2\tilde\epsilon^2}\left[1-\frac{\tilde\epsilon}{2}-\tilde\epsilon\tilde L_1\right],$$

$$I_{10}=\frac{1}{\epsilon(\epsilon+\tilde\epsilon)}\left[1-\frac{\tilde\epsilon}{2}-\frac{\epsilon+\tilde\epsilon}{2}\tilde L_1\right],\quad I_{11}=\frac{1}{2\tilde\epsilon^2}\left[1-\frac{\tilde\epsilon}{2}-\tilde\epsilon L_2\right],$$

$$I_{12}=\frac{\epsilon_d}{\tilde\epsilon(\epsilon+\tilde\epsilon)},\quad I_{13}=\frac{1}{\tilde\epsilon}\left[\int\frac{d^{\perp}q_1(\alpha_R p_1^{\|2})^{1-\tilde\epsilon/2}+p_1^{\|}\to p_2^{\|}}{(\alpha_R p_1^{\|2}+q_1^{\perp 2})^2[\alpha_R p_2^{\|2}+(q_1^{\perp}-\kappa^{\perp})^2]}\right],$$

where

$$L=L(p_i^{\|},p_i^{\perp})=\int_0^1 dx\,\ln[(1-x)x(\alpha p^{\|2}+p^{\perp 2})],$$

$$\tilde L_1=\tilde L_1(p_i^{\|},p_i^{\perp})=\int_0^1 dx\,\ln[\alpha p_1^{\|2}x+\alpha p_2^{\|2}(1-x)+(p_1^{\perp}+p_2^{\perp})^2x(1-x)],$$

$$\tilde L_2=\tilde L_2(p_i^{\|},p_i^{\perp})=\int_0^1 dx\,\ln[\alpha p_1^{\|2}+(p_1^{\perp 2}+p_2^{\perp 2})x(1-x)].$$

$$\bar Z_{\varphi^2}=1+Z_1 g_R+Z_2\delta_R+Z_3 g_R^2+Z_4\delta_R^2+Z_5\delta_R g_R\ . \tag{3.14}$$

The coefficients Z_n will be defined so as to cancel the poles in $\Gamma_R^{(2,1)}$. This is equivalent to

$$\bar Z_{\varphi^2}\Gamma^{(2,1)}=1+\cdots\quad\text{as }\epsilon,\tilde\epsilon\to 0\ , \tag{3.15}$$

where the ellipsis again represents regular terms.

To first order there is no need for α or coupling constant renormalization, and we find

$$Z_1=\frac{1}{2}\frac{m+2}{3}\frac{1}{\epsilon},\quad Z_2=-\frac{4}{\tilde\epsilon}\ . \tag{3.16}$$

To carry the program up to two loops, the α_0 in the one-loop integrals must be expressed in terms of α_R. For the two-point function we replace (3.10) in the one-loop integral:

TABLE III. Contributions to $\Gamma_u^{(4)}$ from graphs in Fig. 5.

(1) $=\frac{m+8}{6}I_1$	(2) $=-24I_7$	(3) $=-\frac{6m+m^2+20}{36}I_1^2$
(4) $=-96I_7^2$	(5) $=-48I_7^2$	(6) $=-96I_8$
(7) $=4\frac{m+8}{3}I_1I_7$	(8) $=2\frac{m+8}{3}I_2$	(9) $=8\frac{m+8}{3}I_{10}$
(10 $=-\frac{5m+22}{9}I_4$	(11) $=16\frac{m+8}{3}I_5$	(12) $=24\frac{m+8}{3}I_{10}$
(13) $=-192I_8$	(14) $=-348I_8$	(15) $=-8\frac{m+8}{6}\frac{I(\nu=\tilde{\epsilon}/2)}{\tilde{\epsilon}}$
(16) $=96I_{13}$		

$$\int\frac{d^\perp q}{\alpha_0 p^{\parallel 2}+q^{\perp 2}}=\int\frac{d^\perp q}{\alpha_R p^{\parallel 2}+q^{\perp 2}}+4\frac{\delta_R}{\tilde{\epsilon}}\alpha_R p^{\parallel 2}\times\int\frac{d^\perp q}{(\alpha_R p^{\parallel 2}+q^{\perp 2})^2}\,. \tag{3.17}$$

In all two-loop graphs we can set $\alpha_0=\alpha_R$, $\delta=\delta_R$ and $g=g_R$, but in the first-order contribution we must use (3.9) and the correction given by (3.17) must be taken into account along with the second-order contributions.

In the $\Gamma^{(4)}$ we have to renormalize α_0 in the two typical integrals:

$$I_1(\alpha_0,p)=\int\frac{d^\parallel q_1 d^\perp q_1}{(\alpha_0 q_1^{\parallel 2}+q_1^{\perp 2})[\alpha_0(q_1^\parallel-p^\parallel)^2+(q_1^\perp-p^\perp)^2]}\,,$$
$$I_2(\alpha_0,p)=\int\frac{d^\perp q_1}{(\alpha_0 p_1^{\parallel 2}+q_1^{\perp 2})[\alpha_0 p_2^{\parallel 2}+(q_1-p^\perp)^2]}\,. \tag{3.18}$$

With the use of (24),

$$I_1(\alpha_0,p)\rightarrow I_1(\alpha_R,p)+8\frac{\delta_R}{\tilde{\epsilon}}I(\nu=0)\,, \tag{3.19}$$

where the function $I(\nu)$ is given in the Appendix. We have

$$I_2(\alpha_0,p)\rightarrow I_2(\alpha_R,p)+4\frac{\delta_R}{\tilde{\epsilon}}\left[\alpha_R p_1^{\parallel 2}\int\frac{d^\perp q_1}{(\alpha_R p_1^{\parallel 2}+q_1^{\perp 2})^2[\alpha_R p_2^{\parallel 2}+(q_1^\perp-p^\perp)^2]}+p_1^\parallel\rightarrow p_2^\parallel\right]\,. \tag{3.20}$$

Using (3.2a), (3.3a), (3.9), and (3.12) and the result for two-loop integrals (Table I), we find:

$$z_1=\frac{2}{\tilde{\epsilon}},\quad z_2=-\frac{2}{\epsilon+\tilde{\epsilon}}\frac{m+2}{3},\quad z_3=\frac{1}{\epsilon}\frac{m+2}{144}\,,$$
$$\alpha_2=\frac{40}{\tilde{\epsilon}^2}-\frac{10}{\tilde{\epsilon}},\quad \alpha_3=-\frac{4}{\tilde{\epsilon}(\epsilon+\tilde{\epsilon})}\frac{m+2}{3},\quad \alpha_4=0\,. \tag{3.21}$$

The contribution of order g_R^2 is canceled by wave-function renormalization because for this graph $\partial/(\partial p^{\parallel 2})=\partial/(\partial p^{\perp 2})$.

