

Onset of convection in dilute superfluid ^3He - ^4He mixtures. I. Unbounded slab

Alexander L. Fetter

*Institute of Theoretical Physics, Department of Physics, Stanford University,
Stanford, California 94305*

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The thermohydrodynamics of a dilute superfluid ^3He - ^4He mixture confined between parallel horizontal planes is studied with the two-fluid equations of Landau and Khalatnikov. When heated from above, downward normal fluid flow concentrates ^3He impurities near the cooler bottom plate, creating a potentially unstable density gradient. The linearized equations for the convective amplitudes contain nonclassical two-fluid terms proportional to $\vec{v}_n^0 \cdot \vec{\nabla}$ and to $\vec{\nabla} \cdot \vec{v}_n^1$, which alter the critical Rayleigh number from that for a classical one-component fluid. This shift is expected to be smaller near $T \approx 1$ K.

I. INTRODUCTION

Over the past decade, cryogenic studies of convection¹ have stimulated new interest in the Rayleigh-Bénard problem of a fluid heated from below. Although pure ^4He acts like a typical classical fluid above the transition temperature T_λ , the appearance of the superfluid component below T_λ means that the corresponding thermal conductivity is effectively infinite. Consequently, heat transport in bulk pure ^4He II has a qualitatively different character from that in normal ^4He I.² The situation changes considerably in dilute mixtures of ^3He in ^4He . This system also becomes superfluid but has a finite effective thermal conductivity.³ Thus even an unbounded slab of the dilute superfluid mixture can support a uniform temperature gradient. In analogy with classical fluids, this conducting state should be stable up to a critical temperature difference, beyond which a spatially periodic steady convective flow should appear. Recent experiments on cylindrical geometries^{4,5} have indeed found such behavior and confirmed the similarity to that of a classical one-component fluid.

The onset of convection in classical fluids has been studied in great detail.^{6,7} For a pure substance, the instability signals the appearance of steady convection, but classical mixtures can also exhibit oscillatory instabilities,⁸ leading to time-dependent flow. In all these classical cases, however, the initial (preconvective) heat-conducting state is stationary. In this regard, superfluids should act differently, owing to the presence of

normal fluid and superfluid counterflow. In particular, an externally applied temperature difference produces a chemical-potential gradient. For pure ^4He , this can be balanced only by a ("fountain") pressure; the resulting dynamical equilibrium has a finite normal flow determined by the viscous Poiseuille drag at the sidewalls. Thus the effective thermal conductivity depends on the geometry,² varying as R^2 for a cylindrical channel of radius R , and becoming infinite as $R \rightarrow \infty$. In a dilute mixture, however, the additional mechanism of a concentration gradient can balance the external temperature gradient, leading to a steady counterflow even for an infinite sample without lateral boundaries. Since the ^3He impurities are part of the normal fluid, they tend to accumulate in the colder regions, which become less dense than the remaining fluid. A gravitational instability is therefore expected to occur when the fluid is heated from *above*, in contrast to the usual classical case.

Superfluid hydrodynamics differ significantly from classical hydrodynamics, for example, in the appearance of two distinct components. Consequently it is interesting to ask whether the onset of convection in dilute superfluid mixtures should also exhibit specific two-fluid effects. The present work examines this question for the realistic case of a convection cell bounded above and below by rigid planes. Section II derives the general nonlinear hydrodynamic equations for convection in dilute superfluid mixtures. These equations are then linearized in Sec. III for the simplest case of an unbounded slab. The principal two-fluid effect

is the appearance of new first-order terms associated with the nonzero normal fluid velocity \vec{v}_n^0 in the initial conducting state⁹ and the nonzero divergence of the velocity perturbation \vec{v}'_n . These correction terms are small, however, and their effect on the onset of convection is treated in Sec. IV with an expansion procedure. A subsequent paper will present a similar analysis of a cylinder, which is the geometry used in recent detailed experiments.⁴

II. BASIC FORMULATION

The fundamental equations of two-fluid hydrodynamics in mixtures express the conservation of mass, momentum, impurities, and entropy, augmented by the dynamical equation of the superfluid.^{3,10-12} A completely general treatment is prohibitive, however, and it is convenient to rewrite the equations in a form that applies for slow hydrodynamic motions. Thus we shall follow the usual Boussinesq approximation in classical one-component fluids,^{6,7} ignoring changes in ρ_n and ρ_s except for their gravitational effects. Furthermore, the irrotational character of the superflow permits the introduction of a velocity potential Φ obeying the relation $\vec{v}_s = -\vec{\nabla}\Phi$. Conse-

quently, the equation of mass conservation ($\vec{\nabla} \cdot \vec{j} = 0$) can be rewritten immediately as

$$\vec{\nabla} \cdot \vec{v}_n = (\rho_s / \rho_n) \nabla^2 \Phi. \quad (1)$$

In contrast to the case for a classical one-component fluid, the normal flow is not, in general, solenoidal.

