PHYSICAL REVIEW B VOLUME 26, NUMBER 3 1 AUGUST 1982

Onset of convection in dilute superfluid ^3He-^4He mixtures. II. Closed cylindrical container

Alexander L. Fetter
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California 94305
(Received 26 March 1982)

The thermohydrodynamics of dilute superfluid ^3He-^4He mixtures is applied to a circular container with insulating sidewalls and rigid isothermal ends. For small downward heat flow, the boundary conditions yield a uniform temperature gradient and superflow but a *nonuniform* normal flow. Linearized perturbations about this conducting state become unstable at a critical Rayleigh number that (at ~ 1 K) differs only slightly from that for a classical one-component fluid in the same container. Despite the preferred direction of the unperturbed normal flow, infinitesimal-amplitude convection can occur in either of two degenerate modes. The splitting of these modes observed by Warkentin, Haucke, Lucas, and Wheatley probably involves nonlinear corrections.

I. INTRODUCTION

Recent experimental studies[1] of a dilute superfluid ^3He-^4He mixture near 1 K have demonstrated its similarity to a classical one-component fluid. For example, the onset of convection in a right circular cylinder with insulating sidewalls and unit aspect ratio (radius and height are each equal to 2.07 cm) occurred at a critical Rayleigh number ~ 1700, close to that for a pure substance in an infinite slab.[2] On the other hand, classical hydrodynamics also predict that the critical Rayleigh number increases with decreasing aspect ratio,[3] rising to ~ 2262 for the geometry used in the experiments. A second unusual feature was the observation of two distinct convective axisymmetric toroidal flows, differing only in the sense of motion. The center-falling pattern was more stable than that with center rising, whereas linearized perturbation theory for a classical one-component fluid predicts full equivalence between the two states.

To analyze these observations, it is important to use the two-fluid hydrodynamics, generalized to include the ^3He impurities.[4] The resulting nonlinear dynamical equations and appropriate boundary conditions have been obtained in a previous paper that studied the onset of convection in an unbounded horizontal slab.[5] Here, the effect of sidewalls is included, with particular attention to a cylindrical configuration. One striking new feature (Sec. II) is that the normal fluid flow is intrinsically nonuniform, even in the preconvective heat-conducting state. As shown in Sec. III, the linearized perturbation equations contain additional two-fluid contributions arising from the unperturbed normal-fluid flow and from the nonzero $\vec{\nabla}\cdot\vec{v}_n'$. Numerical estimates indicate that the critical Rayleigh number for the onset of convection of the dilute mixture in a cylinder near 1 K should be close to that for a pure classical fluid.

II. UNPERTURBED CONDUCTING STATE

The fundamental hydrodynamic variables in superfluid mixtures are the normal and superfluid densities ρ_n and ρ_s and velocities $\vec{v}_n$ and $\vec{v}_s$. If $\vec{v}_n = \vec{v}_s$, the state of a small element of fluid can be characterized by the center-of-mass velocity and a set of internal variables, which can conveniently be taken as the temperature T, the concentration c of ^3He impurities, and the pressure p, all defined in the center-of-mass frame. More generally, the relative velocity $\vec{v}_n - \vec{v}_s$ serves as an additional internal variable, and it is often simpler to treat the two velocity fields separately. Since $\vec{v}_s$ is irrotational, it can be specified by a scalar velocity potential

$$\vec{v}_s = -\vec{\nabla}\Phi \ . \tag{1a}$$

Current conservation then requires

$$\rho_s \nabla^2\Phi = \rho_n \vec{\nabla}\cdot\vec{v}_n \ . \tag{1b}$$

Moreover, for slow motions, the dynamical equation of the superfluid can be rewritten in the approximate form

$$\vec{\nabla} c = -\frac{\gamma c}{T}\vec{\nabla} T + \frac{\zeta_1 - \rho\zeta_3}{(\partial\mu_4/\partial c)_{Tp}}\vec{\nabla}(\vec{\nabla}\cdot\vec{v}_n)\ , \tag{2}$$

where ζ_1 and ζ_3 are second viscosities, μ_4 is the chemical potential of the ^{4}He atoms, and

$$\gamma = -\left[\frac{\partial \ln c}{\partial \ln T}\right]_{p\mu_4} = \frac{T(\partial\mu_4/\partial T)_{cp}}{c(\partial\mu_4/\partial c)_{Tp}} \tag{3}$$

is a dimensionless parameter of order 1 for dilute mixtures near 1 K. In this way, $\vec{v}_s$ can be related to $\vec{\nabla}\cdot\vec{v}_n$, and c can be eliminated from the two-fluid equations to give the following approximate dynamical equations:

$$\nabla^2\vec{v}_n - \frac{1}{\mathscr{P}}\frac{\partial\vec{v}_n}{\partial t} - (\vec{v}_n\cdot\vec{\nabla})\vec{v}_n - \vec{\nabla}P - \hat{z}T - \left[\vec{v}_n - \vec{v}_s - \frac{\rho}{\rho_n}\frac{g\beta_c d}{(\partial\mu_4/\partial c)_{Tp}}\frac{\zeta_1-\rho\zeta_3}{\nu_n}\hat{z}\right]\vec{\nabla}\cdot\vec{v}_n = 0\ , \tag{4}$$

$$\frac{\partial T}{\partial t} + \mathscr{P}\vec{v}_n\cdot\vec{\nabla}T - \nabla^2 T = 0\ , \tag{5}$$

$$\vec{\nabla}\cdot\vec{v}_n - \frac{\gamma}{T}\left[\frac{1}{\mathscr{P}}\frac{\partial T}{\partial t} + \vec{v}_n\cdot\vec{\nabla}T\right] + \frac{D}{\nu_n T}\left[\gamma - \frac{k_T}{c}\right]\nabla^2 T = 0\ , \tag{6}$$

here written in dimensionless variables. To define the appropriate units, we introduce the kinematic viscosity $\nu_n = \eta/\rho_n$ for the normal fluid, the thermal expansion coefficient $\alpha_{p\mu_4}$ at constant μ_4, the coefficient $\beta_c \equiv -\rho^{-1}(\partial\rho/\partial c)_{Tp}$, the effective thermal conductivity κ_{eff}, the "specific heat" $C_{p\mu_4}$ at constant μ_4, and the effective thermal diffusivity $\chi_{\text{eff}} = \kappa_{\text{eff}}/\rho C_{p\mu_4}$. In Eqs. (4)–(6), the units are the cell height d for length, d^2/χ_{eff} for time, ν_n/d for velocity, $\rho_n(\nu_n/d)^2$ for pressure, and $(\rho_n/\rho)(\nu_n^2/|\alpha_{p\mu_4}|gd^3)$ for the temperature. Furthermore, $\mathscr{P} = \nu_n/\chi_{\text{eff}}$ is the Prandtl number, k_T is the thermal diffusion ratio, D is the diffusion coefficient, and P contains several superfluid contributions as well as the actual pressure and the gravitational term.

At a rigid surface with normal $\hat{n}$, the mass current must have zero perpendicular component

$$\hat{n}\cdot(\rho_n\vec{v}_n + \rho_s\vec{v}_s) = 0\ , \tag{7}$$

and a similar condition on the impurity current may be rewritten as

$$\hat{n}\cdot\vec{v}_n + \frac{D}{\nu_n T}\left[\gamma - \frac{k_T}{c}\right]\hat{n}\cdot\vec{\nabla}T = 0\ . \tag{8}$$

For slow motions, the normal fluid sticks to the walls

$$\hat{n}\times\vec{v}_n = 0\ , \tag{9}$$

and the perpendicular component of heat flux $\vec{Q}$ through the surface is given by

$$\hat{n}\cdot\vec{Q} = -\kappa_{\text{eff}}\,\hat{n}\cdot\vec{\nabla}T\ , \tag{10}$$

expressed here in conventional units.

These equations will be applied to a cylinder of radius R and height d, with aspect ratio $\Gamma \equiv R/d$. It is convenient to work in cylindrical polar coordinates (r,θ,z) with $0 \le r \le \Gamma$ and $|z| \le \frac{1}{2}$ in dimensionless units. For definiteness, the bottom and top are assumed to be maintained at temperature T^0 and $T^0 + \Delta T$, respectively, whereas the sides are insulating. Thus $\hat{n}\cdot\vec{\nabla}T$ vanishes for $r = \Gamma$, and Eqs. (8) and (9) then show that $\vec{v}_n = 0$ on the sides. In contrast, v_{nz} (and hence v_{sz}) are nonzero at the top and bottom.

Before the onset of convection, the fluid is in a steady conducting state, with normal and superfluid counterflow induced by the temperature difference ΔT. Since $\Delta T/T^0$ is generally small, it is permissible to omit terms of second order in ΔT in determining the steady flow. In this approximation, Eq. (5) implies that the temperature satisfies Laplace's equation subject to the boundary condition $\partial T/\partial r = 0$ at $r = \Gamma$ and distinct constant values on the top and bottom. Evidently, the unique solution is a linear temperature profile with constant gradient

$$\vec{\nabla}T = \hat{z}\Delta T \tag{11}$$

in dimensionless units. Since $\vec{\nabla}\cdot\vec{v}_n$ will vanish, Eq. (2) gives the corresponding steady concentration gradient, with the negative sign implying the accumulation of ^{3}He in the cooler regions.