Two comments should be made at this point. (1) The Feynman integrals have poles with momentum-dependent residues (the higher-order pole is momentum independent).[13] For the renormalization program to make sense these terms must cancel among the various contributions for the coefficients in the renormalization-group functions to be momentum independent. (2) The renormalization of α_0 in one-loop graphs (3.19) and (3.20) leads to the dependence of these graphs upon α_R instead of α_0. The analysis of Eqs. (3.6) and (3.7) implies a factor $\alpha_R^{-\epsilon_d}$ in front of each coupling u, so that in terms of the dimensionless couplings,

$$\Gamma_u^{(4)}=-g\alpha_0^{\epsilon_d/2}\kappa^{\epsilon}\left[1+b_1\left[\frac{\alpha_0}{\alpha_R}\right]^{\epsilon_d/2}g+b_2\delta+b_3\left[\frac{\alpha_0}{\alpha_R}\right]^{\epsilon_d}g^2+\cdots\right],$$
$$\Gamma_\Delta^{(4)}=12\delta\kappa^{\tilde\epsilon}\left[1+a_1\delta+a_2\left[\frac{\alpha_0}{\alpha_R}\right]^{\epsilon_d/2}g+a_3\delta^2+\cdots\right]. \qquad (3.22)$$

However, the ratio α_0/α_R only depends on δ_R and g_R. Then using (3.8)–(3.10), (3.12), and (3.21),

$$D_3=\frac{256}{\tilde\epsilon^2}-\frac{84}{\tilde\epsilon},\quad D_4=\frac{m+2}{3}\left[\frac{m+4}{4\epsilon^2}\frac{-5}{24\epsilon}\right],$$
$$D_5=-\frac{m+2}{3}\left[\frac{64}{\epsilon\tilde\epsilon}-\frac{32}{\tilde\epsilon(\epsilon+\tilde\epsilon)}-\frac{24}{\epsilon(\epsilon+\tilde\epsilon)}\right]$$
$$+2\frac{m+2}{3}\left[\frac{5}{\epsilon}+\frac{7}{\tilde\epsilon}-\frac{6\tilde\epsilon}{\epsilon(\epsilon+\tilde\epsilon)}-\frac{6\epsilon}{\tilde\epsilon(\epsilon+\tilde\epsilon)}-\frac{2}{(\epsilon+\tilde\epsilon)}\right]+2\frac{m+2}{3}\frac{\epsilon_d}{\tilde\epsilon\epsilon}-2\frac{m+2}{3}\frac{\epsilon_d}{\tilde\epsilon(\epsilon+\tilde\epsilon)},$$
$$F_3=\frac{1}{\epsilon^2}\left[\frac{m+8}{6}\right]^2-\frac{1}{\epsilon}\frac{3m+14}{24}, \qquad (3.23)$$
$$F_4=\frac{480}{\tilde\epsilon^2}-\frac{1}{\tilde\epsilon}\left[64\frac{\epsilon_d}{\tilde\epsilon}+164\right],$$
$$F_5=-\left[\frac{1}{\epsilon\tilde\epsilon}\frac{56m+304}{3}-\frac{20}{\tilde\epsilon(\epsilon+\tilde\epsilon)}\frac{m+8}{3}-\frac{1}{\epsilon(\epsilon+\tilde\epsilon)}\frac{32m+112}{3}\right]$$
$$-\left[-\frac{1}{\epsilon}(5m+16)-\frac{3}{\tilde\epsilon}(m+8)+\frac{\epsilon}{\tilde\epsilon(\epsilon+\tilde\epsilon)}\frac{68+10m}{3}\right.$$
$$\left.+\frac{\tilde\epsilon}{\epsilon(\epsilon+\tilde\epsilon)}\frac{16m+56}{3}+\frac{2}{\epsilon+\tilde\epsilon}\frac{m+2}{3}\right]+\frac{2}{\tilde\epsilon}\frac{m+8}{6}\left[\frac{\epsilon_d}{\epsilon}-\frac{\epsilon_d}{\epsilon+\tilde\epsilon}\right].$$

The procedure followed for $\Gamma^{(4)}$ and $\Gamma^{(2)}$ is applied to $\Gamma^{(2,1)}$ and we find

$$Z_3=\frac{m+2}{6}\left[\frac{m+5}{6\epsilon^2}-\frac{1}{4\epsilon}\right],\quad Z_4=\frac{40}{\tilde\epsilon^2}-\frac{12}{\tilde\epsilon},$$
$$Z_5=-\frac{m+2}{3}\left[\frac{18}{\epsilon\tilde\epsilon}-\frac{12}{\tilde\epsilon(\epsilon+\tilde\epsilon)}-\frac{4}{\epsilon(\epsilon+\tilde\epsilon)}\right] \qquad (3.24)$$
$$+\frac{m+2}{3}\left[\frac{5}{\tilde\epsilon}+\frac{1}{\epsilon}-\frac{4\epsilon}{\tilde\epsilon(\epsilon+\tilde\epsilon)}-\frac{2\tilde\epsilon}{\epsilon(\epsilon+\tilde\epsilon)}+\frac{\epsilon_d}{\epsilon\tilde\epsilon}-\frac{\epsilon_d}{\tilde\epsilon(\epsilon+\tilde\epsilon)}\right].$$

The fact that the factors conspire so as to cancel the momentum-dependent singularities is a check of the renormalizability of the theory and the calculation itself.

