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Frequency dependence of mutual friction in rotating He II

P. Mathieu and Y. Simon
Groupe de Physique des Solides de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cédex 05, France
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Interaction of a second-sound wave with vortices in rotating He II results in an extra-attenuation of the wave and a small decrease Δu_2 of the velocity of second sound. We report accurate measurements, at 1.9 K, of both effects as functions of the angular velocity Ω and the second-sound frequency ω. A resonator, whose fundamental mode was near 50 Hz, was specially designed for this work, with the purpose of extending the frequency range commonly used. As suggested by Mehl, the velocity decrease Δu_2 has been interpreted in terms of an imaginary part of a complex mutual-friction parameter $B=B_1+iB_2$. Thus, our measurements have consisted in determining the frequency dependence (at 1.9 K) of B_1 and B_2. Just as previously pointed out by Mehl *et al.*, in connection with their own data B_2 vs T, it is found that the detailed predictions of the theory of Hall and Vinen, in spite of the crude approximations contained in it, fit surprisingly well our experimental data B_1 and B_2 vs ω. We have recently published a hydrodynamic theory of mutual friction which provides an alternate explanation of our experimental results. The latter theory was originally inspired by the present work, which conversely gives experimental evidence of its validity.

I. INTRODUCTION

He II contained in a uniformly rotating vessel is threaded by a system of quantized vortex lines parallel to the axis of rotation. At angular velocity Ω, the number of lines crossing the unit area perpendicular to the axis of rotation is $2\Omega/\kappa$, where κ the quantum of circulation. A second-sound wave propagating perpendicular to vortices causes them to oscillate at the second-sound frequency ω. The resulting line velocity $\vec{V}_L$ can be expressed in terms of the normal and superfluid velocity fields of the incident wave, $\vec{V}_{n1}$ and $\vec{V}_{s1}$, by the relation[1]

$$\vec{V}_L=\vec{V}_{s1}-\frac{B\rho_n}{2\rho}\vec{\nu}\times(\vec{V}_{s1}-\vec{V}_{n1})-\frac{B'\rho_n}{2\rho}(\vec{V}_{s1}-\vec{V}_{n1}) , \quad (1)$$

where B and B' are the two mutual-friction parameters, first introduced by Hall and Vinen[2]; ρ is the helium density, ρ_n the normal fluid density, and $\vec{\nu}$ is a unit vector along the vortex line. The mutual-friction force $\vec{F}_s$ on the superfluid exerted by unit length of vortex line is related to $\vec{V}_L$ by the Magnus formula:

$$\vec{F}_s=\rho_s\kappa\vec{\nu}\times(\vec{V}_{s1}-\vec{V}_L) , \quad (2)$$

where ρ_s is the superfluid density. The average frictional force per unit volume appearing in the macrosopic equation of superfluid flow is equal to the vortex density $2\Omega/\kappa$ times $\vec{F}_s$. As shown by Bekarevich and Khalatnikov,[1,3] expressions (1) and (2) for the line velocity and the mutual-friction force can be introduced phenomenologically, without relying on any specific model of vortex motion.

Vortex motion is a dissipative process showing up as an additional attenuation of second-sound in rotating He II. The extra-attenuation constant is proportional to the first coefficient B, or more strictly to its real part B_1, if a small imaginary part B_2 has to be considered.[4] B_1 has been extensively measured as function of temperature[2,4–6]; all B_1 vs T data agree (up to 2.1 K) within the common accuracy of measurements (5–10%).

Interaction of second sound with vortices entails, as a side effect, a decrease in rotation of the second-sound velocity u_2. This is a minute effect ($|\Delta u_2/u_2|\sim10^{-5}$ at $\Omega\sim1$ sec^{-1}), which was first observed and discussed by Vidal *et al.*[7–10] and then by Mehl *et al.*[6,11,12] Δu_2 measurements as function of temperature obtained by these two groups exhibit strong discrepancies. To some extent, this explains why they also disagree about their theoretical interpretation. The initial purpose of the present work was to clear up this point.

The effect of velocity decrease is most simply interpreted if the mutual-friction parameter B is taken as a complex quantity, as proposed by Mehl.[12]

In their pioneering work on mutual friction,[2] Hall and Vinen (HV) did find a small imaginary term in their expression for the mutual-friction force, pointing out that such a component of force can change the velocity of sound; they, however, neglected it in later applications. Taking the HV formulas literally, and writing B and B' as B_1+iB_2 and $B'_1+iB'_2$, Mehl has shown that B_2 can be calculated from the known values of B_1 and B'_1.[12] Experimental values of B_2 determined from the Δu_2 data of Miller, Lynall, and Mehl[6] are in good agreement with the calculation of Mehl. The HV theory also predicts a weak dependence of B on second-sound frequency (see Sec. II). Sensitive measurements of B_1, also reported in Ref. 13, have revealed its frequency dependence, consistent with the HV theory. Nevertheless, in view of the experimental scatter (see Fig. 7 in Ref. 6), the frequency range that was used (500 − 1700 Hz) was too narrow for the comparison with theory to be entirely conclusive, especially as the correlative frequency dependence of B_2 was not observed.

For their part Vidal and Lhuillier[10] consider B and B' as real and frequency-independent quantities. Generalizing the equations of Bekarevich and Khalatnikov, they instead introduce a new kinetic coefficient ν coupling the dissipative heat flux and the mutual-friction force. Through their moving-vortex model, ν is related to the transported entropy per unit length of vortex line s_v. The obvious result of this additional term should be a velocity reduction $-\Delta u_2 \propto \nu \propto s_v$. Vidal and Lhuillier then give a rough estimate of the transported entropy s_v, from which they derive a theoretical expression for Δu_2 in reasonable agreement with their experimental results.