The determination of the steady normal velocity field is not as easy because Eq. (8) implies the unusual boundary conditions

$$\left.\begin{aligned} v_{nz} &= -\epsilon_1 \\ v_{nr} &= v_{n\theta} = 0 \end{aligned}\right\} \text{at } z = \pm\tfrac{1}{2} , \qquad \begin{aligned}(12\text{a})\\(12\text{b})\end{aligned}$$

$$\vec{v}_n = 0 \quad \text{at } r = \Gamma , \tag{12c}$$

where

$$\epsilon_1 \equiv (D/\nu_n)(\gamma - k_T/c)(\Delta T/T^0) \lesssim 10^{-2} \tag{13}$$

is a small dimensionless constant. For small ΔT, the quadratic terms in Eqs. (4) and (6) may be omitted, implying

$$\vec{\nabla}\cdot\vec{v}_n = 0 \tag{14}$$

in this approximation. Conservation of mass then shows that $\vec{\nabla}\cdot\vec{v}_s$ also vanishes, so that the corresponding scalar potential satisfies Laplace's equation, subject to the boundary conditions $\partial\Phi/\partial z = -\rho_n\epsilon_1/\rho_s$ at $z=\pm\frac{1}{2}$ and $\partial\Phi/\partial r = 0$ at $r=\Gamma$. The resulting $\vec{v}_s$ must be uniform, in the direction of $\vec{\nabla}T$, with magnitude $\rho_n\epsilon_1/\rho_s$. In contrast, $\vec{v}_n$ is *nonuniform* with finite vorticity.

Equations (12) and (14) suggest introducing a vector potential $\vec{X}$ by the relation

$$\vec{v}_n = -\epsilon_1 \vec{\nabla}\times\vec{X} , \tag{15}$$

and it follows from the curl of Eq. (4) that $\vec{X}$ satisfies the fourth-order equation

$$(\vec{\nabla}\times)^4\vec{X} = 0 , \tag{16}$$

again neglecting terms quadratic in $\Delta T/T^0$. Since the conducting state in a cylinder will be axisymmetric, $\vec{X}$ has the form

$$\vec{X} = \hat{\theta} X(r,z) , \tag{17}$$

and Eq. (16) becomes a single scalar equation

$$\left[\mathscr{D}_r + \frac{\partial^2}{\partial z^2} \right]^2 X(r,z) = 0 , \tag{18}$$

where

$$\mathscr{D}_r = \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} . \tag{19}$$

The corresponding normal velocity field is given by

$$v_{nr} = \epsilon_1 \frac{\partial X}{\partial z} , \tag{20a}$$

$$v_{nz} = -\frac{\epsilon_1}{r}\frac{\partial}{\partial r} rX , \tag{20b}$$

and X therefore obeys the simple boundary conditions

$$X = \tfrac{1}{2} r, \quad \frac{\partial X}{\partial z} = 0 \quad \text{at } z = \pm\tfrac{1}{2} , \tag{21a}$$

$$X = \frac{1}{2}\Gamma, \quad \frac{1}{r}\frac{\partial}{\partial r} rX = 0 \quad \text{at } r = \Gamma . \tag{21b}$$

Equations (18) and (21) constitute an unusual problem in incompressible hydrodynamics, because the condition that v_{nz} be uniform on the top and bottom surfaces precludes both the static solution $\vec{v}_n = 0$ and the familiar parabolic Poiseuille flow. Indeed, the stream lines must meet the top and bottom parallel to z but with no flow along the sidewalls. Thus the streamlines concentrate near the axis of the cylinder. Furthermore, the corners ar $r=\Gamma$ and $z=\pm\frac{1}{2}$ involve nonanalytic behavior similar to, but more complicated than, the temperature near a corner between two surfaces at different specified temperatures. Here, the occurrence of a fourth-order equation with boundary conditions on both X and its derivatives is reminiscent of a thin elastic plate, which cannot be solved in closed form.[6] For that reason, we have resorted to an approximate expansion procedure.

If there were no sidewalls, then the exact solution would be $\frac{1}{2}r$, appropriate for the uniform normal flow in a slab. To incorporate this behavior explicitly, it is convenient to write

$$X = \tfrac{1}{2} r + \bar{X} , \tag{22}$$

where $\bar{X}$ obeys the same equation (18) but with the boundary conditions

$$\begin{aligned} &\bar{X} = \frac{\partial \bar{X}}{\partial z} = 0 \quad \text{at } z = \pm\tfrac{1}{2} , \\ &\bar{X} = 0, \quad r^{-1}\frac{\partial(r\bar{X})}{\partial r} = -1 \quad \text{at } r = \Gamma . \end{aligned} \tag{23}$$

This set of boundary conditions makes explicit that $\bar{X}$ arises solely from the sidewalls.