Anticipating the next section, we will calculate the various renormalization-group functions

$$\beta_g=\kappa\frac{\partial g_R}{\partial\kappa}\bigg|_B,\quad \beta_\delta=\kappa\frac{\partial\delta_R}{\partial\kappa}\bigg|_B,\quad \gamma_\varphi=\kappa\frac{\partial\ln Z_\varphi}{\partial\kappa}\bigg|_B,\quad \xi_{\alpha_R}=\kappa\frac{\partial\alpha_R}{\partial\kappa}\bigg|_B,\quad \bar\gamma_{\varphi^2}=-\kappa\frac{\partial\ln\bar Z_{\varphi^2}}{\partial\kappa}\bigg|_B,\quad \gamma_{\varphi^2}=\bar\gamma_{\varphi^2}+\gamma_\varphi, \qquad (3.25)$$

where the subscript B means that the bare quantities are held fixed. In order to calculate these derivatives, we have to invert the relations (3.2a), (3.2b), and (3.9) and write all renormalized quantities in terms of the bare ones (u,Δ,α_0) using (2.12), take derivatives, and write the result back in terms of renormalized (dimensionless) constants. We find

$$\beta_\delta=-\tilde{\epsilon}\delta_R-16\delta_R^2+\frac{m+2}{3}\delta_R g_R-168\delta_R^3+2\delta_R^2 g_R\frac{m+2}{3}\left[10-\frac{\epsilon_d}{\tilde{\epsilon}}\right]-\frac{5}{12}\frac{m+2}{3}\delta_R g_R^2\;, \tag{3.26a}$$

$$\beta_g=-\epsilon g_R+\frac{m+8}{6}g_R^2-24g_R\delta_R-g_R^3\frac{3m+14}{12}+g_R^2\delta_R\left[\frac{2}{3}(10m+56)+\frac{m-4}{3}\frac{\epsilon_d}{\tilde{\epsilon}}\right]-\delta_R^2 g_R\left[328+128\frac{\epsilon_d}{\tilde{\epsilon}}\right].$$

$$\gamma_\varphi=4\delta_R^2-2\frac{m+2}{3}\delta_R g_R+\frac{m+2}{72}g_R^2\;,$$

$$\xi_\alpha=\alpha_R(-4\delta_R-20\delta_R^2)\;, \tag{3.26b}$$

$$\bar{\gamma}_{\varphi^2}=g_R\frac{m+2}{6}-4\delta_R-\frac{1}{2}\frac{m+2}{6}g_R^2-24\delta_R^2+\frac{m+2}{3}\delta_R g_R\left[6-\frac{\epsilon_d}{\tilde{\epsilon}}\right].$$

Throughout the analysis we have absorbed a factor $2\pi^{d/2}/(2\pi)^d\Gamma(d/2)$ into the definition of u and $2\pi^{\tilde{d}}\Gamma(\tilde{d}/2)$ for Δ. The novel feature in these expressions is the appearance of the ratio $\epsilon_d/\tilde{\epsilon}$ as a consequence of the double $\epsilon,\tilde{\epsilon}$ expansion.[16] [For systems with one typical dimensionality ($\epsilon_d=0$), it can be shown that in the minimal subtraction scheme the renormalization-group functions do not depend on ϵ.]

Scale invariance of the correlation functions will be obtained at the fixed points of the functions β_g and β_δ, so we must look for

$$\beta_g(\delta_R^*,g_R^*)=0,\quad \beta_\delta(\delta_R^*,g_R^*)=0\;. \tag{3.27}$$

Since we want to search for fixed pints in an ϵ,ϵ_d expansion we define

$$\delta_R=\delta_1+\delta_2+\cdots,\quad g_R=g_1+g_2\;\cdots\;, \tag{3.28}$$

where δ_1 and g_1 are of order ϵ,ϵ_d, whereas δ_2 and g_2 are of order $\epsilon^2,\epsilon_d^2,\epsilon\epsilon_d$ and we require Eq. (3.27) to hold for each power of ϵ,ϵ_d independently (ϵ and ϵ_d are regarded to be of the same order). To order ϵ,ϵ_d we find four fixed points.

(1) Gaussian:

$$\delta_R^*=0,\quad g_R^*=0\;.$$

(2) Pure:

$$\delta_R^*=0,\quad g_R^*=\frac{6\epsilon}{m+8}\;.$$

(3) Unphysical: (3.29)

$$\delta_R^*=-\frac{\tilde{\epsilon}}{16},\quad g_R^*=0\;.$$

(4) Random:

$$\delta_R^*=\frac{1}{32}\frac{(4-m)\epsilon+(m+8)\epsilon_d}{(m-1)}\;,$$

$$g_R^*=\frac{3}{2}\frac{\epsilon+3\epsilon_d}{(m-1)}\;.$$

To calculate the eigenvalues, we diagonalize the matrix of the derivatives of β_g,β_δ with respect to δ_R and g_R and evaluate at the fixed points.

(1) Gaussian:

$$-\epsilon \text{ for } g_R,\quad -\tilde{\epsilon} \text{ for } \delta_R\;.$$

(2) Pure:

$$\epsilon \text{ for } g_R\;,$$

$$-\frac{(4-m)\epsilon}{m+8}-\epsilon_d \text{ for } \delta_R\;.$$

(3) Unphysical: (3.30)

$$-\epsilon \text{ for } g_R,\quad \tilde{\epsilon} \text{ for } \delta_R\;.$$

(4) Random: two complex eigenvalues,

$$\frac{1}{8(m-1)}[A\pm(A^2-B)^{1/2}]\;,$$

where[17]

$$A=[(3\epsilon+\epsilon_d)m+8\epsilon_d]\;,$$

$$B=16(m-1)(\epsilon+3\epsilon_d)[(m+8)\epsilon_d+(4-m)\epsilon]\;.$$

For $m<m_c=4(\epsilon+2\epsilon_d)/(\epsilon-\epsilon_d)$ where $\epsilon>\epsilon_d$, the random fixed point (FP) is in the physical region, and randomness is relevant at the pure fixed point, for $m>m_c$ $(\epsilon>\epsilon_d)$ the random FP goes into the unphysical region and the pure FP is stable, so for $\epsilon > \epsilon_d$ there is a change in stability at m_c. For $\epsilon_d > \epsilon$ the only stable fixed point is the random and is always physical. The calculation of the fixed point to order $\epsilon^2, \epsilon\epsilon_d, \epsilon_d^2$ is very tedious and probably uninteresting. To order ϵ, ϵ_d we find at the random fixed point:

$$\hat{\xi}_\alpha^* = \kappa \frac{\partial \ln \alpha_R}{\partial \kappa}\Big|_{\mathrm{FP}} = -4\delta_1^*$$

$$= -\frac{1}{8} \frac{(4-m)\epsilon + (m+8)\epsilon_d}{m-1} ,$$

$$\bar{\gamma}_{\varphi^2}^* = \bar{\gamma}_{\varphi^2}\Big|_{\mathrm{FP}} = \frac{1}{m-1}[\epsilon(\tfrac{13}{8}m + \tfrac{5}{2}) + \epsilon_d(\tfrac{37}{8}m + 8)] .$$

The system of equations (3.27) with (3.26a) is degenerate to order δ_R, g_R for $m=1$ but the degeneracy can be removed if the next order is taken into account. For $m=1$ we define $\delta_R \simeq \delta_1 + \delta_2 + \cdots$ and $g_R \simeq g_1 + g_2 + \cdots$ where δ_1, g_1 are of order $\epsilon^{1/2}$ and δ_2, g_2 are of order ϵ, ϵ_d. We find[18]

$$g_1^* = 16\delta_1^*, \quad \delta_1^* = \left[\frac{3(\epsilon + 3\epsilon_d)}{328 + 576\epsilon_d/\tilde{\epsilon}}\right]^{1/2} , \tag{3.31}$$

to this order:

$$\hat{\xi}^* = -4\delta_1^* = -\left[\frac{12(\epsilon + 3\epsilon_d)}{82 + 144\epsilon_d/\tilde{\epsilon}}\right]^{1/2} ,$$

$$\eta = \gamma_\varphi^* = -\frac{52(\epsilon + 3\epsilon_d)}{328 + 576\epsilon_d/\tilde{\epsilon}} ,$$

$$\bar{\gamma}_{\varphi^2}^* = \left[\frac{12(\epsilon + 3\epsilon_d)}{82 + 144\epsilon_d/\tilde{\epsilon}}\right]^{1/2} .$$

IV. RENORMALIZATION-GROUP EQUATIONS

The invariance of the theory under the RG means that any change in the (arbitrary) momentum scale should be compensated by a change in the parameters of the theory, so as to keep the bare quantities fixed. The statement that the bare correlation functions are independent of this scale is written as

$$\kappa \frac{\partial \Gamma_B^{(N)}}{\partial \kappa}(\{p_i\}, \alpha_0, g_0, \delta_0,) = 0 . \tag{4.1}$$

Then by Eq. (4.8)

$$\left[\kappa \frac{\partial}{\partial \kappa} + \beta_g \frac{\partial}{\partial g_R} + \beta_\delta \frac{\partial}{\partial \delta_R} + \xi_\alpha \frac{\partial}{\partial \alpha_R} - \frac{N}{2}\gamma_\varphi\right] \times \Gamma_R^{(N)}(\{p_i\}, g_R, \delta_R, \alpha_R, \kappa) = 0 . \tag{4.2}$$

The solution of this equation tells us the behavior of the correlation functions under a change of scale. We will solve (4.2) for $N=2$. Dimensional analysis implies[9,11]

$$\Gamma_R^{(2)}(\{p^{\parallel}, p^{\perp}\}, g_R, \delta_R, \alpha_R, \kappa) = \kappa^2 \psi^{(2)}\left[\left\{\frac{\alpha_R^{1/2} p^{\parallel}}{\kappa}, \frac{p^{\perp}}{\kappa}\right\}, g_R, \delta_R\right] , \tag{4.3}$$

where $\psi^{(2)}$ is a dimensionless function, and we used the fact that α_R always appears multiplying $p^{\parallel 2}$, and this is the only way $\psi^{(2)}$ depends on α_R. If we rescale the momenta, Eq. (4.3) allows us to write

$$\Gamma_R^{(2)}(\{\lambda p^{\parallel}, \lambda p^{\perp}\}, g_R, \delta_R, \alpha_R, \kappa) = \lambda^2 \Gamma_R^{(2)}\left[\{p^{\parallel}, p^{\perp}\}, g_R, \delta_R, \alpha_R, \frac{\kappa}{\lambda}\right] . \tag{4.4}$$

Then we can use (4.2) to write

$$\left[\lambda \frac{\partial}{\partial \lambda} - \beta_g \frac{\partial}{\partial g_R} - \beta_\delta \frac{\partial}{\partial \delta_R} - \xi_\alpha \frac{\partial}{\partial \alpha_R} + (\gamma_\varphi - 2)\right] \times \Gamma_R^{(2)}(\{\lambda p^{\parallel}, \lambda p^{\perp}\} \cdots) = 0 . \tag{4.5}$$

The solution to this equation is as follows[19]: Define the "running" couplings by ($\Lambda = \ln\lambda$)

$$\frac{d\bar{g}(\Lambda)}{d\Lambda} = -\beta_g(\bar{g}(\Lambda), \bar{\delta}(\Lambda)) ,$$
$$\frac{d\bar{\delta}(\Lambda)}{d\Lambda} = -\beta_\delta(\bar{g}(\Lambda), \bar{\delta}(\Lambda)) , \tag{4.6}$$
$$\frac{d\bar{\alpha}(\Lambda)}{d\Lambda} = -\xi_\alpha(\bar{\alpha}(\Lambda), \bar{g}(\Lambda), \bar{\delta}(\Lambda))$$
$$\bar{\delta}(0) = \delta_R, \quad \bar{g}(0) = g_R, \quad \bar{\alpha}(0) = \alpha_R .$$