In this paper we report accurate measurements at 1.9 K of both the extra-attenuation and the decrease of u_2 as function of the second-sound frequency. This working temperature has been chosen primarily because optimum performances of our temperature control system are obtained at about 1.9 K.[5,13] Moreover, 1.9 K lies in the temperature range where the coefficient B' is nearly zero; vortex dynamics thereby are simplified, making the comparison with theoretical models easier. Our results turn out to be in very good agreement with predictions of the HV model (Sec. II), in accordance with Mehl's point of view. One cannot, however, adhere entirely to the HV model, which is otherwise unable to explain the actual magnitude of the principal term B_1, as explained in Sec. II. Recently, we formulated a hydrodynamic theory of mutual friction,[14] the conclusions of which are summarized in Sec. IV. This alternative theory also accounts for the phase and frequency-dependent effects investigated in this paper, while predicting the correct values of B_1 from 1.7 to 2.1 K.

II. THE HALL-VINEN MODEL

The original HV theory[2] was refined by Hall[15] and then reexamined by Hillel, Hall, and Lucas.[16] In this paper we shall make use of the results and notations of the theory in its final form, as presented by Mehl.[12]

The HV theory combines a kinetic treatement of roton-vortex collisions with purely hydrodynamic arguments (Magnus effect, dragging of the normal fluid). As a result, rather complicated relationships are obtained, giving B and B' in terms of two collision diameters $\sigma_{||}$ and $\sigma_{\perp}$, which describe the scattering of rotons by vortex lines. The derived expressions for B and B' can be written in the form[16]

$$B=\frac{2\rho}{\rho_n\rho_s\kappa}\frac{X}{X^2+Y^2}\ ,\qquad B'=\frac{2\rho}{\rho_n\rho_s\kappa}\frac{Y}{X^2+Y^2}\ , \tag{3}$$

where X and Y are functions of $\sigma_{||}$ and $\sigma_{\perp}$.

Near 1.9 K, a useful simplification will be achieved by taking into account the relative smallness of the coefficient B'. Measurements of B' utilize the fact that B', together with the Coriolis force, couples two degenerate modes in square[5] or cylindrical[4] cavities. In spite of experimental scatter, it can be stated that $|B'| \lesssim 0.1$ over the temperature range from 1.7 to 2.1 K. Since $B \simeq 1$, it follows from Eqs. (3) that $|Y/X| = |B'/B| \lesssim 0.1$. Therefore, with accuracy better than 1%, B can be written as

$$B=\frac{2\rho}{\rho_n\rho_s\kappa}X^{-1}\ . \tag{4}$$

X is given by an expression of the form

$$X=\frac{D}{D^2+(D'-\rho_n\kappa)^2}+\frac{1}{E}\ , \tag{5}$$

where D and D' are parameters proportional to the collision diameters $\sigma_{||}$ and $\sigma_{\perp}$. We will pay particular attention to the last term $1/E$. The normal fluid velocity at the vortex line denoted as V_R (roton drift velocity) in the original papers, was expected to differ from the normal fluid velocity V_{n1} far from the line. HV related the difference between these velocities to the mutual-friction force through the equation $V_{n1}-V_R=F_s/E$, where

$$E=\frac{-4\pi\eta}{\ln(L/2\delta)+1+i\pi/4} . \tag{6}$$

Here L is the roton-roton mean free path, η is the normal fluid viscosity, and $\delta=(\eta/\rho_n\omega)^{1/2}$ is the viscous penetration depth at the second-sound frequency $\omega/2\pi$. Bringing out the frequency-dependent term and the imaginary part of $1/E$ in Eqs. (4) and (5), we obtain

$$B=B_1+iB_2$$
$$=\left[A-\frac{\rho_n\rho_s\kappa}{16\pi\eta\rho}(\ln\omega+i\pi/2\right]^{-1} . \tag{7}$$

Here A stands for a *real* and *frequency-independent* expression involving $\sigma_{\parallel}$ and $\sigma_{\perp}$. All terms in Eq. (7) are dimensionless quantities; $A\sim B\sim 1$ is the main term, whereas the last two terms represent small corrections. At 1.9 K, $\rho_n\rho_s\kappa/16\pi\eta\rho\simeq 0.05$. It will be these small terms, however, that will be investigated in the present work. Let $x-iy$ be the complex number in the large parentheses of Eq. (7). Since $y<<x$, terms of the order of $(y/x)^2$ can be neglected, so that $B_1\simeq 1/x$ and $B_2\simeq Y/x^2$, or

$$B_1=\left[A-\frac{\rho_n\rho_s\kappa}{16\pi\eta\rho}\ln\omega\right]^{-1} , \tag{8}$$

$$B_2=\frac{\rho_n\rho_s\kappa}{32\eta\rho}B_1^2 . \tag{9}$$

It should be noted that the above expressions for B and B' indeed do not constitute a complete theory of mutual friction. B and B' are only being substituted by the other two unknown parameters $\sigma_{\parallel}$ and $\sigma_{\perp}$. Several attempts have been made to calculate the collision diameters from first principles. However, as shown in the paper of Hillel *et al.*[16] (see their Fig. 1), none of the theoretical values of $\sigma_{\parallel}$ and $\sigma_{\perp}$ derived in the literature can account for the experimental data of B and B'. Thus, it turns out that the only definite conclusions of the theory, irrespective of the σ's, just concern the weak frequency dependence of B_1 and B_2. This frequency dependence, as well as the relation connecting B_2 and B_1, only involves well-known quantities, namely ρ_n, ρ_s, and η. According to Eqs. (8) and (9), at 1.9 K, an imaginary part $B_2\simeq 0.07$ should be associated with a real part B_1 of order unity, and doubling the second-sound frequency should increase B_1 and B_2 by 3.5% and 7%, respectively.

The authors, however, paid no attention to the frequency dependence and phase effects implied from their model. From the set of equations given above it is clear that these effects originate only in the term E as given by Eq. (6). However, Eq. (6) itself results from an approximate calculation, using the naive model of a solid wire dragging an ordinary viscous fluid. As Hall has pointed out,[15] it is not obvious that the formula for E is correct. Thus Eqs. (8) and (9) should be used with due caution, leading at best to qualitative conclusions. Moreover, given the usual scatter of measurements in rotating helium, the expected effects, if any, are too weak to have been detected.

