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Percolation threshold of a two-dimensional continuum system

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We discuss the percolation threshold of a two-dimensional continuum system which has conductor only in those regions where a function $I(\vec{x})$ is less than a chosen cutoff intensity. An experimental realization of such a system, with $I(\vec{x})$ the electric field intensity of a laser speckle pattern, has recently been reported. We carry out a computer study for this case, the results of which are in excellent agreement with the experimental results. Expanding on earlier ideas that it is the saddle points of $I(\vec{x})$ which determine the percolation threshold, we introduce an "equivalent network" which has the same threshold as the continuum system. With the use of this result, the computer study constructs the network and then easily finds its threshold. The computer study also finds the densities of maxima, minima, and saddle points of $I(\vec{x})$ which are in close agreement with the analytic results of a companion paper. Finally, we use the equivalent network in developing an "effective-lattice" estimate for the percolation threshold.

I. INTRODUCTION

Recently, Smith and Lobb[1] developed an experimental technique to study continuum percolation[2] in a system with well-characterized disorder. They expose high-contrast film using a laser speckle pattern whose electric field amplitude is, to a good approximation, a realization of a complex Gaussian random variable. The film is then used as a mask for photolithography, which yields a planar conductor-insulator composite with conductor only in those regions where the speckle pattern electric field intensity $I(\vec{x})$ is less than a given cutoff intensity I^*. This paper is concerned with the determination of the percolation threshold of the composite I_c, i.e., of the smallest value of I^* for which the system conducts to infinity.

The problem addressed in this paper is equivalent to the problem[3,4] of the localization of classical particles in a random potential in two dimensions. If $I(\vec{x})$ describes the potential, and I^* a particle's energy, then the particle is restricted to move only in those regions of the plane where $I(\vec{x}) < I^*$. If $I^* < I_c$, so that all of the allowed regions are finite in extent, then the particle must be localized; while if $I^* > I_c$ the particle is not localized if it is located in the allowed region of infinite extent which appears when the regions percolate.

A better intuitive understanding of the problem can be achieved by considering a more easily visualizable topographic analogy, namely, the determination of the floodplanes of lakes in a mountain range. Viewing $I(\vec{x})$ as describing a mountainous surface over the plane with valleys filled with water to a given height I^*, the water-covered regions correspond to the conducting regions of the continuum system. There exists an interconnected lake of infinite extent if I^* is greater than the percolation threshold, I_c.

For many two-dimensional continuum systems, the exact percolation threshold is determined by a straightforward argument.[3] In these systems, the "conjugate" system, obtained by interchanging the roles of the conducting and insulating regions, is statistically equivalent to the original system with the area fractions of conductor and insulator interchanged. So, if the conductor percolates at an area fraction of conductor f, then the insulator will percolate when *its* area fraction is f, i.e., at an area fraction of conductor of $1-f$. But, in two dimensions, the point at which one component first percolates and that at which the other component last percolates must be the same, because the percolation path of one component blocks the percolation of the other component. Thus, these systems first percolate at a critical area fraction f_c of 50%. However, the system we are considering does not have this simplifying property, so the determination of the percolation threshold is a more difficult problem.

We might mention that, in addition to the system's experimental relevance, it is the simplest such system based on Gaussian random variables (GRV) defined on the plane which has a nontrivial percolation threshold. If $I(\vec{x})$ were a real GRV then its distribution function would be symmetric

between I and $-I$. The system would then have the symmetry between conductor and insulator discussed above, leading to $f_c = 50\%$ and hence $I_c = 0$. Also, if I were the square, or absolute value, of a real GRV then again $I_c = 0$ but with $f_c = 0\%$. To see this, consider the real GRV case with $I^* = 0^-$. In this case, the system is below its percolation threshold $I_c = 0$, and the interconnected lakes are all of finite extent. The area covered as I^* increases is the shoreline of each of these lakes, and when $I^* = 0^+$, exceeding the percolation threshold, some of these areas will connect to each other, joining the lakes they surround into an infinite lake. But notice that the shoreline areas alone will percolate, even without the areas of "deep water," since each lake is completely surrounded by its shoreline, and the interconnections between lakes which lead to percolation are also included in the shoreline area. Thus, $I_c = 0$ in a system where $I(\vec{x})$ is the square of absolute value of a GRV, since the region conducting when $I^* = 0^+$ is just this shoreline area.

In thinking about the percolation threshold of the system, it becomes apparent that it is the saddle points of $I(\vec{x})$ which determine the percolation threshold[3–5] because two conducting regions coalesce into one when I^* becomes greater than the intensity of the saddle point between them. To see this, consider increasing the level of two adjoining lakes at the same rate. They join as the water level reaches the point of lowest height separating the two, which is the saddle point between them. In this paper we show that in fact there is an exact equivalence between the percolation threshold for a continuum system and that for bond percolation on a specific network. The network is completely determined by the surface $I(\vec{x})$: The vertices correspond to the minima (or maxima) of $I(\vec{x})$, and the bonds to paths from minima to minima (or maxima to maxima) through the saddle points. The states (conducting or insulating) of the bonds are set by comparing the heights of the corresponding saddle points to I^*.