The solution to (4.5) is

$$\Gamma_R^{(2)}(\{\lambda p^{\parallel}, \lambda p^{\perp}\}, g, \delta, \alpha, \kappa) = \exp\left[\int_{-t}^{0} (2 - \gamma_\varphi) dt'\right] \Gamma_R^{(2)}(\{p^{\parallel}, p^{\perp}\}, \bar{g}(-t), \bar{\delta}(-t), \bar{\alpha}(-t), \kappa) . \tag{4.7}$$

In the infrared limit $\lambda\to 0$ ($\Lambda\to-\infty$) the running constants $\bar{g}(-\Lambda)$, and $\bar{\delta}(-\Lambda)$ "flow" towards the fixed point g_R^*,δ_R^*. Near to this fixed point we write

$$\bar{g}(-\Lambda)\simeq g_R^*,\ \ \bar{\delta}(-\Lambda)\simeq\delta_R^*,\ \ \gamma_\varphi\simeq\gamma_\varphi^*=\gamma_\varphi(g^*,\delta^*)\ ,$$

and (4.8)

$$\xi_\alpha\simeq\bar{\alpha}(-\Lambda)\hat{\xi}_\alpha^*,\ \ \hat{\xi}^*=\kappa\frac{\partial\ln\alpha_R}{\partial\kappa}\Bigg|_*\ ,$$

where we used

$$\kappa\frac{\partial\alpha_R}{\partial\kappa}=\alpha_R F(g_R,\delta_R)\ . \tag{4.9}$$

Then in the infrared limit,

$$\Gamma_R^{(2)}(\{p^{\|},p^{\perp}\},g,\delta,\alpha,\kappa)\underset{\lambda\to 0}{\sim}C\lambda^{2-\gamma_\varphi^*}\Gamma_R^{(2)}\left[\left\{\frac{p^{\|}}{\lambda},\frac{p^{\perp}}{\lambda}\right\},g^*,\delta^*,\alpha_R\lambda^{\hat{\xi}^*},\kappa\right]\ . \tag{4.10}$$

Setting $p^{\perp}=\lambda\kappa$ and by (4.3)

$$\Gamma_R^{(2)}(p^{\|},p^{\perp})=\left[\frac{p^{\perp}}{\kappa}\right]^{2-\gamma_\varphi^*}\kappa^2\psi^{(2)}\left[\left\{\frac{\alpha_R^{1/2}}{\kappa^{\hat{\xi}^*/2}}\right\}\left[\frac{p_{\|}}{p_{\perp}^{1-\hat{\xi}^*/2}}\right]\right], \tag{4.11}$$

we recognize $\eta=2-\gamma_\varphi^*$. Away from the critical temperature (massless theory) the RG equation obeyed by $\Gamma_R^{(2)}$ is[11]:

$$\left[\kappa\frac{\partial}{\partial\kappa}+\beta_g\frac{\partial}{\partial g_R}+\beta_\delta\frac{\partial}{\partial\delta_R}+\xi_\alpha\frac{\partial}{\partial\alpha_R}+\gamma_{\varphi^2}t\frac{\partial}{\partial t}-\gamma_\varphi\right]\Gamma_R^{(2)}(\{p^{\|},p^{\perp}\}g,\alpha,\delta,\kappa,t)=0\ , \tag{4.12}$$

where $t=(T-T_c)/T_c$. Recalling that t has dimensions of (mass)^2, namely $[\kappa]^2$, we follow the same steps as above with the only addition of a "running" temperature:

$$\frac{d\bar{t}(\Lambda)}{d\Lambda}=(2-\gamma_{\varphi^2})\bar{t}(\Lambda),\ \ \bar{t}(0)=t_R\ . \tag{4.13}$$

The solution of this equation in the asymptotic region $\lambda\to 0$ ($\Lambda\to-\infty$) is

$$\Gamma_R^{(2)}(\{p^{\|},p^{\perp}\},g,\delta,\alpha,t,\kappa)\underset{\lambda\to 0}{\simeq}C\lambda^{2-\gamma_\varphi^*}\Gamma_R^{(2)}\left[\frac{p^{\|}}{\lambda},\frac{p^{\perp}}{\lambda},g^*,\delta^*,\alpha_R\lambda^{\hat{\xi}^*},\frac{t}{\lambda^{2-\gamma_{\varphi^2}^*}},\kappa\right]\ . \tag{4.14}$$

Now since λ is arbitrary, we define: $\lambda=(\kappa^2/t)^{-1/2-\gamma_{\varphi^2}^*}$. Then

$$\Gamma^{(2)}(p^{\|},p^{\perp},g,\delta,\alpha,t,\kappa)\cong C't^{(2-\gamma_\varphi^*)/(2-\gamma_{\varphi^2}^*)}\Phi(p^{\|}\xi^{\|},p^{\perp},\xi^{\perp})\ , \tag{4.15}$$

where

$$\xi^{\|}\propto t^{-(1-\hat{\xi}^*/2)/(2-\gamma_{\varphi^2}^*)}\ , \tag{4.16}$$

$$\xi^{\perp}\propto t^{-1/(2-\gamma_{\varphi^2}^*)}\ ,$$

are the parallel and perpendicular correlation lengths. The susceptibility is given by

$$\chi^{-1}=\Gamma_R^{(2)}(p^{\|}=0,p^{\perp}=0)\propto t^{\gamma}\ . \tag{4.17}$$

Then we recognize

$$\nu_{\|}=\frac{1-\hat{\xi}^*/2}{2-\gamma_{\varphi^2}^*},\ \ \nu_{\perp}=\frac{1}{2-\gamma_{\varphi^2}^*},\ \ \gamma=\nu_{\perp}(2-\eta)\ ,$$

$$\nu_{\|}=z\nu_{\perp}\ \text{ with }\ z=1-\hat{\xi}^*/2\ . \tag{4.18}$$

The specific-heat vertex $\Gamma^{(0,2)}$ is divergent at the tree level and must be additively and then multiplicatively renormalized.[9,11] Following the steps leading to Eq. (4.2) it is easy to verify that $\Gamma_R^{(0,2)}$ obeys an inhomogeneous RG equation.[11]