In view of the uncertainties contained in the term $1/E$, a careful derivation of simplified expressions for B, such as Eqs. (7)–(9), seems somewhat illusive. As a matter of fact, these equations would not be worth writing, if they did not account for our experimental results remarkably well.

III. EXPERIMENT

A. The cavity

Variations due to rotation of the resonant amplitude and frequency of standing waves have been accurately measured. The second-sound frequency ω, which is the physical parameter of interest, is varied by merely investigating different modes of one single resonator. The experimental procedure is quite similar to that described in two previous papers.[5,13]

A resonator, which we shall refer to as the C cavity (Fig. 1), was specially designed for the present work with the purpose of extending the frequency range as low as possible. It is obtained from a torus of rectangular cross section by cutting it and closing up along two meridian cross sections, so that the cavity is singly connected. The axis of the torus coincides with the vertical axis of rotation Oz. With cylindrical coordinates (r,θ,z), the region of space inside the cavity is given by $r_1<r<r_2$, $0<\theta<\theta_0$, $0<z<h$, where $r_1=25$ mm, $r_2=35$ mm, $\theta_0=346°$, $h=13$ mm. The mean length along a circular path of radius $r=30$ mm is $L=181$ mm. The fundamental mode and its first few harmonics are well separated and nondegenerate modes. The resonant frequencies, labeled ω_n, corresponding to these longitudinal modes are close to the eigenfrequencies $n\pi u_2/L$ of a long rectangular cavity of length L. At 1.9 K, the fundamental

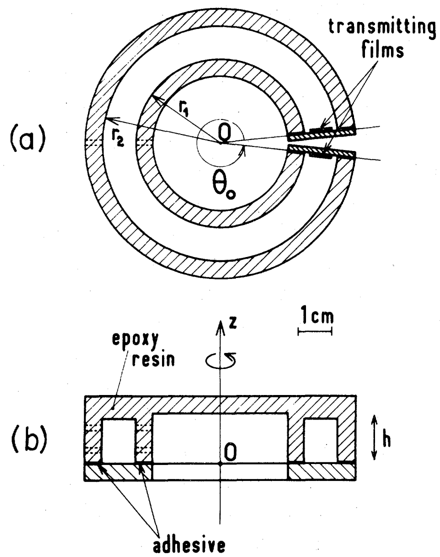

FIG. 1. (a) Cross-section view of the C cavity viewed along the vertical rotation axis Oz. (b) Meridian cross section in a plane through Oz. Dimesions are given in the text.

frequency of the C cavity is $\omega_1/2\pi = 53$ Hz. Harmonics up to the twenty-first (1100 Hz) have been investigated.

Unless placed at nodes of the standing wave, leaks coupling the helium inside the cavity with the external helium bath may strongly affect the second-sound response and lead to unreliable results. This point is discussed in the Appendix. The cavity walls were machined from epoxy resin rods and carefully bonded with an epoxy resin adhesive (Araldite). Small holes were drilled at the midlength of the cavity ($\theta = \theta_0/2$) to allow the helium to flow and the dc heat to escape, these holes being the only links between the cavity and the helium bath. Therefore we only used odd harmonics of the fundamental mode that have a node at $\theta_0/2$.

B. Experimental principle

When the driving frequency is near one of the characteristic frequencies ω_n so that the nth mode is predominantly excited, the calculated amplitude of the temperature field in the cavity, $T_1(\vec{r})e^{i\omega t}$, can be written as

$$T_1(\vec{r}) = \frac{a_n}{1 - i(Q_n/\omega_n)(\omega - \omega_n)} \varphi_n(\vec{r}) , \tag{10}$$

where $\varphi_n(\vec{r}) = f(r)\cos n\pi\theta/\theta_0$ [$f(r) \simeq$ const.] is the corresponding normalized eigenfunction (or normal mode such as that defined in the Appendix). The receiving bolometer, located near an antinode of $\varphi_n(\vec{r})$, measures both amplitude and phase of the temperature oscillation. The bolometric signal $s = X + iY$ is directly plotted on the Argand diagram through the two channels of an X-Y recorder.[5] From Eq. (10), the frequency response $s(\omega)$ takes the form of a classical resonance curve:

$$s(\omega) = \frac{A}{1 - i(Q_n/\omega_n)(\omega - \omega_n)} . \tag{11}$$

If rotating helium is regarded as an homogeneous medium with uniformly distributed vortices, the quality factor Q_n and the resonant frequency ω_n should vary with the angular velocity Ω according to the following simple laws, to first order in the small quantity Ω/ω:

$$\frac{1}{Q_n} = \frac{1}{Q_{n0}} + \frac{B_1\Omega}{2\omega} , \tag{12}$$

$$\omega_n = \omega_{n0} - \frac{B_2\Omega}{2} . \tag{13}$$

Equation (12) is a classical result.[2,5] Equation (13) amounts to

$$\frac{\Delta u_2}{u_2} = \frac{\Delta\omega_n}{\omega} = -\frac{B_2\Omega}{2\omega} , \tag{14}$$

and, in the latter form, follows at once from the wave equation for second sound, provided B is complex, as shown by Mehl.[12] The response of a rotating cavity for any given boundary condition is calculated in the Appendix using a Green's-function method, and Eqs. (10), (12), and (13) are re-derived. We now wish to make two remarks about these results.

First we note that the second coefficient B' does not appear in the expressions for Q_n and ω_n given above. This simple circumstance does not result from any approximation such as that used in Sec. II. As shown in the Appendix, in the case of a *nondegenerate* mode, B', whether small or large, complex or real, cannot affect the response of the rotating resonator in any way.