We report on a computer study of the percolation threshold for the system in which we create a realization of $I(\vec{x})$ on a large lattice of points, and look for the onset of percolation. One method of finding the threshold would be to construct contours of constant intensity; I_c would then be the height of the first contour to span the system. However, rather than constructing contours, we create the equivalent network. This approach has the advantage that once we have the network, finding its percolation threshold is a straightforward task. A possible disadvantage is that in constructing the network it is necessary to find all of the critical points (i.e., the maxima, minima, and saddle points) of $I(\vec{x})$. In our case, this is not a concern since we want to find the critical points for comparison with the analytic results of the companion paper[6] for the densities of these points.

The rest of the paper is organized as follows: Section II reviews the connection between laser speckle patterns and GRV's showing that the correlation function of the GRV is simply related to the intensity distribution of the laser spot. In Sec. III we construct the network mentioned above, and argue that it has the same percolation threshold as the original system. Section IV describes the computer study of the threshold of the system; the results are found to be in excellent agreement with the experimental results of Smith and Lobb. Finally, an effective-medium-type of argument, using the equivalence between the continuum system and the network, is presented in Sec. V.

II. LASER SPECKLE PATTERNS

A laser speckle pattern[7] occurs when a laser beam impinges upon a rough scattering surface. If the scattered light is linearly polarized[8] then $E(\vec{x})$, the electric field amplitude at a point $\vec{x}$ on a given "image plane," is the sum of the amplitudes of a large number of essentially independent electric fields due to the scattering of the laser beam by the many microscopic elements of the rough surface. Thus, calling upon the central limit theorem, $E(\vec{x})$ is, to a very good approximation, a complex Gaussian random variable (CGRV) defined on the plane. The correlation function is

$$G_E(\vec{x},\vec{y}) = \langle E(\vec{x})E^*(\vec{y})\rangle \propto e^{i\kappa(x^2-y^2)} \int d^2x_0 P(\vec{x}_0) e^{-2i\kappa\vec{x}_0\cdot(\vec{x}-\vec{y})}, \tag{1}$$

where $\langle\ \rangle$ denotes an average over the random variable, κ is a geometric factor, and $P(\vec{x}_0)$ is the intensity distribution of the laser spot on the scattering surface. Introducing $A(\vec{x}) = \exp(-i\kappa x^2)E(\vec{x})$, and assuming that the spot is rotationally symmetric, i.e., $P(\vec{x}_0) = P(|\vec{x}_0|)$, we have

$$G(|\vec{x}-\vec{y}|) = \langle A(\vec{x})A^*(\vec{y})\rangle \propto \int d^2x_0 P(|\vec{x}_0|) e^{-2i\kappa\vec{x}_0\cdot(\vec{x}-\vec{y})}. \tag{2}$$

Notice that the Fourier transform of G is trivially related to $P(x_0)$: $\bar{G}(k) \propto P(k/2\kappa)$.

Thus, in the experiment of Smith and Lobb, the electric field intensity to which the film responds is $I(\vec{x}) \equiv |E(\vec{x})|^2 = |A(\vec{x})|^2$, where $A(\vec{x})$ is a homogeneous, isotropic CGRV defined on the plane. Since the film is of very high contrast, there is a sharp cutoff I^* in electric field intensity between the transparent and the opaque regions of the developed film; I^* is controllable by varying the exposure time and laser power. Using the film as a mask for photolithography results in conductor being removed only in those regions where $I(\vec{x}) > I^*$. The critical intensity I_c is then the minimum value of I^* for which the conducting regions span the system, giving rise to a macroscopic conductivity. I_c is simply related to the quantity actually measured in the experiment, the area fraction at the percolation threshold, $f_c = 1 - \exp(-I_c/I_0)$, where $I_0 \equiv \langle I(\vec{x}) \rangle = G(0)$.

In the experiment, the intensity of the laser spot on the rough surface is essentially Gaussian in form, leading to a Gaussian correlation function $G(r)$, which is the form used in the computer study described below. In addition to studying the isotropic case, Smith and Lobb found it easy to introduce anisotropy into the system by tilting the scattering surface, thereby making the laser spot noncircular. They investigated the effect of anisotropy on the conductivity of the system above the percolation threshold. In this paper only the isotropic case shall be considered; simply stretching the sytem in one direction will leave its topology unchanged and thus will not change its percolation threshold.