To proceed, it is convenient to introduce two separate sets of eigenfunctions: $\cos(2m-1)\pi z$ and $J_1(j_m x)$, where $x \equiv r/\Gamma$ and j_m is the mth zero of J_1. These functions vanish at $z=\pm\frac{1}{2}$ and $r=\Gamma$, respectively. If $\kappa_m = j_m/2\Gamma$, then the product $J_1(j_m x) Z_m(z)$ can satisfy Eq. (18) with $Z_m(z)$ a linear combination of $\cosh(2\kappa_m z)$ and $z\sinh(2\kappa_m z)$ that obeys the boundary conditions $Z_m = 0$ and $dZ_m/dz = \pm 1$ at $z=\pm\frac{1}{2}$. Similarly, if $\nu_m = (2m-1)\pi\Gamma$, the product $F_m(x)\cos(2m-1)\pi z$ can satisfy Eq. (18) with $F_m(x)$ a linear combination of $I_1(\nu_m x)$ and $x I_0(\nu_m x)$ that obeys the boundary conditions $F_m = 0$ and $x^{-1} d(xF_m)/dx = 1$ at $x = 1$. As a result, the unknown function $\bar{X}(r,z)$ may be expanded in these sets of functions[6]

$$\bar{X}(r,z)=\sum_{m=1}^{\infty}[a_m J_1(j_m x)Z_m(z)+b_m F_m(x)\cos(2m-1)\pi z] \ . \tag{24}$$

By construction, $\bar{X}$ obeys the correct fourth-order differential equation (18) and vanishes at the boundaries. The remaining conditions (23) on the derivatives of $\bar{X}$ at the boundaries lead to the following pair of relations:

$$\sum_{m=1}^{\infty}[a_m J_1(j_m x)+b_m(-1)^m(2m-1)\pi F_m(x)]=0 \ , \tag{25a}$$

$$\sum_{m=1}^{\infty}[a_m j_m J_0(j_m)Z_m(z)+b_m\cos(2m-1)\pi z]=-\Gamma \ . \tag{25b}$$

Since $J_1(j_m x)$ are orthogonal and complete on the interval [0,1], Eq. (25a) readily yields a set of homogeneous linear equations for the unknown coefficients a_m and b_m, and all the integrals can be evaluated analytically. Similarly, the orthogonality of the trigonometric functions provides a second set of linear algebraic equations for a_m and b_m. These coupled equations have been solved numerically for various truncated sets containing up to the first 20 terms in Eq. (24).

The corresponding flow pattern of the normal fluid is easily determined. Owing to the axial symmetry, the motion is confined to a given vertical plane at fixed azimuthal angle θ; the corresponding streamlines satisfy the differential equation

$$\frac{dr}{v_{nr}}=\frac{dz}{v_{nz}} \ , \tag{26}$$

and use of Eq. (20) immediately yields the condition

$$rX(r,z)=\text{const} \ . \tag{27}$$

Figure 1 illustrates the streamlines for the configuration ($\Gamma=1$) used in the experiments in Ref. 1. As anticipated, the flow is nonuniform, with the streamlines assuming an "hourglass" form. In particular, v_{nz} is even and v_{nr} is odd under reflection about the midplane. As noted previously, $\vec{v}_s$ is uniform, leaving a net local mass flow $\vec{j}$ throughout the container, even though $\vec{\nabla}\cdot\vec{j}=0$ by construction.

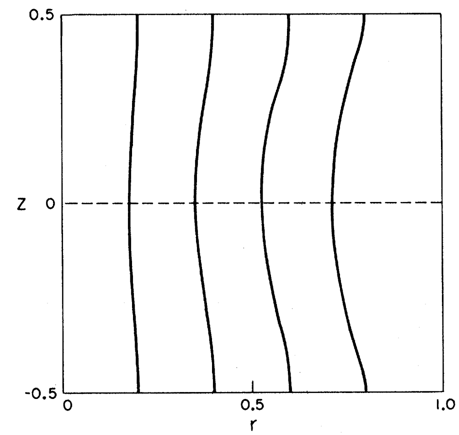

FIG. 1. Streamlines for unperturbed normal fluid flow (in heat-conducting state) in a cylindrical cell with unit aspect ratio.