In the limit of number of replicas $n\to 0$, the only diagrams contributing to $\Gamma^{(0,2)}$ are those of order n; these diagrams carry only one internal free-replica index and are associated with loops where a parallel component of momentum is integrated over (Fig. 6). This allows us to write

$$\Gamma_R^{(0,2)}(p,t,g_R,\alpha_R,\kappa,\delta_R) = \kappa^{-\epsilon}\alpha_R^{-\epsilon_d/2}\psi\left[\frac{p}{\kappa},\frac{t}{\kappa^2},g_R,\delta_R\right] . \qquad (4.19)$$

The solution of the RG for this vertex is given by

$$\Gamma_R^{(0,2)}(p=0,t) \underset{t\to 0}{\approx} t^{-\alpha} , \qquad (4.20)$$

where α is given by

$$\alpha = 2-(d-\epsilon_d)\nu_\perp - \epsilon_d \nu_\| . \qquad (4.21)$$

V. RELATION TO QUANTUM SYSTEMS

It is known that the critical properties of d-dimensional quantum spin systems are equivalent to those of a $(d+1)$-dimensional classical system.[20] For simplicity let us consider the Ising model in a transverse field.

The Hamiltonian of the quantum (spin-$\frac{1}{2}$) Ising model in a transverse field is

$$H = -\Gamma\sum_i \sigma_i^x - \sum_{i;j} J_{ij}\sigma_i^z\sigma_j^z , \qquad (5.1)$$

where the σ's are Pauli spin matrices, Γ is the transverse field, and J_{ij} is the interaction between adjacent spins. This system in d dimensions is equivalent to the $(d+1)$-dimensional classical Ising model whose Hamiltonian is

$$\mathscr{H} = \sum_{\langle ij\rangle} g_{ij}S_iS_j , \qquad (5.2)$$

where g_{ij} is a function of Γ and J_{ij} of (5.1), and S's are classical variables ($S=\pm 1$). The "extra" dimension is infinite in extent if the original quantum system is considered at zero temperature. However, if the initial system is considered at finite temperature, the extra dimension for the classical system has width $\beta = 1/kT$.[21] Two different situations arise in this case. When the correlation length ξ is much smaller than β ($\xi \ll T^{-1}$) the system behaves as if the extra dimension were actually infinite in extent. The effects of finite temperature are negligible; the quantum system behaves as if it were at $T=0$, namely, purely quantum behavior.

On the other hand, when $\xi \gg T^{-1}$ in the $d+1$ classical system there is a crossover to d-dimensional behavior; that is, the quantum system behaves classically. The crossover region is characterized by the condition

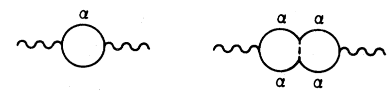

FIG. 6. Contributions for $\Gamma^{(0,2)}$ in the limit $n \to 0$. The wavy lines stand for a $\frac{1}{2}\varphi^2(x)$ insertion.

$$|U-U_c|^{-\nu(d+1)} \sim T^{-1} , \qquad (5.3)$$

where U is the coupling [$U=J/\Gamma$ in the Ising case (5.1)] of the initial quantum system and $\nu(d+1)$ is the correlation-length critical exponent of the $d+1$ classical system.

If we introduce quenched random-point impurities in our quantum system, for instance, defining in (5.1),

$$J(r) = \widetilde{J}(r) + \delta J(r) , \qquad (5.4a)$$

$$\langle\langle \delta J(r)\delta J(r')\rangle\rangle \propto \Delta\delta^d(r-r') , \qquad (5.4b)$$

this model then is equivalent to the $(d+1)$-dimensional classical Ising model with impurities totally correlated along the extra dimension forming thus an impurity "line."[22] This situation reduces to the problem we studied throughout the previous sections with $\epsilon_d = 1$, if we identify then the extra dimension of the classical system with the parallel direction of the problem we studied. Various situations can arise depending on whether the finite-size (finite-temperature) effects are more important than the randomness or vice versa.

Hereafter we shall concentrate on the case of (relatively) strong randomness. By (2.15), the crossover from pure, quantum behavior to random quantum behavior will take place when

$$\left|\frac{U-U_c}{U_c}\right| \sim \left[\frac{\Delta}{\widetilde{J}^{1/2}}\right]^{1/[\alpha(d+1,P)+\nu(d+1,P)]} , \qquad (5.5)$$

where the α and ν are those of the pure $(d+1)$-dimensional classical system. On the other hand, the crossover from pure quantum behavior to pure classical behavior would take place when

$$\left|\frac{U-U_c}{U_c}\right| \sim \left[\frac{kT}{\widetilde{J}}\right]^{1/\nu(d+1,P)} . \qquad (5.6)$$

The case of strong randomness then corresponds to the right-hand side of (5.5) being much larger than that of (5.6). In this situation, the crossover to classical behavior will not be described by (5.6), but by

$$\left|\frac{U-U_c}{U_c}\right| \sim \left[\frac{kT}{\tilde{J}}\right]^{1/\nu_{\parallel}(d+1,R)}, \tag{5.7}$$

where $\nu_{\parallel}(d+1,R)$ is just the exponent defined in the previous sections. Note that $\nu_{\parallel}$ enters, not $\nu_{\perp}$. We thus identify $\nu_{\parallel}^{-1}$ as the quantum classical crossover exponent for the random system. We can also write this as $\nu_{\parallel}^{-1}=1/z\nu_{\perp}$. This agrees with the result $2/z$ obtained by Hertz,[21] since he was studying a system for which $\nu_{\perp}=\frac{1}{2}$.