The equation for momentum conservation contains the gravitational force density, which may be expanded to first order in the temperature and concentration changes to give

$$\rho g \hat{z} [1 - \alpha_{pc}(T - T_0) - \beta_c(c - c_0)], \quad (2)$$

where T_0 and c_0 are fixed reference values of the temperature and concentration. Here

$$c = n_3 m_3 / (n_3 m_3 + n_4 m_4)^{-1} \quad (3)$$

is the mass fraction of ³He atoms, $\alpha_{pc} = -\rho^{-1}(\partial\rho/\partial T)_{pc}$ is the usual thermal expansion coefficient (denoted β_T in Ref. 4) and $\beta_c = -\rho^{-1}(\partial\rho/\partial c)_{Tp}$ is of order unity (~ 0.6 for the experiments in Ref. 4). Equation (2) omits an additional small term arising from the pressure change. Use of Eq. (1) allows us to cast the momentum-conservation relation in the form

$$\rho_n \left[\frac{\partial \vec{v}_n}{\partial t} + (\vec{v}_n \cdot \vec{\nabla}) \vec{v}_n \right] + \vec{\nabla} P + \rho_n (\vec{v}_n - \vec{v}_s) \vec{\nabla} \cdot \vec{v}_n = \eta \nabla^2 \vec{v}_n + \rho g \hat{z} [\alpha_{pc}(T - T_0) + \beta_c(c - c_0)], \quad (4)$$

where

$$P = p - \rho_s \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho_s v_s^2 - \left(\frac{1}{3} \eta + \zeta_2 - \rho \zeta_1 \right) \vec{\nabla} \cdot \vec{v}_n + \rho g z, \quad (5)$$

and p is the ordinary pressure.

It is helpful next to consider the equation for the concentration of ³He impurities; it may be written

$$\frac{\partial(\rho c)}{\partial t} + \vec{\nabla} \cdot (\rho c \vec{v}_n + \vec{i}) = 0, \quad (6)$$

where \vec{i} is the impurity flux (denoted \vec{g} by Khalatnikov³), and \vec{v}_n appears explicitly because the ³He atoms are part of the normal fluid. To first order in the gradients, \vec{i} may be expanded to yield

$$\vec{i} = -\rho D [\vec{\nabla} c + (k_T/T) \vec{\nabla} T], \quad (7)$$

with D the diffusion coefficient and k_T the thermal diffusion ratio. Once again, an unimportant hydrostatic pressure term has been omitted. The previous approximations simplify Eq. (6) consider-

ably to give

$$\frac{\partial c}{\partial t} + \vec{v}_n \cdot \vec{\nabla} c + c \vec{\nabla} \cdot \vec{v}_n - D \left[\nabla^2 c + \frac{k_T}{T} \nabla^2 T \right] = 0. \quad (8)$$

As in pure ⁴He II, the dynamical equation of superfluid motion involves the gradient of the chemical potential, here in the combination $\mu - Zc/\rho + \frac{1}{2} v_s^2$, where μ and Z are related by the thermodynamic identity

$$d\mu = -s dT + \rho^{-1} dp + \rho^{-1} Z dc - \rho^{-1} \vec{j}_0 \cdot d(\vec{v}_n - \vec{v}_s). \quad (9)$$

Here, we use Khalatnikov's notation,³ with μ and s the chemical potential and entropy per unit mass of solution, and $\vec{j}_0 = \rho_n (\vec{v}_n - \vec{v}_s)$. For small $|\vec{v}_n - \vec{v}_s|$, the last term can be included explicitly by writing

$$\mu = \mu_0(T, c, p) - (\rho_n/2\rho) (\vec{v}_n - \vec{v}_s)^2. \quad (10a)$$

The derivative with respect to c and some straightforward manipulations then give the desired quantity

$$\begin{aligned} \mu - \frac{Zc}{\rho} &= \mu_0 - c \frac{\partial \mu_0}{\partial c} - \frac{1}{2} \left(\frac{\rho_n}{\rho} - c \frac{\partial}{\partial c} \frac{\rho_n}{\rho} \right) (\vec{v}_n - \vec{v}_s)^2 \\ &= \mu_4(T, c, p) \\ &\quad - (2\rho)^{-1} (\rho_{n0} - c\rho_n \beta_c) (\vec{v}_n - \vec{v}_s)^2, \end{aligned} \quad (10b)$$

where, as shown in Ref. 3, $\mu_4 = \mu_0 - c \partial \mu_0 / \partial c$ is the chemical potential of the ${}^4\text{He}$ atoms in stationary fluid ($\vec{v}_n = \vec{v}_s = 0$), and $\rho_{n0} = \rho_n - m_3^* c$ is the normal fluid density for pure ${}^4\text{He}$ at the same temperature and pressure. In this way, the dynamical equation for the superfluid takes the form

$$\begin{aligned} \frac{\partial \vec{v}_s}{\partial t} + \vec{\nabla} \left[\mu_4(T, c, p) - \frac{1}{2} \left(\frac{\rho_{n0}}{\rho} - \frac{c\rho_n}{\rho} \beta_c \right) (\vec{v}_n - \vec{v}_s)^2 \right. \\ \left. + \frac{1}{2} v_s^2 + (\rho \zeta_3 - \zeta_1) \vec{\nabla} \cdot \vec{v}_n \right] = -g \hat{z}. \end{aligned} \quad (11)$$