III. EQUIVALENT NETWORK

In this section we show that the percolation threshold for a two-dimensional continuum system with conductor in those regions where a function $I(\vec{x})$ is less than I^*, and with insulator where $I(\vec{x})$ is greater than I^*, is related to the percolation threshold of a network which we shall construct in the argument. $I(\vec{x})$ is an arbitrary function, except that we shall require that its gradients be continuous, and that it be suitably random in order to allow us to argue that certain special configurations (which are unstable to infinitesimal deformations of the surface) are of measure zero, and thus can be ignored.

Following Longuet-Higgins,[9] who studied the scattering of light from random surfaces, we partition the plane into a polyhedron with vertices at the minima of $I(\vec{x})$, edges along paths from minimum to minimum through the saddle points, and with each face containing a single maximum (see Fig. 1). Consider a point on the surface which is not a maximum, minimum, or saddle point of $I(\vec{x})$ and which does not lie on any of the (two per saddle) paths of steepest descent from a saddle point, and imagine following the path of steepest ascent which passes through the point. This path must eventually lead to a maximum; the point belongs to the face containing the maximum which has been reached. In general, a point in the neighborhood of the original point will also belong to the same face, i.e., the path of steepest ascent passing through the new point will lead to the same maximum. In this way, the set of all points which make up a face of the polyhedron is determined.

Now, at the edge of the face there will be neighboring points which belong to two different maxima. Lying between these points is a separatrix which divides flows to different maxima. The separatrix is well defined since $\vec{\nabla} I$ is continuous, and is a path of steepest ascent itself, since it is bounded on either side by other paths of steepest ascent. By its very nature this bounding curve cannot end on a maximum; rather, it ends at the saddle point between the two maxima. It might seem that the point at which the separatrix ends could be a higher-order critical point (such as a "monkey saddle") rather than a simple saddle point, but for our suitably random function defined

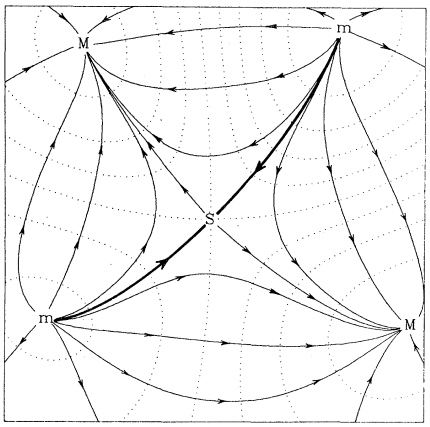

FIG. 1. Construction of an edge of the polyhedron. The points marked S, M, and m are saddle points, maxima, and minima of $I(\vec{x})$, respectively. The dotted lines are contours of constant intensity, and the solid lines are paths of steepest ascent from minima to maxima. The heavy line is an edge of the polyhedron, separating paths of steepest ascent which end on different maxima.

on the plane such points can be ignored.[10]

Thus, the faces are bounded by the separatrices for the flows to different maxima. The separatrices are the paths of steepest ascent ending on the saddle points, each saddle point having two such paths. These paths must start on a minimum. The only other possibility, a separatrix starting at another saddle point, will not occur because it would require that the path of steepest ascent ending at a saddle point also be one of the (only two per saddle) paths of steepest ascent beginning on another saddle, an occurrence of measure zero.

Thus, we have constructed a polyhedron, the equivalent network, with vertices at the minima of $I(\vec{x})$ and edges following the paths of steepest ascent from minimum to saddle and then back to minimum, and a single maximum contained in each face. We can also immediately prove a sum rule[11] connecting the numbers of maxima, minima, and saddle points of a random function with continuous gradients in two dimensions. Imposing periodic boundary conditions on the function, so that it is defined on the surface of a torus, the polyhedron we have constructed will also lie on the surface of a torus. Then, we can apply Euler's sum rule,[12] $N_{\text{edge}}=N_{\text{vertex}}+N_{\text{face}}$ on a torus, to find:

$$N_{\text{sad}}=N_{\text{max}}+N_{\text{min}} , \tag{3}$$

where we have used the correspondence between the vertices, edges, and faces of the polyhedron, and the minima, saddles, and maxima of $I(\vec{x})$, respectively.

For the continuum system with $I(\vec{x})$ viewed as a surface over the plane, it is sufficient that the lake be at least as high as the intervening saddle point in order that one minimum be covered by the same lake as a neighboring minimum. The two vertices of the network corresponding to the two minima are connected by the edge through the intervening saddle point. In order that the two vertices be connected, it is sufficient that this edge be conducting. Thus, if we consider bond percolation on the network, with the bonds conducting only if I^* is greater than the associated saddle height, then the network will first percolate when $I^*=I_c$, the same value as that at which our original continuum system described by $I(\vec{x})$ first percolates. In the next section we use this result to determine the percolation threshold of a number of realizations of the continuum system. The network is constructed, and then its bond percolation threshold is easily determined.