On the other hand, Eqs. (12) and (13) describe the effects that are expected with a uniform distribution of vortices filling the cavity. As a better approximation, we must allow for the narrow vortex-free region existing along the walls of the cavity. The effect of missing vortices is equivalent to reducing the apparent value of B, such as that given from experiment through uncorrected equations (12) and (13). In other words, B must be replaced in the equations by the product $B\gamma$, where γ is a filling factor less than unity, depending on both the vortex state and the excited mode. This question has been studied in detail in Ref. 13, both by theory and experiment. At the thermodynamic equilibrium, the vortex-free region has a uniform width expressed in mm [see Eq. (23) of Ref. 13]

$$d_0=0.29\Omega^{-1/2}\ , \tag{15}$$

which is independent of the shape of the boundaries. As the radial variation of the wave field in longitudinal modes of the C cavity is negligibly small, we may use the same theoretical expression for γ as in a rectangular $l\times L$ cavity:

$$\gamma=1-\frac{2d_0}{l}=1-0.057\Omega^{-1/2}\ , \tag{16}$$

l now standing for the radial width of the C cavity ($l=r_2-r_1$). Thus, for example, the second-sound attenuation should not yield the parameter B_1 directly, but instead the product $B_1\gamma$:

$$B_1\gamma=\frac{2\omega}{\Omega}\left[\frac{1}{Q_n}-\frac{1}{Q_{n0}}\right]. \tag{17}$$

At angular velocity $\Omega=1\ \text{sec}^{-1}$, Eq. (16) gives $\gamma\simeq0.94$. If B_1 is to be measured with an accuracy of 1% or so, clearly the correcting factor γ cannot be disregarded.

C. Experimental results

At a given angular velocity Ω the frequency response $s(\omega)$ of the investigated mode is plotted by points for about 20 discrete values of the exciting frequency. The accuracy and stability of recorded points was about 0.1% of the maximum signal A. This obviously required that all involved parameters be carefully controlled. The rotational speed was measured and regulated to a few parts in 10^4. During a run, we also ascertained that the ac input power to the transmitter was stable to better than 10^{-4}. Moreover, the signal is very sensitive to small fluctuations of the driving frequency and, since u_2 is temperature dependent, of the bath temperature. At 1.9 K, and for typical $Q\sim10^3$, a variation $|\Delta s|\sim10^{-3}A$ should follow from $\Delta\omega/\omega\sim10^{-6}$ or $\Delta T\sim10^{-6}$ K ($\Delta u_2/u_2\sim10^{-6}$). While taking advantage of the high-frequency stability of a synthesizer ($\sim10^{-8}$), particular attention must be paid to regulating the bath temperature.

Our temperature control system utilizes an auxiliary second-sound resonator of high quality factor ($Q\sim30.000$ at 30 kHz). When driven at fixed frequency, this resonator can be used as a sensitive thermometer, the rapid variation of phase near resonance providing an error signal for regulation. As the control resonator is a part of the rotating system, it is important to note that the second sound in it propagates parallel to vortices, and consequently, is not affected by rotation. Even with the assumption of the existence of a residual effect $\Delta u_2/u_2\propto\Omega/\omega$ of the form given by Eq. (14), it still would be unimportant because of the relatively high frequency used (30 kHz). As pointed out by Miller *et al.*,[6] u_2 is also pressure dependent, so that measurements should need corrections for changes in the helium bath level ($\Delta u_2/u_2\sim10^{-6}$/cm He). In our experiment, however, such corrections proved unnecessary, since the physical parameter we are actually regulating is not the temperature but u_2 itself, i.e., the *second-sound velocity in stationary helium*. At 1.9 K, the long-term stability, expressed in terms of temperature drift (at constant helium level), was better than 10^{-8} K for periods greater than 1 h.

The perfect circular shape of the resonance curve, in accordance with Eq. (11), ensures that we actually observe a well-separated nondegenerate mode. It also guarantees the absence of nonlinear distortion at resonance. The small background signal due to both electrical crosstalk at second-sound frequency and weak excitation of distant modes (see the Appendix) was negligibly small in our experiment, so that the frequency-response curves could be fitted to the theoretical resonance formula directly, without background correction.[6]

The quality factors Q_n or Q_{n0} were determined from resonance curves with 0.5% accuracy. The quantity $B_1\gamma$ then was calculated using Eq. (17). With the use of the data with the fundamental mode of the C cavity, Fig. 2 shows the typical dependence of $B_1\gamma$ on angular velocity, which is well fitted by the straight line

$$B_1\gamma=0.823(1-0.053\Omega^{-1/2})\ ,$$

in good agreement with the predicted value of γ as that given by Eq. (16). This behavior is consistent with earlier results obtained with rectangular cavities.[13] It should be noted that the easy occurrence of metastable states may considerably scatter the observed values of $B_1\gamma$.[13] The data reported in Fig. 2 were obtained by systematically achieving thermodynamic equilibrium, as explained in Ref. 13.

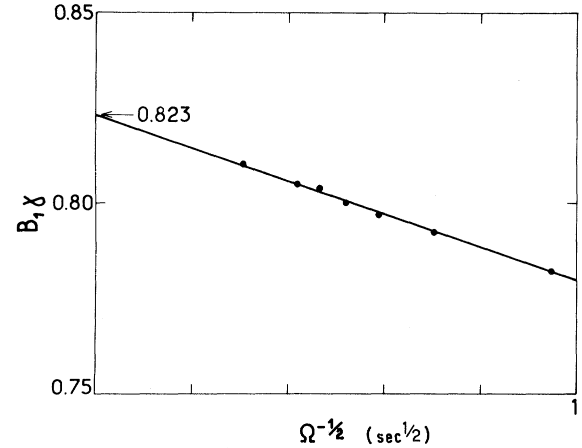

FIG. 2. Product of the mutual-friction parameter B_1 and the filling factor γ, as function of the angular velocity Ω. As explained in the text, $B_1\gamma$ was directly determined from measurements of Q factors, first in the stationary, then in the rotating resonator. This figure shows the data obtained at 1.9 K in the C cavity (Fig. 1) driven on its fundamental mode, $\omega_{1/2\pi}=53$ Hz. $B_1\gamma$ is plotted as function of $\Omega^{-1/2}$, showing the linear dependence of γ on $\Omega^{-1/2}$ in agreement with Eq. (16). On extrapolating the fitting line to $\Omega\rightarrow\infty$, we obtain the value of B_1 at 53 Hz, which will be used in Fig. 4.