In the steady unperturbed heat-conducting state before the onset of convection, the normal and superfluid velocities are small, and the chemical potential μ_4 therefore obeys the approximate relation^{10,11}

$$\vec{\nabla} \mu_4(T, c, p) = -g \hat{z}. \quad (12)$$

The hydrostatic condition $\vec{\nabla} p \approx -\rho g \hat{z}$ and a little algebra show that the associated unperturbed temperature T and concentration c obey the relation

$$\vec{\nabla} c = -\frac{\gamma c}{T} \vec{\nabla} T - \frac{c \beta_c g}{(\partial \mu_4 / \partial c)_{Tp}} \hat{z}, \quad (13)$$

where

$$\gamma \equiv - \left(\frac{\partial \ln c}{\partial \ln T} \right)_{p\mu_4} = \frac{T}{c} \frac{(\partial \mu_4 / \partial T)_{cp}}{(\partial \mu_4 / \partial c)_{Tp}} \quad (14)$$

is a dimensionless parameter characterizing the superfluid. In the dilute limit, μ_4 may be written approximately as^{3,4,12,13}

$$\begin{aligned} \mu_4 &\approx \mu_{40} + (k_B T / m_4) \ln(1-x) \\ &\approx \mu_{40} - k_B T x / m_4, \end{aligned} \quad (15a)$$

where μ_{40} is the chemical potential for pure ${}^4\text{He}$ II and $x = n_3(n_3 + n_4)^{-1}$ is the mole fraction of ${}^3\text{He}$. For dilute solutions, x and c are related by $c \approx m_3 x / m_4$, so that Eq. (15a) becomes

$$\mu_4 \approx \mu_{40} - k_B T c / m_3. \quad (15b)$$

It follows immediately that

$$\left(\frac{\partial \mu_4}{\partial c} \right)_{Tp} \approx \frac{-k_B T}{m_3} \quad (16a)$$

and

$$\gamma = 1 + s_{40} m_3 / k_B c, \quad (16b)$$

which is of order 1 for the experimental situation considered in Ref. 4. Equations (13) and (16a) then yield the relative concentration change across a cell of thickness d :

$$\frac{\Delta c}{c} = -\gamma \frac{\Delta T}{T} + \frac{\beta_c g m_3 d}{k_B T}. \quad (17)$$

Experimental values from Ref. 4 indicate that $\Delta T / T$ is of order $10^{-2} - 10^{-3}$ at onset of convection; in contrast, the buoyancy correction¹³ (the last term) is of order

$$\left| \frac{gd}{(\partial \mu_4 / \partial c)_{Tp}} \right| \approx \frac{g d m_3}{k_B T} \approx 3 \times 10^{-5}, \quad (18)$$

and hence negligible. As a result, the equilibrium concentration and temperature variations occur at constant μ_4 and pressure.¹⁰

More generally, Eq. (11) can be expanded to relate the deviations in c and T as

$$c - c_0 \approx -\frac{\gamma c}{T} (T - T_0) + \frac{\zeta_1 - \rho \zeta_3}{(\partial \mu_4 / \partial c)_{Tp}} \vec{\nabla} \cdot \vec{v}_n, \quad (19)$$

apart from terms that lead to smaller corrections of the same order as in Eq. (18). The last term in Eq. (19) (proportional to $\vec{\nabla} \cdot \vec{v}_n$) arises from the viscous forces on the superfluid and provides the leading correction to the condition $\mu_4 = \text{const.}$ ¹¹ In some cases, this term can be neglected. For example, a combination of Eqs. (1), (8), and (19) yields the approximate equation

$$\begin{aligned} \frac{\rho_s}{\rho_n} \nabla^2 \Phi = \vec{\nabla} \cdot \vec{v}_n \\ = \frac{\gamma}{T} \left[\frac{\partial T}{\partial t} + \vec{v}_n \cdot \vec{\nabla} T \right] - \frac{D}{T} \left[\gamma - \frac{k_T}{c} \right] \nabla^2 T, \end{aligned} \quad (20)$$

that relates the temperature directly to $\vec{\nabla} \cdot \vec{v}_n$ and $\nabla^2 \Phi$, apart from corrections of relative order

$$(\zeta_1 - \rho \zeta_3) D [d^2 (\partial \mu_4 / \partial c)_{Tp}]^{-1}.$$

Steinberg¹¹ estimates that the ratio $(\xi_1 - \rho\xi_3)/v_n$ (which he denotes m) depends significantly on temperature and concentration, being of order 1 for the experiments in Ref. 4. Here $v_n \equiv \eta/\rho_n$ is the "normal" kinematic viscosity, and $D \approx v_n \approx 10^{-2} \text{ cm}^2 \text{ s}^{-1}$ are comparable in magnitude.^{4,14}

Use of Eq. (16a) therefore shows that the corrections to Eq. (20) are of order 10^{-12} .