The same arguments will clearly apply if we consider paths of steepest descent instead of ascent (imagine turning the system over). In this case, the vertices are at the maxima of the random function, and each face contains one minimum. This new network is the dual of the original network, and will have the same I_c as the original network if the bonds are now taken to be conducting only when I^* is less than the associated saddle heights. The difference is that, for the original network, as I^* increases through I_c the system becomes conducting, while for the dual network the system stops conducting as I^* increases through I_c.

In Fig. 2 we illustrate two special cases, not explicitly considered in the above argument, which merit further discussion, although the conclusions reached above remain unchanged. It is possible for a minimum to have only a single path of steepest descent from a saddle reaching it, and also for a saddle to have its two paths of steepest descent both end on the same minimum. In the first case, the path of steepest descent from the saddle acts as a separatrix for flows to the same maximum which go to either side of the minimum, while in the second the separatrix completely surrounds a maximum. The sum rule still applies; the more general polyhedra generated when these configurations are allowed have vertices with coordination number of one, and have bonds with both ends attached to the same vertex, but still satisfy Euler's sum rule. (A

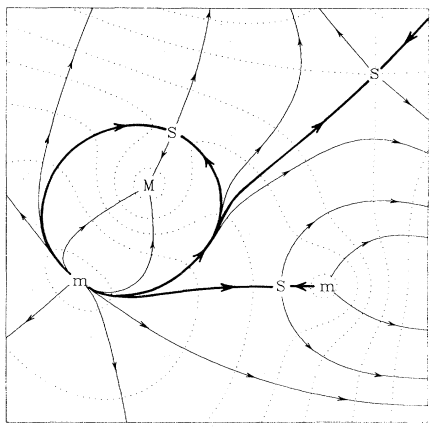

FIG. 2. Two special configurations of the surface; one leading to a vertex of coordination number of unity, the other to an "edge" which begins and ends on the same vertex. The points marked *S*, *M*, and *m* are saddle points, maxima, and minima of $I(\vec{x})$, respectively. The dotted lines are contours of constant intensity and the solid lines are paths of steepest ascent from minima to maxima. The heavy lines are the edges of the more general polyhedron which is generated.

more general polyhedron is reduced to a polyhedron without the special cases by removing edges and vertices in pairs, and edges and faces in pairs.) The special edges and vertices of our equivalent network are irrelevant to the determination of its percolation threshold, since they do not affect the long-range connectivity of the clusters. The corresponding special configurations of the continuum system can also be seen to be irrelevant to the determination of its threshold, they result only in isolated lakes or mountains as the water level is raised. Thus, the above argument relating the percolation threshold of the continuum system to that of the network remains unaffected.

IV. COMPUTER STUDY

In this section a computer study of the percolation threshold is discussed. A realization of a complex Gaussian random variable (CGRV) defined on a square lattice of points is created and its squared modulus is found. A continuous function with everywhere continuous gradients is then fit to the points and the function's maxima, minima, and saddle points are found. Since the gradient is continuous, the sum rule, Eq. (3), applies, and serves as a useful check on the algorithm used to find the critical points. Following the paths of steepest ascent from each of the saddle points until they end on maxima, the equivalent (dual) network is created. It is then an easy matter to find the percolation threshold of the network, and thus, as argued above, to identify the percolation threshold of the continuum system.

A realization of a homogeneous CGRV with periodic boundary conditions defined on a lattice of points can be created by taking the inverse Fourier transform of independent CGRV's defined on the reciprocal lattice. If $A(\vec{n})$ is a CGRV defined on a square lattice of points, $\vec{n}=(i,j)$ $(i,j=1,\ldots,N)$, with correlation function

$$G(\vec{n}-\vec{n}')=\langle A(\vec{n})A^*(\vec{n}')\rangle \tag{4}$$

then the Fourier transform of $A(\vec{n})$,

$$\bar{A}(\vec{k})=\sum_{\vec{n}} e^{i\vec{k}\cdot\vec{n}}A(\vec{n}) ,$$

is a CGRV with correlation function

$$\langle \bar{A}(\vec{k})\bar{A}^*(\vec{k}')\rangle = N^2\bar{G}(\vec{k})\delta_{\vec{k},\vec{k}'} , \tag{5}$$

where $\bar{G}(\vec{k})$ is the Fourier transform of $G(\vec{n})$. Thus, the $\bar{A}(\vec{k})$ are independent CGRV's, and there are well-known techniques to create a realization of them,[13] and to take the inverse Fourier transform to find $A(\vec{n})$. The intensity is then just $I(\vec{n})=|A(\vec{n})|^2$.

To allow us to compare the results of this study with the experimental ones obtained by Smith and Lobb, we take the correlation function to be Gaussian (see Sec. II above),

$$G(\vec{n})=I_0 e^{-n^2/2\sigma^2} . \tag{6}$$

It is convenient to set the intensity scale to unity, $I_0=1$. Also, a choice of $\sigma=8$ was found to be a good tradeoff between two competing requirements: (1) Since σ determines the typical length scale for variations of the intensity, σ should be large compared to the grid spacing (of 1) so that we can adequately approximate the true intensity by interpolating between the values of the intensity on the grid, $I(\vec{n})$. (2) In order to accurately determine the percolation threshold, the system should contain many uncorrelated regions. Correlations in the intensity extend over distances of order σ, so σ should be small compared to the system dimensions. The system size of N^2 points is limited by computer time and memory constraints to a maximum of $N=512$, which is the value used here.