On the other hand, the decrease in the resonance frequency $\delta\omega_n=\omega_{n0}-\omega_n$ yields the parameter B_2. From Eq. (13), after correcting for the factor $\gamma(\Omega)$, we have

$$B_2=\frac{1}{\gamma}\frac{2\delta\omega_n}{\Omega} . \tag{18}$$

Experimental results of B_2 vs Ω for the fundamental mode are given in Fig. 3. The frequency shifts $\delta\omega_n$ were measured with a precision of a few percent. Averaging the data for different angular velocities provides an even more accurate value of B_2.

Similar measurements were made for odd harmonics of the C cavity up to the twenty-first. We obtained in this way values of B_1 and B_2 for frequencies ranging from 50 to 1600 Hz. These results are shown in Figs. 4 and 5. To test Eq. (8), B_1^{-1} has been plotted against $\ln\omega$ (Fig. 4). A straight line with adjustable zero intercept A but having the predicted slope $-\rho_n\rho_s\kappa/16\pi\eta\rho$ $=-5.27\times10^{-2}$ (at 1.9 K) closely fits the experimental results. According to Eq. (9), values of B_2 could have been deduced from those measured for B_1. Taking the linear fit from Fig. 4 as a smooth experimental curve $B_1(\omega)$ and substituting in Eq. (9), we obtain the solid line $B_2(\omega)$ shown in Fig. 5.

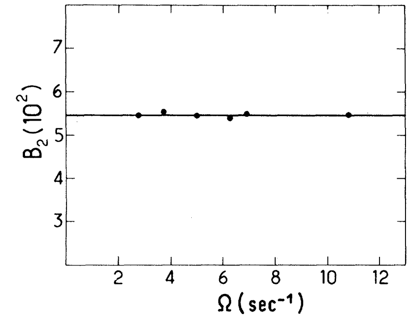

FIG. 3. Mutual-friction parameter B_2 as function of the angular velocity Ω. B_2 was calculated for each Ω, using the measured resonant-frequency shift and correcting for the factor γ. This figure shows a set of data with the fundamental mode of the C cavity at 1.9 K. By averaging these data we obtain the experimental value of B_2 at 53 Hz (solid line), which will be used in Fig. 5.

Experimental points of B_2 vs ω are in excellent agreement with this theoretical prediction. It is to be emphasized that the calculation of the curve $B_2(\omega)$ from the B_1 vs ω data involves no free parameter.

The frequency dependence of B_2 has escaped detection in the experiment of Miller *et al.*[6] However, although the three values of B_2 at 1.9 K, reported in Fig. 3 of Ref. 6, unexpectedly decrease with increasing frequency, they are roughly consistent with our own data, whereas somewhat higher values of B_2 follow from the Δu_2 data of

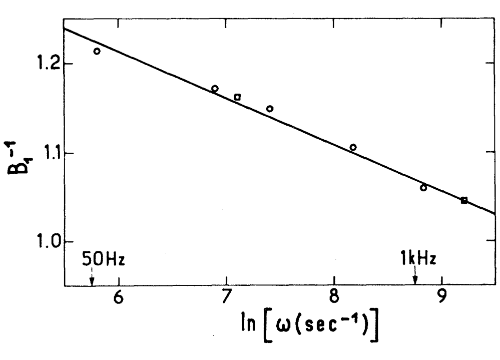

FIG. 4. Frequency dependence of B_1 at 1.9 K. Open circles correspond to data taken with the fundamental mode and 3rd, 5th, 11th, and 21st harmonics of the C cavity. Open squares correspond to data of Ref. 13 taken with the x and y fundamental modes of a rectangular cavity 49×6 mm^2 (194 and 1593 Hz). The data are fitted to Eq. (8) by plotting B_1^{-1} vs $\ln\omega$. The straight line has the calculated slope $-\rho_n\rho_s\kappa/16\pi\eta\rho=5.27\times10^{-2}$ (at 1.9 K) and the adjusted zero intercept $A=1.53$. Note that this value of A is obtained by taking $\epsilon=0.7$ Å in Eq. (21).

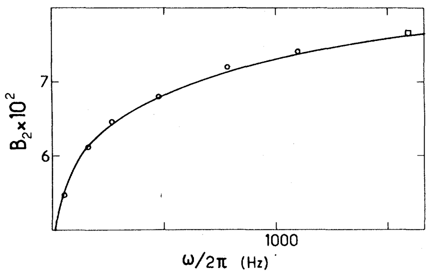

FIG. 5. Frequency dependence of B_2 at 1.9 K. The experimental data (open circles) were taken with harmonics 1, 3, 5, 9, 15, and 21 of the C cavity. The last point at 1593 Hz (open square) was taken with the y fundamental mode of the rectangular cavity 49×6 mm^2. The solid line is the theoretical curve $B_2(\omega)$ calculated from experimental values of B_1 by using the linear fit of Fig. 4 as a smooth experimental curve $B_1(\omega)$ and substituting in Eq. (9).

Vidal and Lhuillier.[10] In addition, the measurements of Vidal and Lhuillier show a linear dependence of u_2 with Ω at small Ω with a more rapid decrease above $\Omega=4$ sec^{-1} (the maximum angular velocity used in the work of Miller *et al.*). According to Eq. (14) such a nonlinear dependence of u_2 on Ω would be equivalent in the present data to B_2 increasing with Ω. Nevertheless, we observe no significant variation of B_2 in the entire experimental range of angular velocities ($\Omega \lesssim 10$ sec^{-1}; Fig. 3).

Miller *et al.* suggest that the presence of a large electronic pickup in the experiments of Vidal *et al.* might have given rise to spurious changes in the resonant frequency, accounting for both the disagreement about the experimental results of Δu_2 vs T, and the observed nonlinear dependence of u_2 on Ω. We disagree with this interpretation. Electronic pickup is a steady background that can be easily distinguished from the second-sound signal and compensated for by a simple change in the origin of the complex plane. Vidal and Lhuillier state that in some experiments, one wall of their cavity was removed entirely.[10] Under such conditions, actual but unforeseen changes in the resonance frequency were more likely to result from a strong coupling between the cavity and the external helium bath, as explained in the Appendix.