In other cases, however, the last term of Eq. (19) can be significant. A combination of Eq. (4) and (19) yields

$$v_n \nabla^2 \vec{v}_n - \frac{\partial \vec{v}_n}{\partial t} - (\vec{v}_n \cdot \vec{\nabla}) \vec{v}_n - \frac{1}{\rho_n} \vec{\nabla} P + \frac{\rho}{\rho_n} g \hat{z} \alpha_{p\mu_4} (T - T_0) - (\vec{\nabla} \cdot \vec{v}_n) \left[\vec{v}_n - \vec{v}_s - \frac{\rho g \beta_c}{\rho_n} \hat{z} \frac{\xi_1 - \rho \xi_3}{(\partial \mu_4 / \partial c)_{Tp}} \right] = 0, \quad (21)$$

where $\alpha_{p\mu_4} = -\rho^{-1}(\partial \rho / \partial T)_{p\mu_4}$ is the thermal expansion coefficient at constant μ_4 . Some elementary thermodynamics show that

$$\alpha_{p\mu_4} = \alpha_{pc} - \gamma c \beta_c / T, \quad (22)$$

thus relating it to measurable quantities. In the present case of dilute mixtures at low temperature, $\alpha_{p\mu_4}$ is invariably negative, reflecting the physical fact that heat flush tends to move the light ³He impurities toward cooler regions.

Before studying the equation for the conservation of entropy, it is helpful to rewrite Eqs. (20)

and (21) in dimensionless form, with distance in units of the thickness d , time in units of d^2/χ_{eff} , v_n in units of $v_n/d = \eta/\rho_n d$, and T in units of $(v_n^2 / |\alpha_{p\mu_4}| d^3 g)(\rho_n/\rho)$, where χ_{eff} is the effective thermal diffusivity (defined in detail below). Typical values⁴ are $\chi_{\text{eff}} \approx 0.2 \text{ cm}^2 \text{ s}^{-1}$, $\rho_n/\rho \approx 10^{-2}$, $v_n \approx 10^{-2} \text{ cm}^2 \text{ s}^{-1}$, and $|\alpha_{p\mu_4}| \approx 5 \times 10^{-3} \text{ K}^{-1}$.

Thus for $d \approx 1 \text{ cm}$, the characteristic units are 2 s for t , $10^{-2} \text{ cm}^2 \text{ s}^{-1}$ for v_n , and $2 \times 10^{-7} \text{ K}$ for T . Temporarily using an asterisk to denote a dimensionless dependent variable, we obtain the corresponding dimensionless form of Eqs. (20) and (21):

$$\vec{\nabla} \cdot \vec{v}_n^* - \frac{\gamma}{T^*} \left[\frac{1}{\mathcal{P}} \frac{\partial T^*}{\partial t} + \vec{v}_n^* \cdot \vec{\nabla} T^* \right] + \frac{D}{v_n T^*} \left[\gamma - \frac{k_T}{c} \right] \nabla^2 T^* = 0, \quad (23)$$

$$\nabla^2 \vec{v}_n^* - \frac{1}{\mathcal{P}} \frac{\partial \vec{v}_n^*}{\partial t} - (\vec{v}_n^* \cdot \vec{\nabla}) \vec{v}_n^* - \vec{\nabla} P^* + \hat{z} \text{sgn}(\alpha_{p\mu_4})(T^* - T_0^*) - \left[\vec{v}_n^* - \vec{v}_s^* - \frac{\rho}{\rho_n} \frac{g d \beta_c}{(\partial \mu_4 / \partial c)_{Tp}} \frac{\xi_1 - \rho \xi_3}{v_n} \hat{z} \right] \vec{\nabla} \cdot \vec{v}_n^* = 0, \quad (24)$$

where P^* is measured in units of $\rho_n (v_n/d)^2$ and^{4,10}

$$\mathcal{P} = v_n / \chi_{\text{eff}} \quad (25)$$

is the appropriate Prandtl number (~ 0.05 for the values in Ref. 4). Apart from the last term, which is a specific two-fluid contribution proportional to $\vec{\nabla} \cdot \vec{v}_n^*$, Eq. (24) is essentially the same as for a one-component fluid.

We now consider the entropy equation, which can be written as

$$\frac{\partial(\rho s)}{\partial t} + \vec{\nabla} \cdot \left[\frac{\vec{Q}}{T} \right] = 0. \quad (26)$$

Here s is the entropy density, \vec{Q} is the entropy flux vector, and we have omitted the entropy creation

terms from the right-hand side. As shown by Khalatnikov,³ \vec{Q} is given by

$$\vec{Q}/T = \rho s \vec{v}_n + (\vec{q} - \vec{i} Z/\rho) T^{-1}, \quad (27)$$

where

$$\vec{q} = -\kappa \vec{\nabla} T - T^2 \vec{i} \left[\frac{\partial}{\partial T} \left[\frac{Z}{\rho T} \right] - \frac{k_T}{T} \frac{\partial}{\partial c} \left[\frac{Z}{\rho T} \right] \right], \quad (28)$$

and κ is the diffusive thermal conductivity of the normal component, here associated with rotons, phonons, and the ³He impurities.¹⁵ In Eq. (28), the partial derivatives are taken keeping fixed the remaining variables of the set (T, c, p) , and the fluid is assumed to be stationary with $\vec{v}_n = \vec{v}_s = 0$.