III. LINEARIZED EQUATIONS FOR ONSET OF CONVECTION

The preceding section determined the steady conducting state, with the dimensionless normal fluid velocity $-\vec{\epsilon}_1\equiv-\epsilon_1\vec{\nabla}\times(X\hat{\theta})$ and temperature $T^0+\Delta T(z+\frac{1}{2})$. We now expand the dynamical equations (4)–(6) to first order about this steady state, writing

$$\vec{v}_n=-\vec{\epsilon}_1+\vec{v}_n' \ ,$$
$$T=T^0+\Delta T(z+\tfrac{1}{2})+T' \ ,$$
$$P=P^0+P' \ ,$$

where the explicit form of the original "pressure" P^0 is not needed in the present work. The resulting linear equations for the primed variables become

$$\nabla^2\vec{v}_n'-\frac{1}{\mathscr{P}}\frac{\partial\vec{v}_n'}{\partial t}+(\vec{\epsilon}_1\cdot\vec{\nabla})\vec{v}_n'+(\vec{v}_n'\cdot\vec{\nabla})\vec{\epsilon}_1-\vec{\nabla}P'-\hat{z}T'+\left[\vec{\epsilon}_1+\left[\frac{\rho_n}{\rho_s}\epsilon_1-\epsilon_3\right]\hat{z}\right]\vec{\nabla}\cdot\vec{v}_n'=0 \ , \tag{28}$$

$$\nabla^2T'-\frac{\partial T'}{\partial t}+\mathscr{P}\vec{\epsilon}_1\cdot\vec{\nabla}T'-Rv_{nz}'=0 \ , \tag{29}$$

$$\vec{\nabla}\cdot\vec{v}_n'-\epsilon_2 v_{nz}'=0 \ . \tag{30}$$

Here,

$$\epsilon_2=\gamma\Delta T/T^0\ , \tag{31a}$$

$$\epsilon_3=\frac{\rho}{\rho_n}\frac{gd\beta_c}{|(\partial\mu_4/\partial c)_{Tp}|}\frac{\zeta_1-\rho\zeta_3}{\nu_n}\ , \tag{31b}$$

are small parameters of order $10^{-2}-10^{-3}$, comparable with ϵ_1, and

$$R=|\alpha_{p\mu_4}|d^3g\Delta T/\nu\chi_{\text{eff}} \tag{32}$$

is the Rayleigh number, now expressed in conventional dimensional quantities. Since Eq. (28) indicates that $\vec{v}_n'$ and T' have similar dimensionless magnitudes, Eq. (30) has been simplified by omitting several terms of relative order T^{-1} or smaller. In addition, the Prandtl number is no larger than 0.1. Thus the third term in Eq. (29) is smaller than the other normal fluid contributions and will be omitted for simplicity.

This set of equations does not constitute a self-adjoint system owing to the explicit two-fluid contributions (proportional to $\Delta T/T$). Consequently, the time dependence is not necessarily described by real exponentials of the form $\exp(\sigma t)$, which would be the case for a one-component fluid ($\epsilon_i=0$). Since these parameters are small, however, is it reasonable to expect the onset of convection to occur through a steady instability, when σ first vanishes for some solution of these linear equations. Thus we assume that the critical Rayleigh number is determined by the eigenvalue problem

$$\nabla^2\vec{v}_n'-\hat{z}T'-\vec{\nabla}P'+(\vec{\epsilon}_1\cdot\vec{\nabla})\vec{v}_n'+(\vec{v}_n'\cdot\vec{\nabla})\vec{\epsilon}_1+\left[\vec{\epsilon}_1+\left[\frac{\rho_n}{\rho_s}\epsilon_1-\epsilon_3\right]\hat{z}\right]\epsilon_2 v_{nz}'=0\ , \tag{33}$$

$$-v_{nz}'+R^{-1}\nabla^2T'=0\ , \tag{34}$$

$$\vec{\nabla}\cdot\vec{v}_n'-\epsilon_2 v_{nz}'=0\ , \tag{35}$$

subject to the boundary conditions

$$\vec{v}_n'=T'=0 \text{ on } z=\pm\tfrac{1}{2}\ , \tag{36a}$$

$$\vec{v}_n'=\frac{\partial T'}{\partial r}=0 \text{ on } r=\Gamma \tag{36b}$$

on the walls of the container. If ϵ_i all vanished, the solution would have v_{nz}' and T' even under inversion about the midplane, but the two-fluid corrections introduce a small odd component. In this way, the true solutions do not have a definite parity. Nevertheless, it is evident that any solution $(\vec{v}_n,T',P')$ will be accompanied by another one $(-\vec{v}_n,-T',-P')$ that also satisfies the same equations. Thus the theory predicts that linearized small-amplitude convective flow patterns occur in pairs (differing in the direction of $\vec{v}_n'$) with the same Rayleigh number, even though the unperturbed normal flow $-\vec{\epsilon}_1$ provides a preferred sense. In particular, the observed preference in a cylinder with unit aspect ratio for a toroidal convection pattern with center falling[1] cannot be explained within the framework of a linearized description.