Once $I(\vec{n})$ has been created, we wish to determine a continuous function $I(x,y)$, defined on the torus $\{x,y$ in $[0,N]$ with $I(x,y)$ continuous at the edges$\}$ such that $I((x,y)=\vec{n})=I(\vec{n})$ and $\vec{\nabla}I(x,y)$ is everywhere continuous. These conditions can be satisfied by using a polynomial interpolation of the form

$$I(x,y)=\sum_{n=0}^{3}\sum_{m=0}^{3} a_{nm}(\bar{x},\bar{y})(x')^n(y')^m , \tag{7}$$

where $\bar{x}$ is the integer part of x, x' is the fractional part of x, and similarly for y. For each square [labeled by $(\bar{x},\bar{y})$], the 16 coefficients $a_{nm}(\bar{x},\bar{y})$ are determined by 16 values of $I(\vec{n})$, the values at the four corners of the square and at the 12 next-nearest points of the grid. These values are substituted into 16 equations arising from the constraints on the function $I(x,y)$. Four of the constraints are simply that $I(x,y)$ must attain the correct value at the four corners of the square; the rest arise from fixing the first and second derivatives of $I(x,y)$ at the four corners by the values of $I(\vec{n})$ at the 16 points in such a way that $I(x,y)$ and $\vec{\nabla}I(x,y)$ are continuous from square to square.

The algorithm used to find the zeros of $\vec{\nabla}I(x,y)$ in each square involves finding the zeros of I_x along the curves defined by $I_y=0$ (where

$I_a = \partial I/\partial a$). $I_y(x,y)=0$ is a quadratic equation for $y(x)$, and is thus easily solved. The zeros of $I_x(x,y(x))$, as a function of x, are determined by finding the value of the function on a discrete set of trial values of x and noting where it changes sign. Newton's method is then applied to accurately determine the zero. By only considering those trial values of x for which $[x,y(x)]$ is in the square, we avoid having to find the potentially large number of points at which $\vec{\nabla} I = 0$ but which are not in the square, and thus are not of interest. In addition, computation time is saved by searching a square for the zeros of $\vec{\nabla} I$ only if curves along which $I_x = 0$ and curves along which $I_y = 0$ both enter the square. It is extremely unlikely to have a point in the square at which $\vec{\nabla} I = 0$ (where the two types of curves intersect) without having both types of curve go through an edge of the square, since σ, which sets the characteristic length of variation of the function, is large compared to the size of a square. The number of trial values of x required was reduced to 20 along each square by developing algorithms to explicitly deal with certain special cases. An example of such a special case is illustrated in Fig. 3.

Once a point at which $\vec{\nabla} I = 0$ is found, it is classified as a maximum, minimum, or saddle point by considering the matrix of second derivatives, I_{ij}. The eigenvalues of I_{ij} describe the principal curvatures at the point, so we need only examine the determinant and the trace of I_{ij} to determine the signs of the curvatures, and, hence, to classify the points.

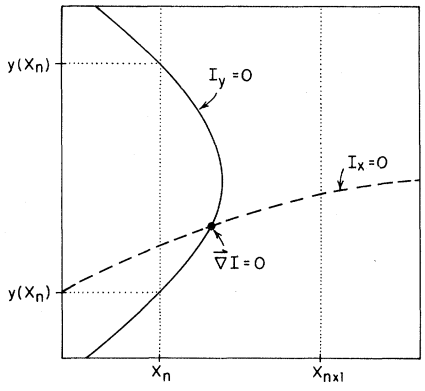

FIG. 3. A special case which must be dealt with by a special algorithm. The curves along which $I_x = 0$ and along which $I_y = 0$ are shown, as are two consecutive trial values of x: x_n and x_{n+1}. The algorithm must recognize that there is a zero of $\vec{\nabla} I$ between the two trial values of x even though $y(x_{n+1})$, the solution of $I_y(x_{n+1},y)=0$, does not exist.

The sum rule, Eq. (3), applies to this system, since $I(x,y)$ is defined on a torus and I and $\vec{\nabla} I$ are continuous, and provides a useful consistency test of our procedure. The sum rule is satisfied very well, to within less than six out of a total number of saddle points on the order of 2000, for each of the realizations created. Comparing the results of the exact calculation of the companion paper and those of the computer simulation of this paper, we find that for intensities >0.2 the densities of saddle points and of maxima plus minima are in good agreement. However, our computer simulation has many more saddle points and minima at small intensity than the exact result predicts for the true continuum system. Due to these excess points the total density of saddle points (and maxima plus minima) is on the order of 10% too high, and varies from one realization to another. In Fig. 4 we show a comparison between the exact and the computer results (averaged over four realizations) for $R_{\rm sad}(I)$ and $R_{\rm max}(I)+R_{\rm min}(I)$, the densities of saddle points and of maxima plus minima, respectively, which are at intensity greater than I. As in the companion paper, we plot, as a function of the scaled intensity I/I_0, the number of points per unit characteristic area $l^2 = -2G(0)/\nabla^2 G(0)$ which is equal to σ^2 in our case.