Equations (10b) and (9) together give the relations

$$\frac{\partial}{\partial c} \left[\frac{Z}{\rho} \right] = \frac{\partial}{\partial c} \left[\frac{\mu - \mu_4}{c} \right] = -\frac{1}{c} \frac{\partial \mu_4}{\partial c}, \quad (29)$$

$$\frac{\partial}{\partial T} \left[\frac{Z}{\rho} \right] = \frac{\partial}{\partial T} \left[\frac{\mu - \mu_4}{c} \right] = -\frac{s}{c} - \frac{1}{c} \frac{\partial \mu_4}{\partial T} = -\frac{\partial s}{\partial c}, \quad (30)$$

where the last form arises from a Maxwell relation [see Eq. (9)]. A combination of Eqs. (27)–(30) readily yields

$$\frac{\vec{Q}}{T} = \frac{s}{c} (\rho c \vec{v}_n + \vec{i}) + \frac{\vec{i}}{T} \frac{\partial \mu_4}{\partial c} \left[\gamma - \frac{k_T}{c} \right] - \frac{\kappa \vec{\nabla} T}{T}. \quad (31)$$

In taking the divergence of \vec{Q}/T , we shall drop terms quadratic in the gradients, consistent with the omission of entropy-creation terms in Eq. (26). Use of Eq. (6) then gives

$$c \frac{\partial}{\partial t} \left[\frac{s}{c} \right] + c \vec{v}_n \cdot \vec{\nabla} \left[\frac{s}{c} \right] + \frac{\partial \mu_4}{\partial c} \left[\gamma - \frac{k_T}{c} \right] \frac{\vec{\nabla} \cdot \vec{i}}{\rho T} - \frac{\kappa \nabla^2 T}{\rho T} = 0. \quad (32)$$

The quantity s/c may be considered a function of T , c , and p , and the pressure dependence is again negligible. The relation (19) allows us to eliminate c explicitly to give

$$Tc \left[\frac{\partial}{\partial T} \left[\frac{s}{c} \right] \right]_{p\mu_4} \left[\frac{\partial T}{\partial t} + \vec{v}_n \cdot \vec{\nabla} T \right] - \frac{\kappa_{\text{eff}}}{\rho} \nabla^2 T + (\xi_1 - \rho \xi_3) \left[\gamma \left[\frac{\partial}{\partial t} + \vec{v}_n \cdot \vec{\nabla} \right] - D \left[\gamma - \frac{k_T}{c} \right] \nabla^2 \right] \vec{\nabla} \cdot \vec{v}_n = 0, \quad (33)$$

where

$$\kappa_{\text{eff}} = \kappa - \frac{\partial \mu_4}{\partial c} \frac{\rho D c}{T} \left[\gamma - \frac{k_T}{c} \right]^2. \quad (34)$$

Equation (16a) shows that $\partial \mu_4 / \partial c$ is intrinsically negative, so that κ_{eff} exceeds κ . The additional contribution to the thermal conductivity arises from the hydrodynamic transport of entropy by the normal fluid. It is natural to introduce a modified specific heat

$$C_{p\mu_4} = Tc \left[\frac{\partial}{\partial T} \left[\frac{s}{c} \right] \right]_{p\mu_4}, \quad (35)$$

and Eq. (33) then becomes

$$\frac{\partial T}{\partial t} + \vec{v}_n \cdot \vec{\nabla} T - \chi_{\text{eff}} \nabla^2 T + \frac{\xi_1 - \rho \xi_3}{C_{p\mu_4}} \left[\gamma \left[\frac{\partial}{\partial t} + \vec{v}_n \cdot \vec{\nabla} \right] - D \left[\gamma - \frac{k_T}{c} \right] \nabla^2 \right] \vec{\nabla} \cdot \vec{v}_n = 0, \quad (36)$$

where

$$\chi_{\text{eff}} = \kappa_{\text{eff}} / \rho C_{p\mu_4}$$

is the effective thermal diffusivity. In terms of the dimensionless variables introduced previously, it has the form

$$\left[\frac{\partial}{\partial t} + \mathcal{P} \vec{v}_n^* \cdot \vec{\nabla} - \nabla^2 \right] T^* + \frac{|\alpha_{p\mu_4}| dg}{C_{p\mu_4}} \frac{\rho}{\rho_n} \frac{\xi_1 - \rho \xi_3}{v_n} \left[\gamma \left[\frac{1}{\mathcal{P}} \frac{\partial}{\partial t} + \vec{v}_n^* \cdot \vec{\nabla} \right] - \frac{D}{v_n} \left[\gamma - \frac{k_T}{c} \right] \nabla^2 \right] \vec{\nabla} \cdot \vec{v}_n^* = 0. \quad (37)$$