Solving (3.27) for $\epsilon=0$ we predict for the three-dimensional quantum system (for $m\neq 1$):

$$\begin{aligned} \eta &= \frac{\epsilon_d^2}{(m-1)^2}\frac{m+2}{32}\left[9-23\frac{m+2}{8}\right], \\ z &= 1-\frac{1}{2}\left[-\frac{1}{8}\frac{m+2}{m-1}\epsilon_d+\frac{\epsilon_d^2}{4096(m-1)^3}(-284m^3+2208m^2+21312m+13376)\right], \\ \nu_{\perp}^{-1} &= 2-\left[\frac{5}{8}\epsilon_d\frac{m+2}{m-1}-\frac{\epsilon_d^2}{4096(m-1)^3}(692m^3+11472m^2+15552m+8896)\right], \end{aligned} \tag{5.8}$$

and for $m=1$:

$$\begin{aligned} z &= 1+0.2\epsilon_d^{1/2}, \\ \eta &= -0.173\epsilon_d, \\ \nu_{\perp}^{-1} &= 2-0.4\epsilon_d^{1/2}. \end{aligned} \tag{5.9}$$

The expansion in ϵ_d does not seem to be very reliable and requires some resummation method.

VI. CONCLUSIONS

We have studied the effect of impurities correlated along ϵ_d directions and at random in $d-\epsilon_d$ dimensions ($d=4-\epsilon$) on the critical behavior of magnets with $O(m)$ symmetry. The presence of this impurity line introduces anisotropy and nonlocality in the theory, which implies a richer divergence structure.

The quantities of interest were computed in a double expansion in terms of ϵ and ϵ_d, and in order to expose the full structure of the theory the calculation has been carried through two loops. The renormalizability of the theory has been proven to this order.

The renormalization group has been used to approach the critical region and scaling of the correlation function emerges at the fixed points. For $m\neq 1$ nontrivial pure and random fixed points of order $a\epsilon+b\epsilon_d$ were found, while for $m=1$ the nontrivial random FP is of order $(A\epsilon+B\epsilon_d)^{1/2}$. The eigenvalues (for $m\neq 1$) at the random fixed point turn out to be complex, bringing oscillatory corrections to scaling.

A main feature of the double expansion in terms of ϵ and ϵ_d is the fact that all the RG functions depend on the regular ratio $\epsilon_d/(\epsilon+\epsilon_d)$, and as a result the critical exponents depend on this ratio.

The RG equations were solved for the spin-spin correlation function $\Gamma^{(2)}$ and its scaling form is given at T_c by

$$\Gamma^{(2)} \sim \left[\frac{p_{\perp}}{\kappa}\right]^{2-\eta}\kappa^2\psi^{(2)}\left[\frac{p_{\parallel}}{p_{\perp}^z}\right], \tag{6.1}$$

while away from T_c,

$$\Gamma^{(2)} \sim t^{\gamma}\Phi(p^{\parallel}\xi^{\parallel},p_{\perp}\xi^{\perp}), \quad t=\frac{T-T_c}{T_c} \tag{6.2}$$

where $p_{\perp}$ and $p_{\parallel}$ are the components of momenta perpendicular and parallel to the impurity line, respectively. $\xi^{\parallel}\sim t^{-\nu_{\parallel}}$ and $\xi^{\perp}\sim t^{-\nu_{\perp}}$ are the parallel and perpendicular correlation lengths, $\nu_{\parallel}=z\nu_{\perp}$ and $\gamma=\nu_{\perp}(2-\eta)$. The exponent z is a result of the anisotropy of the system. The solution for the specific heat yields

$$\alpha=2-(d-\epsilon_d)\nu_{\perp}-\epsilon_d\nu_{\parallel}. \tag{6.3}$$

To one-loop order we find for $m\neq 1$

$$\begin{aligned} z &= 1+\frac{1}{16(m-1)} \\ &\quad \times[(4-m)\epsilon+(m+8)\epsilon_d], \\ \nu_{\perp}^{-1} &= 2-\frac{1}{m-1}[\epsilon(\tfrac{13}{8}m+\tfrac{5}{2})+\epsilon_d(\tfrac{37}{8}m+8)], \\ \eta &\sim O(\epsilon^2), \end{aligned} \tag{6.4}$$

and for $m=1$

$$z=1+\frac{1}{2}\left[\frac{12(\epsilon+3\epsilon_d)}{82+144\epsilon_d/(\epsilon+\epsilon_d)}\right]^{1/2},$$
$$\nu_\perp^{-1}=2-\left[\frac{12(\epsilon+3\epsilon_d)}{82+144\epsilon_d/(\epsilon+\epsilon_d)}\right]^{1/2}, \tag{6.5}$$
$$\eta=\frac{-52(\epsilon+3\epsilon_d)}{328+576\epsilon_d/(\epsilon+\epsilon_d)}.$$

We pointed out the relation between a d-dimensional quantum system with impurities distributed at random and the $(d+1)$-dimensional classical system with impurities totally correlated in the "extra" dimension and at random in the remaining d dimensions i.e., an impurity "line." When the quantum system is at finite temperature we expect a crossover from quantum to classical behavior. For the random system we predict the quantum-classical crossover exponent to be $\nu_{\|}^{-1}=1/z\nu_\perp$.

ACKNOWLEDGMENTS

We thank R. L. Sugar for useful discussions and encouraging comments. This work was supported by the National Science Foundation under Grant No. PHY 80-18938.

APPENDIX: CALCULATION OF A TYPICAL TWO-LOOP INTEGRAL

Some useful formulas for the computation of integrals are the following:

$$\int dq\,F(q+k)=\int dq\,F(q),\quad \int d^{\|}q\,d^{\perp}q\,F(\lambda^{\|}q^{\|},\lambda^{\perp}q^{\perp})=|\lambda^{\|}|^{-\epsilon_d}|\lambda^{\perp}|^{-(d-\epsilon_d)}\int d^{\|}q\,d^{\perp}q\,F(q^{\|},q^{\perp}), \tag{A1}$$

$$\int\frac{d^dq}{(q^2+2kq+m^2)^\alpha}=\frac{\Gamma(d/2)\Gamma(\alpha-d/2)}{2\Gamma(\alpha)}(m^2-k^2)^{d/2-\alpha}, \tag{A2}$$

where a factor $2\pi^{d/2}/(2\pi)^d\Gamma(d/2)$ is absorbed in the definition of the coupling constant

$$\frac{1}{A^\alpha B^\beta}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dx\frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax+B(1-x)]^{\alpha+\beta}}. \tag{A3}$$