We believe that the excess minima and saddle points at low intensity are due to the polynomial interpolation between the points at which the in-

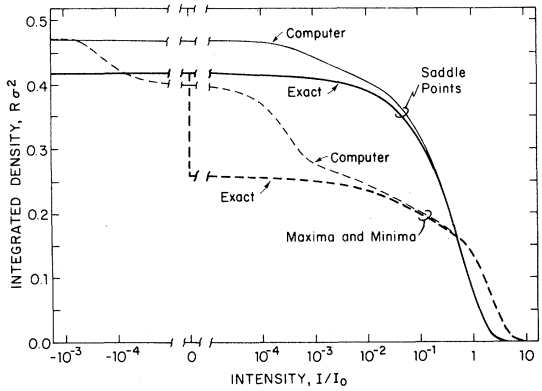

FIG. 4. Scaled density of saddle points $\sigma^2 R_{\rm sad}(I)$ (solid curves) and of maxima plus minima $\sigma^2[R_{\rm max}(I) + R_{\rm min}(I)]$ (dashed curves) that are at intensity greater than I. The heavy lines are the exact results of the companion paper, the light lines are the computer results of this paper averaged over four realizations. Notice that the exact and computer results differ only for small values of the intensity. The exact $R_{\rm max}(I) + R_{\rm min}(I)$ has a step at $I=0$, while in the computer result the step has been rounded for $|I/I_0| < 5\times 10^{-4}$.

tensity is known. At low intensity, the system exhibits long chains of minima and saddle points which are very close together (often closer than the grid spacing). Thus, we should not expect the polynomial fit to the grid of points to be a good approximation to the true intensity of a continuum CGRV in these regions. We carried out two tests to verify that the interpolation scheme in fact introduces excess minimum—saddle-point pairs. Realizations were made with a number of different values of σ. As σ is increased, so that the grid is effectively made finer, the number of excess points at low intensity decreases. This is to be expected if the excess critical points are due to our interpolation scheme, since the interpolation is bound to be more accurate as the grid is made finer. In addition, we took a realization (with $\sigma=8$) for which we had determined the critical points using the standard interpolation scheme, and found the intensity on a finer mesh in the vicinity of some of the critical points. The intensity on the finer mesh was found by taking the inverse Fourier transform evaluated at each point; fast-Fourier-transform techniques could not be used to do this, so the number of points studied was limited by computer time to about fifty of each kind. Interpolating from the finer mesh, we found that indeed a number of minimum—saddle-point pairs along low intensity chains disappeared using this more accurate determination of the intensity. The number of pairs which disappeared is about the number expected if the excess densities of minima and saddle points is due to these pairs.

In the companion paper only $R_{\max}+R_{\min}$ was calculated, but we can combine the exact results of the companion paper and the computer results found here to accurately determine the separate densities. Except at low intensity, we find excellent agreement with the exact results for $R_{\rm sad}(I)$ and $R_{\max}(I)+R_{\min}(I)$. This close agreement gives us confidence that the computer results for $R_{\max}(I)$ and $R_{\min}(I)$ separately are accurate except at low intensity; but there are very few maxima at low intensity, so we expect that the computer result for $R_{\max}(I)$ is correct. Figure 5 displays the separate densities of critical points: the exact result for $R_{\rm sad}(I)$, the computer result for $R_{\max}(I)$, and $R_{\min}(I)$ obtained by subtracting $R_{\max}$ from the exact $R_{\max}+R_{\min}$. As in Fig. 4, in Fig. 5 we plot the scaled density $\sigma^2R(I)$ versus the scaled intensity I/I_0.

We now construct the equivalent network for the system as discussed in Sec. III. Because there are fewer maxima than minima, and the maxima are

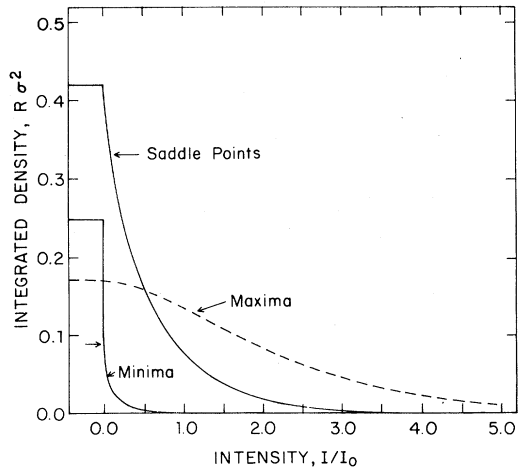

FIG. 5. Separate scaled densities of saddle points $\sigma^2R_{\rm sad}(I)$, maxima $\sigma^2R_{\max}(I)$, and minima $\sigma^2R_{\min}(I)$ that are at intensity greater than I. The arrow indicates the bottom of the step in $R_{\min}$ at $I=0$.