A combination of the previously given parameters with $C_{p\mu_4} \approx 10^6 \text{ erg (gK)}^{-1}$ shows that the overall coefficient in the last term is $\sim 2.5 \times 10^{-5}$. Since $\vec{\nabla} \cdot \vec{v}_n^{(0)} = 0$ in the unperturbed state and the perturbation in $\vec{\nabla} \cdot \vec{v}_n^*$ will turn out to be ~ 0.01 times

that in T^* , Eq. (37) can generally be approximated as

$$\left[\frac{\partial}{\partial t} + \mathcal{P} \vec{v}_n^* \cdot \vec{\nabla} - \nabla^2 \right] T^* = 0. \quad (38)$$

The coupled nonlinear equations (23), (24), and (38) form the basis for our subsequent work.

It is also essential to consider the appropriate boundary conditions. If \hat{n} is a unit vector perpendicular to a rigid impenetrable surface, then the mass current and impurity current must have zero perpendicular component

$$(\rho_n \vec{v}_n + \rho_s \vec{v}_s) \cdot \hat{n} = 0, \quad (39)$$

$$(\rho c \vec{v}_n + \vec{i}) \cdot \hat{n} = 0. \quad (40)$$

This latter condition can be rewritten with Eqs. (7) and (19) as

$$\hat{n} \cdot \vec{v}_n^* + (D/\nu_n)(\gamma - k_T/c)(\hat{n} \cdot \vec{\nabla} T^*) T^{*-1} = 0, \quad (41)$$

apart from negligible corrections of relative order

$$D(\xi_1 - \rho \xi_3) \left| cd^2 \frac{\partial \mu_4}{\partial c} \right|^{-1}.$$

Furthermore, the tangential component of the normal fluid velocity must match that of the wall, so that

$$\hat{n} \times \vec{v}_n = 0 \quad (42)$$

at a stationary surface. Finally, the heat current at the boundary follows from Eq. (31), which may be rewritten in terms of κ_{eff} as

$$\vec{Q} = \frac{T_s}{c} (\rho c \vec{v}_n + \vec{i}) - \kappa_{\text{eff}} \vec{\nabla} T, \quad (43)$$

omitting corrections of order 10^{-5} [see the discussion below Eq. (37)]. Use of Eq. (40) then implies that

$$\hat{n} \cdot \vec{Q} = -\kappa_{\text{eff}} \hat{n} \cdot \vec{\nabla} T. \quad (44)$$

Thus the heat flow through the boundary follows the usual phenomenological description with κ_{eff} as the thermal conductivity. Equations (41) and (44) together imply that $\hat{n} \cdot \vec{v}_n = 0$ at a perfectly insulating wall, but it need not vanish in more general situations.

III. LINEARIZED EQUATIONS FOR SMALL-AMPLITUDE PERTURBATIONS

The rest of this paper will consider an unbounded horizontal slab of thickness d , with the lower and upper surfaces maintained at temperatures T^0 and $T^0 + \Delta T$, respectively. Since the unperturbed velocity \vec{v}_n^0 arises from a small applied temperature difference ΔT , we shall neglect quadratic terms in studying the initial unperturbed steady state. Thus the second term in Eq. (38) can be omitted, implying that $T^0(z)$ has a linear profile, with constant $\vec{\nabla} T^0 = \hat{z} \Delta T/d$. In addition, the normal fluid velocity \vec{v}_n^0 is constant throughout the fluid, with dimensionless magnitude

$$(v_{nz}^0)^* = -\frac{D}{\nu_n} \left[\gamma - \frac{k_T}{c} \right] \frac{\Delta T}{T^0}, \quad (45a)$$

determined by Eq. (41). As expected, v_n^0 is directed opposite to the temperature gradient. The corresponding equilibrium concentration gradient is

$$\vec{\nabla} c^0 = -(\gamma c^0/T^0) \vec{\nabla} T^0, \quad (45b)$$

which follows from Eq. (19) because $\vec{\nabla} \cdot \vec{v}_n^0$. Note also that the steady downward flow of ^3He impurities (with a flux $\rho c^0 \vec{v}_n^0$) is precisely cancelled by a steady upward diffusion with flux

$$\vec{i}^0 = \rho c^0 D (\gamma - k_T/c^0) \Delta T^0/T^0.$$

As seen previously, $D \approx 10^{-2} \text{ cm}^2 \text{ s}^{-1}$ for $x \approx 0.0047$ and $T \approx 1 \text{ K}$,¹⁴ and $\gamma - k_T/c$ is of order 1, so that $(v_{nz}^0)^*$ is small, of order $\Delta T/T < 10^{-2}$ in the experiments of Ref. 4. Thus the neglect of quadratic terms in Eq. (38) is well justified for the initial state.