IV. DISCUSSION

Summarizing our experimental results, we can state that the slight frequency dependence at 1.9 K of both the increase, due to rotation, of the attenuation of second sound and the decrease $-\Delta u_2$ of the second-sound velocity is remarkably accounted for by Eqs. (8) and (9). On the other hand, the actual magnitude of Δu_2, as well as can be deduced with no adjustable parameter from known values of B_1 through Eqs. (9) and (14), agrees surprisingly well with experimental data.

All these results are contained in a condensed form in the HV expression (7) for the mutual-friction parameter B. We might therefore be tempted to say, like Mehl, that the above experiments support the detailed correctness of the HV theory. This opinion, however, should be moderated. From experiment we may strictly infer that B at 1.9 K is a complex quantity of the form given by Eq. (7). As its main term reads as an adjustable real constant A (at some given temperature), Eq. (7) expresses nothing but a minor consequence of the HV theory. Furthermore, as we have seen in Sec. II the hydrodynamic calculation leading to Eq. (7) was regarded by HV as a crude approximation, which was introduced for lack of a rigorous kinetic treatment of the motion of rotons in the neighborhood of a vortex line. Nevertheless, the approximate model comparing a moving vortex to an oscillating wire did predict the correct frequency dependence of B as $(\rho_n\rho_s\kappa/\eta\rho)\ln\omega$. On the strength of our experiments, we thus concluded that the physical picture of a vortex dragging the normal fluid like an ordinary viscous fluid was essentially correct. It seemed to us, however, that the correct numerical factor $1/16\pi$ in Eq. (7) was obtained somewhat by accident.

These remarks incited us to work out a purely hydrodynamic theory of mutual friction, which was likely to bear out the role of the normal fluid viscosity. Although it originates with the present work, this theory has already been published in a previous paper.[14] Making use of the usual linear approximation of acoustics, we were able to find a solution of the Landau-Khalatnikov two-fluid equations[1] describing the small oscillations of a vortex line subject to the superfluid and normal velocity fields, $\vec{V}_{s1}$ and $\vec{V}_{n1}$, of a second-sound wave. This solution has the simple following properties: (i) incompressible flow ($\rho=$const), (ii) rigid transport at velocity $\vec{V}_L$ of the superfluid vortex field, and (iii) the normal flow around the vortex is perturbed over distances of the order of the viscous penetration depth δ. $\vec{V}_n=\vec{V}_{s1}$ at the core, $\vec{\nabla}\cdot\vec{V}_n=0$, and $\vec{\nabla}\times\vec{V}_n=\vec{u}$ obey the diffusion equation

$$\rho_s\frac{\partial\vec{u}}{\partial t}-\eta\nabla^2 u=\vec{\nabla}\times\vec{\varphi}\ , \tag{19}$$

where $\vec{\varphi}$ is a density of force localized at the vortex core. The total force per unit length acting on the normal fluid,

$$\vec{F}_n = \int\int \vec{\varphi}\, d^2r \, ,$$

is found to be equal (except for its sign) to the mutual-friction force $\vec{F}_s$ such as that given by Eq. (2).[17] In the similar problem of an ordinary viscous fluid submitted to a localized density of force, Eq. (19) would immediately follow from the Navier-Stokes equation. In solving Eq. (19), we obtained the required relation between $\vec{V}_L$, $\vec{V}_{n1}$, and $\vec{V}_{s1}$ in the form given by Eq. (1), with $B'=0$ and

$$B^{-1} = \frac{\rho_n \rho_s \kappa}{16\pi\eta\rho}\left[\ln\left[\frac{\eta}{\rho_n \omega \epsilon^2}\right] - i\pi/2\right] . \qquad (20)$$

Here ϵ is a cutoff radius (denoted as l in Ref. 14) of the order of 1 Å.

Firstly, we note that our expression for B can be rewritten in exactly the same form as Eq. (7) by taking explicitly

$$A = \frac{\rho_n \rho_s \kappa}{16\pi\eta\rho}\ln(\eta/\rho_n\epsilon^2) \, . \qquad (21)$$

Consequently, the hydrodynamic theory provides an alternative and more reliable interpretation of our experimental results. Moreover, Eq. (20), in contrast with the HV theory (see Sec. II), makes it possible to calculate B in terms of known macroscopic parameters, in particular the normal fluid viscosity η. According to the theory, the precise value of ϵ in Eq. (20) depends on the unknown detailed structure of the vortex core. Nevertheless, since ϵ cannot differ greatly from 1 Å and in addition appears in the logarithmic factor, it should not be regarded as a free parameter, except in fitting very accurate measurements; for instance, the best fit in Fig. 4 is obtained for $\epsilon = 0.7$ Å. Thus letting $\epsilon = 1$ Å, it was found that Eq. (20) well accounted for the actual magnitude and temperature dependence of B_1 from 1.7 to 2.1 K.[14] This temperature interval roughly coincides with the one where B' is found experimentally to be nearly zero, also as predicted.

It is not surprising that our hydrodynamic picture of mutual friction fails at low temperature, as well as in the λ region. Below about 1.6–1.7 K the roton-roton mean free path becomes larger than the vortex core ($L \gtrsim 10$ Å), so that there is no escaping a kinetic treatment of rotons-vortex collisions, just as that proposed by Hall and Vinen or by Goodman.[18] On the other hand, the Landau-Khalatnikov equations must be corrected as the λ point is approached to include effects of the relaxation of the superfluid density.[19]