more widely separated than the minima, it is convenient to construct the dual network of paths of steepest ascent from saddle points to maxima, rather than the network of paths of steepest descent to minima. In any case, both networks have the same percolation threshold I_c. At each saddle point, we step off in the direction of the eigenvector of I_{ij} corresponding to the positive eigenvalue, and then follow the direction of $\vec{\nabla}I(x,y)$ until a maximum is reached. Then, starting at the same saddle point, we step off in the direction opposite to the previous one and again follow $\vec{\nabla}I$ to reach a (generally different) maximum. We have thus constructed a bond connecting the two maxima. Following this procedure for all of the saddle points of the system, we construct the dual network.

It is now an easy matter to find the intensity at which the dual network first percolates. I_c is the *largest* value of I^* for which a conducting path spans the system, where bonds are taken conducting only if I^* is *less* than the intensity of the corresponding saddle point. First we "prune" the network, removing irrelevant bonds in order to make the determination of the percolation threshold more efficient. If two sites are connected by more than a single bond (a common occurrence when there are a number of saddle points near to each other), then only the saddle point with the highest intensity needs to be kept since it will short circuit all of the other bonds. In this way, the total number of bonds in the network which need to be considered to determine the percolation threshold is reduced by about 20%. Taking two opposite edges

of the system to be conducting, so that a bond that crosses one of the edges is connected to that edge, and taking the other two edges to be insulating, we search for paths of conducting bonds connecting the two conducting edges. Starting at one of the conducting edges, we follow conducting bonds until either the other side is reached (the system percolates) or all of the accessible conducting bonds have been explored (the system does not percolate). By choosing different values of I^*, we can precisely determine I_c, the percolation threshold of the system. In Fig. 6 we show the pruned equivalent network constructed for one realization; Fig. 6(a) shows the complete network, and 6(b) the bonds conducting when the system is just percolating in both directions.

Table I shows the results for the threshold for percolation from left to right and from top to bottom of the four realizations created. Averaging the I_c's for the two directions and for the four realizations we find an estimate for the intensity at the percolation threshold of $I_c = 0.519$. This value of I_c corresponds to a critical area fraction for the conducting regions of the original continuum system of $f_c = 40.5\%$, in excellent agreement with the experimental result of $f_c = 40.7\%$ quoted by Smith and Lobb. If we were to assume that the variations in the I_c's in the two directions and for the different realizations were all independent, we would be led to assign an uncertainty of ± 0.03 to our estimate for I_c. In the third column of Table I, $\overline{I_c}$, the average of the thresholds for the two directions, exhibits little variation over the four realizations. This is true because the thresholds for the two directions of a single realization are not independent. For a given realization the two thresholds tend to vary in opposing directions. If there are more "easy" bonds in one direction, leading to a lower than normal threshold in that direction, then there will tend to be less "easy" bonds in the other direction, leading to a higher threshold in that direction. Thus, the uncertainty in our estimate for I_c must be less than the value quoted above, and is more likely to be about the same as the variation in $\overline{I_c}$: ± 0.002.

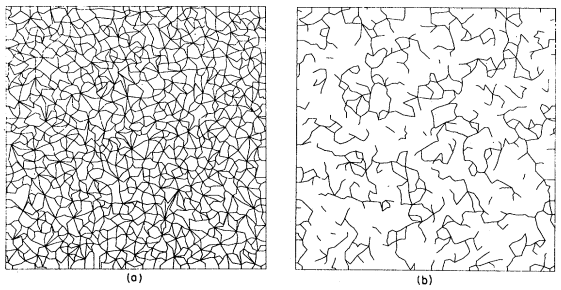

FIG. 6. (a) Pruned equivalent (dual) network for one realization. (b) The bonds conducting when the system just percolates in both directions.

TABLE I. The percolation thresholds for the four realizations. I_c^{LR} and I_c^{TB} are the thresholds for percolation from left to right and from top to bottom, respectively, and $\overline{I_c}$ is the average of the thresholds in the two directions.

Realization	I_c^{LR}	I_c^{TB}	$\overline{I_c}$
1	0.512	0.525	0.519
2	0.516	0.529	0.523
3	0.531	0.505	0.518
4	0.464	0.571	0.518

V. "EFFECTIVE-LATTICE" ESTIMATE

In this section we introduce an effective-medium-type of argument to determine the percolation threshold of a continuum system.[14] The argument makes use of the equivalence between the continuum problem and that of the network developed in Sec. III. While we show that it is possible to construct systems for which the result of the argument is a poor approximation, the argument is a rather successful predictor of the observed percolation threshold for the system studied in this paper.