The critical Rayleigh number may be determined as a perturbation series in the small parameters ϵ_i. In zero order, the problem is that of a classical one-component fluid in a cylinder with insulating sidewalls. Theoretical investigations[3] have shown that the resulting flow is axisymmetric for $\Gamma\geq 0.8$, and we shall therefore assume that $\vec{v}_n'$ retains that character even when the two-fluid corrections are included. With the four-component vector $U=(v_{nr}',v_{nz}',T',P')$, Eqs. (33)–(35) can be written compactly as

$$LU=0\ , \tag{37}$$

and the perturbation expansion then follows by writing $R=R^{(0)}+R^{(1)}+R^{(2)}+\cdots$, $L=L^{(0)}+L^{(1)}+L^{(2)}+\cdots$, and $U=U^{(0)}+U^{(1)}+U^{(2)}+\cdots$. It is easily verified that $L^{(0)}$ is self-adjoint. Furthermore, the inner product $I\equiv(U^{(0)},L^{(0)}U^{(0)})$ provides a variational basis for the zero-order problem.

The actual determination of $U^{(0)}$ proceeds by noticing that the corresponding velocity field is solenoidal and can be represented with a vector potential

$$\vec{v}_n^{(0)}=\vec{\nabla}\times\hat{\theta}\psi^{(0)}(r,z)\ . \tag{38}$$

The resulting velocity components are

$$v_{nr}^{(0)}=-\frac{\partial\psi^{(0)}}{\partial z}\ , \tag{39a}$$

$$v_{nz}^{(0)}=\frac{1}{r}\frac{\partial}{\partial r}r\psi^{(0)} . \tag{39b}$$

Given $\psi^{(0)}$, the temperature $T^{(0)}$ can be found with a Green's function $G(\vec{r},\vec{r}\,')$ that satisfies the differential equation

$$\nabla^2 G(\vec{r},\vec{r}\,')=-\delta(\vec{r}-\vec{r}\,') \tag{40}$$

and the boundary conditions

$$G=0 \quad \text{for } z=\pm\tfrac{1}{2} ,$$
$$\frac{\partial G}{\partial r}=0 \quad \text{for } r=\Gamma . \tag{41}$$

A straightforward calculation yields the explicit solution

$$G(\vec{r},\vec{r}\,')=(2\pi)^{-1}\sum_{l=-\infty}^{\infty}\sum_{p=1}^{\infty}\exp[il(\theta-\theta')]w_p(z)w_p(z')G_{lp}(r,r') , \tag{42}$$

where

$$w_p(z)=\begin{cases}\sqrt{2}\cos p\pi z, & p \text{ odd}\\ \sqrt{2}\sin p\pi z, & p \text{ even}\end{cases} \tag{43}$$

and

$$G_{lp}(r,r')=I_l(p\pi r_<)\left[K_l(p\pi r_>)-\frac{I_l(p\pi r_>)K_l'(p\pi\Gamma)}{I_l'(p\pi\Gamma)}\right] . \tag{44}$$

As a result,

$$T^{(0)}(r,z)=-R^{(0)}\int d^3r' G(\vec{r},\vec{r}\,')v_{nz}^{(0)}(r',z') . \tag{45}$$

A little manipulation then shows that the variational functional I can be rewritten wholly in terms of $\psi^{(0)}$ as

$$I=-\int d^3r\,\psi^{(0)}\left[\mathscr{D}_r+\frac{\partial^2}{\partial z^2}\right]^2\psi^{(0)}+R^{(0)}\int d^3r\,d^3r' v_{nz}^{(0)}(\vec{r})G(\vec{r},\vec{r}\,')v_{nz}^{(0)}(\vec{r}\,') , \tag{46}$$

where $\mathscr{D}_r$ is defined in Eq. (19) and $v_{nz}^{(0)}$ is given in Eq. (39b).