Moreover, the expression for B given above does not apply to arbitrarily low frequencies. In particular, let B_0 be the zero-frequency value of B appropriate to steady-state conditions as required in the study of mutual friction in a dc heat current.[20,21] According to Eq. (20), $B(0)$ should become zero at zero frequency, whereas from experiment B_0 is known to be again of order unity. By measuring in rotating thermal counterflow the chemical potential gradient associated with vortex motion, Yarmchuk and Glaberson[21] obtained values of B_0 at different temperatures; at 1.9 K they found $B_0 \simeq 0.73$. The linear approximation used in the hydrodynamic calculation of B requires the vortex oscillation to be small compared to the viscous penetration depth δ, i.e., $v/\omega << \delta$ or $\omega >> \rho_n v^2/\eta$, where v is the order of magnitude of the velocities V_{n1}, V_{s1}, and V_L.[14] For instance, at 1.9 K for velocities of practical interest [$v \sim 0.1$ cm/sec (Ref. 21)] this assumption breaks down at $\omega \lesssim 50$ sec^{-1}. If indeed the nonlinear terms in the hydrodynamic equations must be taken into account, the calculation of the line velocity $\vec{V}_L$ in a steady-state counterflow becomes rather involved. Again considering the classical analogous problem of the solid wire in an ordinary viscous fluid, Vinen noted that the effect of the nonlinear terms at zero frequency in the expression of the viscous drag was the same as that of an effective penetration depth $\delta_0 = 2\eta/\rho U$,[20] where ρ is the fluid density and U is the velocity of the uniform stream relative to the cylinder. Following Vinen, it may be assumed that B is approximately given by Eq. (20), after dropping the imaginary term and replacing $\delta = (\eta/\rho_n\omega)^{1/2}$ by $\delta_0 = 2\eta/\rho_n \mid \vec{V}_{n1} - \vec{V}_L \mid$, or ω by $\omega_0 = \rho_n(\vec{V}_{n1} - \vec{V}_L)^2/4\eta$. We thus obtain a velocity-dependent value of B. However, this velocity dependence will be very small, since ω_0 appears in a logarithm. At 1.9 K, for $v \sim 0.1$ cm/sec, the effective frequency $\omega_0 \sim 10$ and Eq. (20) yields $B_0 = B_1(\omega_0) \simeq 0.71$, in good agreement with the experimental result of Yarmchuk and Glaberson,[21] $B_0 \simeq 0.73$. In reducing the frequency from $\omega = 10^4$ (the last point in Fig. 4) to zero, the value of B is therefore decreased by about 25%.

Concerning the much debated question of the physical origin of Δu_2,[6–12] our experimental results clearly appear in favor of Mehl's main arguments. In connection with the theory of Vidal and Lhuillier,[10] it is perhaps worth noting that the normal fluid flow around the moving vortex, such as that described by our hydrodynamic solution, entails no entropy drag at all. The transported entropy s_v (see Sec. I) thus should reduce to the negligibly small core contribution. If it is so, Vidal and Lhuillier have estimated the resulting Δu_2 to be 7 orders of magnitude too small.

In conclusion, the experiments reported in this paper, though dealing with very small and somewhat minor effects occurring in rotating second-sound resonators, have led to a consistent theory of mutual friction, using a purely hydrodynamic approach. Conversely, our experiments, together with available B_1 vs T data, support the correctness of the hydrodynamic theory; in particular, they provide experimental evidence that the dissipative mechanism in mutual friction (at the temperatures concerned) is none other than the viscosity of the dragged normal fluid.

APPENDIX

In this appendix we wish to consider the second-sound response of a rotating resonator for rather general boundary conditions. The resonator is assumed to be a deformed cylinder of arbitrary cross section. Its inner walls are coated with one or several transmitting films, which are shaped as vertical strips (two-dimensional problem). In order to generate a second-sound wave at frequency ω, the transmitters are driven at $\frac{1}{2}\omega$. Let $q(\vec{r}\,')e^{i\omega t}$ be the heat input at a point $\vec{r}\,'$ on the boundary surface: $q(\vec{r}\,')$ equals the ac component of the Joule effect on the transmitting film and zero elsewhere.

In setting down the equations of motion in the rotating frame, the two-dimensional temperature field $T_1(\vec{r})e^{i\omega t}$ of a second-sound wave propagating perpendicular to vortices is found to obey the wave equation[5]

$$\nabla^2 T_1 + k^2 T_1 = 0 \,, \qquad \text{(A1)}$$

$$k^2 = \frac{\omega^2}{u_2^2}(1-i\beta) \,.$$

To first order in the small quantity Ω/ω, β can be written as[5]

$$\beta = \beta_0 + \frac{B\Omega}{\omega} = \beta_1 + i\beta_2 \,. \qquad \text{(A2)}$$

Here the small real term β_0 refers to the residual attenuation in the cavity at rest, and u_2 *is* the second-sound velocity in stationary helium.

By equating the heat source q to the inward pointing normal component of the energy flux vector, we find[5] (always to first order in Ω/ω):

$$\vec{\nabla} T_1 \cdot \vec{n} = \frac{\partial T_1}{\partial n} = \alpha(1-i\beta)q + \vec{\beta}\,' \cdot (\vec{n} \times \vec{\nabla} T_1) \,, \qquad \text{(A3)}$$

where

$$\alpha = i\omega z_\infty / u_2 \,, \quad \vec{\beta}\,' = i(2-B')\frac{\vec{\Omega}}{\omega} \,.$$

Here $\vec{n}$ is the outward normal and z_∞ is the characteristic impedance of helium. We are faced with the problem of solving the Helmholtz equation (A1) in a bounded region of space with the prescribed boundary condition (A3). We introduce the Green's function for Eq. (A1) that satisfies $\partial G/\partial n = 0$ on the walls to obtain the integral equation for $T_1(\vec{r})$:

$$T_1(\vec{r}) = -\int\int_{\text{wall}} g(\vec{r}\,')G(\vec{r}/\vec{r}\,')d^2r' \,, \qquad \text{(A4)}$$

where $g(\vec{r}\,')$ stands for the right-hand side of Eq. (A3). The response function $T_1(\vec{r})$ can be expanded in a series of orthonormal eigenfunctions of Eq. (A1), namely the functions $\varphi_i(\vec{r})$ satisfying

$$\nabla^2 \varphi_i + k_i^2 \varphi_i = 0$$

($\partial\varphi_i/\partial n = 0$ on the walls). The corresponding eigenfrequencies are defined as $\omega_{i0} = k_i u_2$. Let c_i be the coefficients of expansion of $T_1(\vec{r})$. Using the so-called bilinear expansion for $G(\vec{r}/\vec{r}\,')$,