We evaluate the typical integral:

$$I(p_i)=\int\frac{d^{\|}q_1d^{\perp}q_1d^{\perp}q_2}{(\alpha q_1^{\|2}+q_1^{\perp 2})[\alpha q_1^{\|2}+(q_1-\kappa^{\perp})^2][\alpha q_1^{\|2}+(q_1+q_2-p^{\perp})^2](\alpha p^{\|2}+q_2^{\perp 2})}. \tag{A4}$$

We define $Q=q_i-p^{\perp}$ and use (A1):

$$I(p_i)=\int\frac{d^dq_1d^{\perp}q_2\alpha^{-\epsilon_d/2}}{q_1^2(q_1-\kappa^{\perp})^2(Q+q_2^{\perp})^2(\alpha p^{\|2}+q_2^{\perp 2})}. \tag{A5}$$

The integral over $d^{\perp}q_2$ yields, using (A3) and (A2),

$$\int d^{\perp}q_2\cdots=\Gamma(\tilde d/2)\frac{\Gamma(\tilde\epsilon/2)}{2}\int_0^1 dx[\alpha p^{\|2}x+Q^2x(1-x)]^{-\tilde\epsilon/2}. \tag{A6}$$

Introducing two more parameters, using (A1), and following the steps as above we arrive at

$$I(p_i)=\frac{\Gamma(\tilde d/2)}{2}\Gamma\left[2+\frac{\tilde\epsilon}{2}\right]\frac{\Gamma(\tilde d/2)}{2}\frac{\Gamma((\epsilon+\tilde\epsilon)/2)}{\Gamma(2+\tilde\epsilon/2)}$$
$$\times\int_0^1dx\int_0^1dy\int_0^1dz\,z^{\tilde\epsilon/2-1}(1-z)[1-z+zx(1-x)]^{-\tilde d/2}$$
$$\times\left\{\kappa^2y(1-z)+\alpha p^{\|2}xz+p^2x(1-x)z-\frac{[\kappa y(1-z)+px(1-x)z]^2}{1-z+zx(1-x)}\right\}^{-(\epsilon+\tilde\epsilon)/2}. \tag{A7}$$

The integral is divergent for $z=0$, so that we write

$$\{z\}^{-(\epsilon+\tilde{\epsilon})/2}=\{z=0\}^{-(\epsilon+\tilde{\epsilon})/2}+\{z\}^{-(\epsilon+\tilde{\epsilon})/2}-\{z=0\}^{-(\epsilon+\tilde{\epsilon})/2}=\{z=0\}^{-(\epsilon+\tilde{\epsilon})/2}-\left[\frac{\epsilon+\tilde{\epsilon}}{2}\right]\ln\left[\frac{\{z\}}{\{z=0\}}\right], \tag{A8}$$

where

$$\{z=0\}^{-(\epsilon+\tilde{\epsilon})/2}=[\kappa^{\perp 2}y(1-y)]^{-(\epsilon+\tilde{\epsilon})/2}. \tag{A9}$$

The term proportional to the ln does not contribute to the singular part. The integral over z is of the form:

$$\int_0^1 dz\, z^{d-1}(1-z)^{\mu-1}(1-\beta z)^{-\nu} = B(\lambda,\mu)\,{}_2F_1(\nu,\lambda;\lambda+\mu;\beta)\,, \tag{A10}$$

where $\lambda=\tilde{\epsilon}/2$, $\mu=2$, $\nu=\tilde{d}/2=2-\tilde{\epsilon}/2$, $\beta=1-x(1-x)$, $B(\lambda,\mu)$ is the beta function, and ${}_2F_1$ is the hypergeometric function

$${}_2F_1(2-\tilde{\epsilon}/2,\tilde{\epsilon}/2;2+\tilde{\epsilon}/2;\beta) \simeq 1+\frac{\tilde{\epsilon}}{2}\left[\beta+\frac{\beta^2}{2}+\frac{\beta^3}{3}+\cdots+\frac{\beta^k}{k}\right] \tag{A11}$$

to leading order in $\tilde{\epsilon}$.

We recognize that Eq. (A11) equals

$$1-\frac{\tilde{\epsilon}}{2}\ln(1-\beta)\approx[x(1-x)]^{-\tilde{\epsilon}/2}.$$

Using

$$B(r,s)=\int_0^1 x^{r-1}(1-x)^{s-1}dx\,,$$
$$B(r,s)=\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}\,, \tag{A12}$$
$$\Gamma(1+x)=x\Gamma(x)\,.$$

The result then is

$$I(p_i)=\frac{1}{\tilde{\epsilon}(\epsilon+\tilde{\epsilon})}\left[1-\frac{\epsilon}{2}-\left[\frac{\epsilon+\tilde{\epsilon}}{2}\right]L(0,\kappa^{\perp})\right], \tag{A13}$$

where

$$L(\kappa^{\parallel},\kappa^{\perp})=\int_0^1 dy\,\ln[y(1-y)(\alpha\kappa^{\parallel 2}+\kappa^{\perp 2})]\,. \tag{A14}$$

The function $I(\nu)$ is given by

$$I(\nu)=\int\frac{d^{\parallel}q_1 d^{\perp}q_1}{(\alpha_R q_1^{\parallel 2}+q_1^{\perp 2})^2[\alpha_R(q_1^{\parallel}-\kappa^{\parallel})^2+(q_1^{\perp}-\kappa^{\perp})^2](\alpha_R q_1^{\parallel 2})^{\nu-1}}\,. \tag{A15}$$

Following the steps as above we find

$$I(\nu)\simeq\frac{\alpha_R^{-\epsilon_d}}{2}\left\{\frac{\epsilon_d}{4(\nu+\epsilon/2)}+B(\kappa^{\parallel},\kappa^{\perp})\right\}, \tag{A16}$$

where

$$B(\kappa^{\parallel},\kappa^{\perp})=\int_0^1 dx(1-x)(-\alpha_R\kappa^{\parallel 2}x^2)[(\alpha_R\kappa^{\parallel 2}+\kappa^{\perp 2})x(1-x)]^{-(\nu+\epsilon/2+1)}\,.$$

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[17]These results coincide up to this order with those of Ref. 5.

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