The small-amplitude perturbations are obtained by linearizing about this nonclassical heat-conducting state with finite \vec{v}_n^0 . It is convenient to use only the dimensionless equations and henceforth omit the asterisks. To first order in the small quantities (denoted by primes), Eqs. (24) and (23) become

$$\nabla^2 \vec{v}'_n - \frac{1}{\mathcal{P}} \frac{\partial \vec{v}'_n}{\partial t} - v_{nz}^0 \frac{\partial \vec{v}'_n}{\partial z} - \vec{\nabla} P' + \hat{z} \text{sgn}(\alpha_{p\mu_4}) T' + \left[\frac{\rho}{\rho_n} \frac{gd\beta_c}{(\partial \mu_4/\partial c) T_p} \frac{\xi_1 - \rho \xi_3}{\nu_n} - v_{nz}^0 + v_{sz}^0 \right] \hat{z} \vec{\nabla} \cdot \vec{v}'_n = 0, \quad (46)$$

$$\vec{\nabla} \cdot \vec{v}'_n - \frac{\gamma}{T^0} \left[\frac{1}{\mathcal{P}} \frac{\partial T'}{\partial t} + \Delta T v'_{nz} + v_{nz}^0 \frac{\partial T'}{\partial z} \right] + \frac{D}{\nu_n T^0} \left[\gamma - \frac{k_T}{c} \right] \nabla^2 T' = 0. \quad (47)$$

It follows from the first and fifth terms of Eq. (46) that \vec{v}'_n and T' are comparable in magnitude. Since $v'_{nz} \lesssim 10^{-2}$ and $T' \lesssim 10^7$ in dimensionless units, Eq. (47) can be simplified to give

$$\vec{\nabla} \cdot \vec{v}'_n - \frac{\gamma \Delta T}{T^0} v'_{nz} = 0. \quad (48)$$

Thus $\vec{\nabla} \cdot \vec{v}'_n$ is smaller than v'_{nz} by a factor of $\sim 10^{-2} - 10^{-3}$, justifying the simplification introduced in obtaining Eq. (38). In a similar way, Eq. (38) becomes

$$\frac{\partial T'}{\partial t} + \mathcal{P} v_{nz}^0 \frac{\partial T'}{\partial z} + \mathcal{P} \Delta T v'_{nz} - \nabla^2 T' = 0, \quad (49)$$

and the dimensionless quantity ΔT is just the relevant Rayleigh number

$$R = \frac{|\alpha_{p\mu_4}| g d^3 \Delta T}{\chi_{\text{eff}} \nu}. \quad (50)$$

Here ΔT is the temperature difference in conventional units and $\nu = \eta / \rho = \nu_n \rho_n / \rho$. As a result, Eq. (49) becomes

$$\frac{\partial T'}{\partial t} + \mathcal{P} v_{nz}^0 \frac{\partial T'}{\partial z} - \nabla^2 T' + R v'_{nz} = 0. \quad (51)$$

The corresponding boundary conditions follow directly from Eqs. (41) and (42). In principle, the second term of Eq. (41) implies that the perturbation \vec{v}'_n has a small perpendicular component, but its overall coefficient is of order 10^{-7} . Thus all components of \vec{v}'_n can be taken to vanish at both surfaces. In addition, T' vanishes at the top and bottom plates because they are maintained at a fixed temperature.

These linearized equations evidently have solutions with exponential time dependence $e^{\sigma t}$. Although the presence of the normal-fluid flow \vec{v}'_n renders the equations non-self-adjoint, the correction terms are small, and we shall assume that the onset of convection occurs when σ first vanishes for some normal mode of the system. Thus the equations constitute an eigenvalue problem (we now drop the primes on the small perturbations):

$$\nabla^2 \vec{v}_n - \hat{z} T - \vec{\nabla} P + \epsilon_1 \frac{\partial \vec{v}_n}{\partial z} + \left[\epsilon_1 \frac{\rho}{\rho_s} - \epsilon_3 \right] \epsilon_2 \hat{z} v_{nz} = 0, \quad (52)$$

$$R^{-1} \nabla^2 T - v_{nz} = 0, \quad (53)$$

$$\vec{\nabla} \cdot \vec{v}_n - \epsilon_2 v_{nz} = 0, \quad (54)$$

where ϵ_i are small parameters of order $10^{-2} - 10^{-3}$, given by

$$\epsilon_1 = -v_{nz}^0, \quad (55a)$$

$$\epsilon_2 = \gamma \Delta T / T^0, \quad (55b)$$

$$\epsilon_3 = \frac{\rho}{\rho_n} \frac{g d \beta_c}{|(\partial \mu_4 / \partial c)_{Tp}|} \frac{\xi_1 - \rho \xi_3}{\nu_n}. \quad (55c)$$

Here Eq. (54) has been used to eliminate $\vec{\nabla} \cdot \vec{v}_n$ from Eq. (52), and a smaller correction term $\mathcal{P} \epsilon_1 \partial T / \partial z$ has been omitted from Eq. (53). Although these three small dimensionless parameters all reflect two-fluid effects, each represents a distinct physical contribution. The product $\epsilon_1 \partial / \partial z$ characterizes the convective transport by normal fluid flow, ϵ_2 is a measure of the normal fluid compressibility [see Eq. (54)], and the third parameter arises from dissipative forces on the superfluid [see Eq. (11)].