It is evident that $\psi^{(0)}$ satisfies the boundary conditions

$$\psi^{(0)}=\frac{\partial\psi^{(0)}}{\partial z}=0 \quad \text{on } z=\pm\tfrac{1}{2} ,$$
$$\psi^{(0)}=\frac{1}{r}\frac{\partial}{\partial r}r\psi^{(0)}=0 \quad \text{on } r=\Gamma . \tag{47}$$

These constraints are readily incorporated by introducing the eigenfunctions $C_m(z)$ and $f_m(x)$ that obey the equations (as before, $x\equiv r/\Gamma$)

$$\frac{d^4C_m}{dz^4}=\lambda_m^4 C_m ,$$
$$\mathscr{D}_x^2 f_m=\alpha_m^4 f_m , \tag{48}$$

and the boundary conditions from Eq. (47). Chandrasekhar[7] tabulates the eigenvalues λ_m, and the radial ones α_m have been found numerically ($\alpha\approx 4.611$, $\alpha_2\approx 7.799$, etc.). We therefore write $\psi^{(0)}$ as a series

$$\psi^{(0)}(r,z)=\sum_{mn}A_{mn}f_m(x)C_n(z) . \tag{49}$$

Substitution in Eq. (46) gives a quadratic form in the parameters A_{mn}, and minimization of I produces a set of linear homogeneous algebraic equations with coefficients that can be evaluated analytically. The critical Rayleigh number $R^{(0)}$ follows from the determinant of coefficients. Figure 2 shows the resulting dependence on the aspect ratio obtained in the simple approximation or retaining only the two terms $A_{m,1}$ and $A_{m+1,1}$ where m is of order Γ. This curve is qualitatively similar to that in Ref. 3, obtained with more elaborate trial functions. The basic new feature is the use of the Green's function in solving for $T^{(0)}$ instead of introducing a separate expansion with a second set of parameters. For $\Gamma=1$, our approximation gave $R^{(0)}=2285$ and $A_{21}/A_{11}=7.66\times 10^{-3}$, which exceeds the value 2262 found in Ref. 3 by only 1%. In addition, an improved choice that retained A_{11} and A_{12} gave $R^{(0)}=2265$ (with $A_{12}/A_{11}=2.63\times 10^{-2}$). The accuracy of this latter result illustrates the power of the variational techniques.

The remaining steps in the perturbation expansion are now easily performed. Given the vector potential $\psi^{(0)}$, the velocity field $\vec{v}_n^{(0)}$ and the tem-

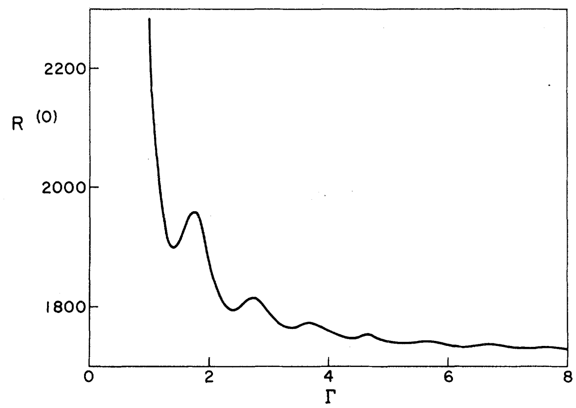

FIG. 2. Unperturbed critical Rayleigh number $R^{(0)}$ for onset of convection in cylindrical cell with aspect ratio $\Gamma = R/d$.

perature $T^{(0)}$ are fully determined and the pressure $P^{(0)}$ follows from the radial component of Eq. (33). Thus the components of $U^{(0)}$ are known explicitly. The first-order terms in the basic equation (37) give the condition

$$L^{(0)}U^{(1)} = -L^{(1)}U^{(0)} , \tag{50}$$

and the requirement that the right-hand side be orthogonal to $U^{(0)}$ implies that there is no first-order shift in the critical Rayleigh number,

$$R^{(1)} = 0 . \tag{51}$$

Similarly, the second-order contribution of Eq. (37) becomes

$$L^{(0)}U^{(2)} = -L^{(1)}U^{(1)} - L^{(2)}U^{(0)} , \tag{52}$$

and the same orthogonality condition now provides an explicit equation for $R^{(2)}$ in terms of inner products of $U^{(0)}$ and $U^{(1)}$.

It is first necessary to evaluate $U^{(1)}$ by solving Eq. (50). Since the corresponding normal fluid flow is no longer solenoidal, it cannot be represented with a vector potential. Nevertheless, it is possible to assume that $v_{nz}^{(1)}$ has a form similar to Eq. (39b)

$$v_{nz}^{(1)} = \frac{1}{r}\frac{\partial}{\partial r} r\psi^{(1)}(r,z) , \tag{53}$$

where $\psi^{(1)}$ satisfies the same boundary conditions as in Eq. (47). The corresponding $v_{nr}^{(1)}$ follows from Eq. (35) as

$$v_{nr}^{(1)} = -\frac{\partial}{\partial z}\psi^{(1)} + \epsilon_2 \psi^{(0)} , \tag{54}$$