For uncorrelated bond percolation on a regular lattice in d spatial dimensions there is an estimate for the percolation threshold[15] which works remarkably well[16]:

$$zp_c = d/(d-1) , \tag{8}$$

where z is the coordination number of the lattice, and p_0 is the fraction of bonds conducting at the percolation threshold. We now apply this estimate to our equivalent network, replacing z, the coordination number of the regular lattice, by $\bar{z}$, the average coordination number of the equivalent network. We are modeling the network by an effective regular lattice with the same average coordination number. Of course, $\bar{z}$ is generally not an integer, so it would be rather difficult to actually construct the lattice; we are continuing the estimate, Eq. (8), to noninteger z. The estimate for the equivalent network becomes $\bar{z}p_c = 2$. Using the equivalence between the continuum system and the

network, we have the fraction of saddle points covered at I_c, $p_c = \bar{R}_{\text{sad}}(I_c)/N_{\text{sad}}$, where $N_{\text{sad}} = R_{\text{sad}}(0)$ is the total density of saddle points, and $\bar{R}_{\text{sad}}(I) = N_{\text{sad}} - R_{\text{sad}}(I)$ is the density of saddle points covered with intensity *less* than I. Also, the average coordination number of the network with vertices at the minima of $I(\vec{x})$ is $\bar{z} = 2N_{\text{sad}}/N_{\text{min}}$, where $N_{\text{min}} = R_{\text{min}}(0)$ is the total density of minima. Thus, the estimate becomes an implicit equation for I_c,

$$\bar{R}_{\text{sad}}(I_c) = N_{\text{min}} \ . \tag{9a}$$

The continuum system is predicted to percolate when the number of saddle points covered is equal to the total number of minima. Equivalently, using the sum rule, we may write (9a) as

$$R_{\text{sad}}(I_c) = N_{\text{max}} \ , \tag{9b}$$

where N_{max} is the total density of maxima.

This estimate cannot be a good one for all systems. Imagine, for instance, taking a system for which the estimate is exact and deforming the surface $I(\vec{x})$ at intensity well above I_c to produce new minimum—saddle-point pairs (pairs so as to preserve the sum rule). In the new system the percolation threshold and the number of saddle points covered at the threshold would be the same as for the original system, but the estimate for the percolation threshold would change, since the total number of minima would increase with the deformation. Similarly, the true threshold is unchanged but the estimate changes when extra maximum—saddle-point pairs are added at intensity well below I_c.

We now apply the estimate derived above to the laser speckle pattern system. As discussed in Sec. IV, the exact calculation of the companion paper only finds the density of maxima plus minima, but because there is excellent agreement between the exact results and the computer study of this paper for $R_{\text{sad}}(I)$ and $R_{\text{max}}(I) + R_{\text{min}}(I)$ except at small values of I, we accurately determined R_{max} and R_{min} separately using a combination of the exact and the computer results. We find that the minima make up 60% of the total number of minima plus maxima [which is equal to the total number of saddle points, by the sum rule, Eq. (3)]. Therefore, we estimate that the system percolates when 60% of the saddle points are covered, corresponding to $I_c = 0.490$ and $f_c = 38.7\%$. In fact, the experimental system percolates when 62.5% of the saddle points are covered, corresponding to $I_c = 0.523$ and $f_c = 40.7\%$. Thus, for this system the estimate agrees quite well with the experimental result.

Also, notice that the "conjugate" system, obtained by interchanging the role of conductor and insulator [i.e., the system with conductor only in those regions where $I(\vec{x}) > I^*$], has an equivalent network which is simply the dual of the original system's network. The original network and the dual network both percolate at the same value I_c, so our estimate (9) which focuses on I_c is as good for the conjugate system as for the original system. This result is in contrast with estimates which focus solely on f_c.[17] Such estimates cannot be good for both a system and its conjugate except for the special cases for which $f_c = 50\%$.

VI. CONCLUSION

We have considered a general class of two-dimensional continuum percolation systems described by a characteristic surface over the plane, and have introduced an equivalent network determined by the geometry of the surface. Taking the state of the bonds of the network to be directly related to the height of the corresponding saddle point, we find that the network has the same percolation threshold as the continuum system. We thus have formalized the intuition that the saddle points of the random surface determine the percolation threshold. We have also utilized the network in developing a technique to find the percolation threshold of a system in which the surface is the intensity of a laser speckle pattern, and have found excellent agreement with experimental results for this system. In addition, we have determined the separate densities of maxima and minima for this case, which was not calculated in the companion paper. An effective-medium-type argument, which is based on the equivalent network, was introduced to estimate the percolation threshold. The resulting estimate compares fairly well with the experimental result.

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