$$G(\vec{r}/\vec{r}\,') = \sum_i \frac{\varphi_i(\vec{r})\varphi_i(\vec{r}\,')}{k^2 - k_i^2} \,,$$

and substituting into Eq. (A4), we obtain the set of simultaneous linear equations

$$c_i(k^2 - k_i^2) = \alpha(1-i\beta)q_i + \vec{\beta}\,' \cdot \left[\sum_j \vec{\lambda}_{ij} c_j\right] \,, \qquad \text{(A5)}$$

where

$$q_i = \oint\!\!\oint q_i(\vec{r}\,')\varphi_i(\vec{r}\,')d^2r' \,,$$

is a transmitting factor and

$$\vec{\lambda}_{ij} \oint\!\!\oint [\vec{n} \times \vec{\nabla}\varphi_j(\vec{r}\,')]\varphi_i(\vec{r}\,')d^2r' \qquad \text{(A6)}$$

is a vectorial coupling coefficient. The effects of rotation on the second-sound response arise in Eqs. (A5) through the two first-order terms β and β', involving B and B', respectively.

Assume now the exciting frequency ω to be close to one of the characteristic frequencies, such as

ω_{mo}. This mode is presumed not to be degenerate or quasidegenerate, so that $c_m\varphi_m(\vec{r})$ is the single resonant term predominating in the series. At resonance $c_m \sim 1/\beta$, and, if $\Delta\omega$ denotes the half-width of the common frequency-response curve, the quantity $1/Q_m = \Delta\omega/\omega_{m0}$ is of the same order of smallness as β (or Ω/ω). In a first approximation, the c_i's are determined from the zero-order parts of Eq. (A5):

$$c_m(k^2-k_m^2-\vec{\beta}'\cdot\vec{\lambda}_{mm})=\alpha q_m\ ,\quad i=m$$
$$c_i=\alpha q_i/(k_m^2-k_i^2)\ ,\quad i\neq m\ . \tag{A7}$$

From Eq. (A6) the antisymmetry relation $\vec{\lambda}_{ij}=-\vec{\lambda}_{ji}$ can be easily inferred. As $\lambda_{mm}=0$ in Eq. (A7), we therefore reach the conclusion stated in Sec. III that the response function does not involve the coefficient B'. Furthermore, dropping second-order terms in $k^2-k_m^2$,

$$k^2-k_m^2\simeq k_m^2\left[-i\beta_1+\beta_2+\frac{2(\omega-\omega_{mo})}{\omega}\right],$$

we obtain

$$c_m=a_m\left[1-i\frac{Q_m}{\omega_m}(\omega-\omega_m)\right]^{-1}, \tag{A8}$$

where $a_m=z_\infty q_m/\beta k_m$, $Q_m=2/\beta_1$, and $\omega_m=\omega_{mo}(1-\beta_2)$. Equations (12) and (13) follow immediately.

A better approximation would require, to be consistent, that the terms of second order in Ω/ω should not be neglected in the basic equations. This would lead to rather cumbersome expressions for k^2 or β.[10] However, the perfect circular shape of the observed resonance curve in the complex plane, in accordance with Eq. (A8), as well as the absence of quadratic effects in our experiment B vs Ω bear out the adequacy of the above approximation.

In the series expansion of T_1,

$$T_1=c_m\varphi_m(\vec{r})+\sum_{i\neq m}c_i\varphi_i(\vec{r})\approx c_m\varphi_m(\vec{r})\ , \tag{A9}$$

the sum of the nonresonant modes constitutes a small constant background $b(\vec{r})$, which generally proves to be of no importance in the bolometric signal. As a practical example, consider the C cavity operating on its fundamental mode ω_{10} ($Q_1 \sim 10^3$). The transmitting film and the bolometer are applied to opposite vertical walls, $\theta=0$ and θ_0, respectively (Fig. 1). The first main terms in $b(\vec{r})$ correspond to the harmonics $n=2,3,\ldots$ ($\omega_{n0}=n\omega_{10}$). Then taking $\varphi_n(\vec{r})\propto\cos n\pi\theta/\theta_0$ we have $q_n=q_1$ and $\varphi_n=(-1)^{n+1}\varphi_1$ on the receiving film. Thus b reads as an alternating series of decreasing terms, the sum of which is not greater than the first term $c_2\varphi_2$:

$$\left|\frac{b}{T_1}\right|<\left|\frac{c_2}{c_1}\right|=\frac{\beta_1 k_1^2}{k_2^2-k_1^2}=\frac{\beta_1}{3}\lesssim 10^{-3}\ .$$

The presence of undesirable leaks (or superleaks), coupling the cavity to the external helium bath, may strongly affect the parameters of resonance. This is easily demonstrated by using the abovementioned formalism. Suppose a leak opens into the cavity at any point $\vec{r}'_0$. The local second-sound amplitude $T_1(\vec{r}'_0)$ acts as a small temperature difference across the leak, between the cavity and the helium bath, giving rise to two-fluid (or superfluid) motion through the leak at driving frequency. Denoting by $\underline{a}$ the leak admittance, a concentrated heat input $\underline{a}\ T_1(\vec{r}'_0)\ \delta(\vec{r}'-\vec{r}'_0)$ must be added in the boundary condition (A3) to heat sources $q(\vec{r}')$. Instead of Eq. (A7) we now obtain

$$c_m[k^2-k_m^2-\alpha\underline{a}\varphi_m^2(\vec{r}'_0)]=\alpha q_m\ . \tag{A10}$$

Letting $\underline{a}$ be complex we see that both the quality factor and the resonant frequency can be modified owing to coupling term. Therefore, a possible dependence of coupling on rotation would invalidate any analysis of experimental data by means of Eqs. (12) and (13). This explains acoustic anomalies occurring in untight cavities, such as poor multiplication of harmonics, or the paradoxical first increase of the quality factor, sometimes observed as Ω is raised from zero. In any second-sound resonator there must be provision for the dc heat input to escape at places where there is normally a temperature node $[\varphi_m(\vec{r}'_0)]=0$.

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