with the last term representing the small nonzero divergence. Finally, the curl of Eq. (33) and use of the Green's function from Eq. (40) to "solve" Eq. (34) yield a single inhomogeneous equation for $\psi^{(1)}$ in terms of the zero-order solution $\psi^{(0)}$ and the original normal fluid flow that characterizes the conducting state [Eq. (15)] before the onset of convection. Since $\psi^{(1)}$ is odd under reflection about the midplane, it can be expanded in a series of the form[7]

$$\psi^{(1)}(r,z) = \sum_{mn} B_{mn} f_m(x) S_n(z) , \tag{55}$$

and the orthogonality relations then give a set of linear equations for the B_{mn}'s in terms of the (assumed known) A's from Eq. (49) and various matrix elements of the function X from Eq. (24). Since the shift $R^{(2)}$ in the critical Rayleigh number is expected to be small (and perhaps unobservable, based on the value[5] found for an unbounded slab), we considered only the approximation $X \approx \frac{1}{2}r$, which replaces the nonuniform normal flow of Fig. 1 by a uniform vertical flow. Although this simpler expression violates the boundary conditions, it should provide a qualitative guide for aspect ratios $\Gamma \gtrsim 0.8$. The resulting $R^{(2)}$ depends only weakly on the aspect ratio Γ and approaches the value found previously for an unbounded slab[5] as $\Gamma \to \infty$. The small values of the parameters ϵ_i for $T \approx 1$ K and molar concentration 0.0047 indicate that current experiments are unlikely to detect such shifts.

IV. DISCUSSION

The present work has direct relevance to recent experiments of Warkentin, Haucke, Lucas, and Wheatley[1] on superfluid ^{3}He-^{4}He mixtures. First, the accepted theory of two-fluid hydrodynamics[4] for dilute mixtures predicts that the critical Rayleigh number (32) for the onset of convection near 1 K should be close to that for a classical one-component fluid in a cell of the same aspect ratio. Numerical calculations by Charlson and Sani[3] and those shown in Fig. 2 indicate that R should be $\sim 2.3 \times 10^3$ for $\Gamma = 1$, falling nonmonotonically toward $\sim 1.7 \times 10^3$ for large Γ. In contrast, the observations[1] for $\Gamma = 1$ found $\sim 1.7 \times 10^3$ in the interval 0.8 K $\leq T \leq$ 1.0 K for molar concentration 0.0047. Experimental studies for cells with larger Γ might help resolve this discrepancy.

The second point of interest is the form of the convective flow. In the present approximation of azimuthal symmetry, the normal fluid velocity for $0.8 \lesssim \Gamma \lesssim 1.7$ is a single toroidal roll with the coefficient $\tilde{A}_{11}$ in Eq. (49) much larger than the others. Near the local maximum in Fig. 2 at $\Gamma \simeq 1.8$, however, the coefficient A_{21} becomes comparable, and the interval $1.8 \lesssim \Gamma \lesssim 2.7$ corresponds to convective flow with two toroidal rolls.[3,8] Thus for $\Gamma = 1$, the two-fluid hydrodynamics predicts a single toroidal roll, and this pattern appears to be seen experimentally.[1] On the other hand, the linearized perturbation theory also predicts that the critical Rayleigh number is identical for the two convective states that differ only in the sense of the small-amplitude flow, in contrast to the observed preference[1] for that with center falling. Nonlinear calculations for classical one-component fluids[9] have shown that finite-amplitude effects can stabilize one or the other of the two patterns. In the present case of two-fluid hydrodynamics for superfluid mixtures, the problem is considerably more complicated owing to the presence of parity-breaking normal fluid flow, even at threshold. This interesting problem deserves further study in connection with the experimental observations.

ACKNOWLEDGMENT

This work was supported in part by the NSF Grant No. DMR 78-25258.

---

[1]P. A. Warkentin, H. J. Haucke, and J. C. Wheatley, Phys. Rev. Lett. 45, 918 (1980); P. A. Warkentin, H. J. Haucke, P. Lucas, and J. C. Wheatley, Proc. Natl. Acad. Sci. USA 77, 6983 (1980).

[2]S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, Oxford, 1961), Chap. II.

[3]G. S. Charlson and R. L. Sani, Int. J. Heat Mass Transfer 13, 1479 (1970); 14, 2157 (1971).

[4]I. M. Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965), Sec. 24.

fluidity (Benjamin, New York, 1965), Sec. 24.

[5]A. L. Fetter, preceding paper, Phys. Rev. B 26, 1164 (1982).

[6]A. E. H. Love, *The Mathematical Theory of Elasticity* (Dover, New York, 1944), pp. 491–496.

[7]S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Ref. 2, App. V.

[8]See also, K. Stork and U. Muller, J. Fl. Mech 71, 231 (1975).

[9]G. S. Charlson and R. L. Sani, J. Fl. Mech. 71, 209 (1975).