Translational invariance on the xy plane indicates that all quantities can be assigned a plane-wave dependence e^{iqx} . It is easy to see that v_{ny} vanishes in this case. The remaining four variables v_{nx} , v_{nz} , T , and P depend only on z and can be taken to form the elements of a vector U . Correspondingly, the coupled differential equations may be written

$$LU = 0 \quad (56)$$

in an obvious short-hand notation.

If $\epsilon_i = 0$, these equations reduce to the familiar Rayleigh-Bénard problem for a classical one-component fluid. Since the ϵ 's are small, it is natural to solve Eq. (56) perturbatively. The zero-order problem becomes $L^{(0)} U^{(0)} = 0$, where

$$L^{(0)} = \begin{bmatrix} \partial^2 - q^2 & 0 & 0 & -iq \\ 0 & \partial^2 - q^2 & -1 & -\partial \\ 0 & -1 & R^{(0)-1}(\partial^2 - q^2) & 0 \\ iq & \partial & 0 & 0 \end{bmatrix} \quad (57)$$

is manifestly self-adjoint and ∂ denotes d/dz . Although this zero-order problem can be solved exactly, it is more convenient for our purposes to use a different procedure⁶ that also applies to the more general case of a container with sidewalls. The solenoidal velocity field described by $U^{(0)}$ can be obtained from a vector potential through the prescription

$$\vec{v}_n^{(0)} = \vec{\nabla} \times [e^{iqx} \psi^{(0)}(z) \hat{y}], \quad (58)$$

with

$$v_{nx}^{(0)} = -\partial\psi^{(0)}, \quad (59a)$$

$$v_{nz}^{(0)} = iq\psi^{(0)}. \quad (59b)$$

Evidently, $\psi^{(0)}$ and $\partial\psi^{(0)}$ both must vanish at the upper and lower surfaces ($z = \pm \frac{1}{2}$). The corresponding pressure follows from Eq. (52) as

$$P^{(0)} = -(iq)^{-1}(\partial^2 - q^2)\partial\psi^{(0)}, \quad (60)$$

while the temperature $T^{(0)}$ can be found from Eq. (53) with a Green's function $G(z, z')$ that satisfies the equation

$$(\partial^2 - q^2)G(z, z') = -\delta(z - z'), \quad (61)$$

and the boundary condition $G(\pm \frac{1}{2}, z') = 0$. An elementary calculation yields

$$G(z, z') = -\frac{\sinh[q(z < + \frac{1}{2})]\sinh[q(z > - \frac{1}{2})]}{q \sinh q}. \quad (62)$$

In this way, $T^{(0)}$ may be written as

$$T^{(0)}(z) = -R^{(0)} \int dz' G(z, z') v_{nz}^{(0)}(z'). \quad (63)$$

It is easy to verify that the inner product

$$\begin{aligned} I &\equiv (U^{(0)}, L^{(0)} U^{(0)}) \\ &\equiv \int_{-1/2}^{1/2} dz U^{(0)\dagger}(z) L^{(0)} U^{(0)}(z) \end{aligned}$$

provides a variational basis for this problem.

When expressed in terms of the single real function $\psi^{(0)}(z)$, it takes the simple form

$$\begin{aligned} I &= - \int dz \psi^{(0)}(z) * (\partial^2 - q^2) \psi^{(0)}(z) \\ &\quad + q^2 R^{(0)} \int \int dz dz' \psi^{(0)}(z) * G(z, z') \psi^{(0)}(z'). \end{aligned} \quad (64)$$

The boundary conditions on $\psi^{(0)}$ suggest an expansion in the complete set¹⁶ of even eigenfunctions $C_l(z)$ that vanish and have vanishing slope at $z = \pm \frac{1}{2}$ and obey the fourth-order differential equation

$$\partial^4 C_l(z) = \lambda_l^4 C_l(z). \quad (65)$$

Thus we can write

$$\psi^{(0)} = (iq)^{-1} \sum_l A_l C_l(z) \quad (66)$$

with undetermined coefficients A_l . They obey coupled algebraic equations obtained by minimizing I . A straightforward analysis gives

$$\begin{aligned} \sum_m [-\delta_{lm} (\lambda_l^4 + q^4) - 2q^2 (C_l | C'_m) \\ + q^2 R^{(0)} (l | m)] A_m = 0, \end{aligned} \quad (67)$$

where

$$(C_l | C'_m) = \int dz \partial C_l \partial C_m, \quad (68a)$$

$$(l | m) = \int \int dz dz' C_l(z) G(z, z') C_m(z'). \quad (68b)$$

Note that $T^{(0)}$ and $v_{nz}^{(0)}$ are even in z , whereas $v_{nx}^{(0)}$ and $P^{(0)}$ are odd. Keeping only the single term A_1 yields the approximate critical Rayleigh number $R_c^{(0)} \approx 1719$ and $q_c \approx 3.102$, whereas a two-term approximation yields the improved estimates 1709 and 3.114, respectively. These agree well with the known⁶ values 1708 and 3.117, indicating that the variational procedure converges rapidly.

IV. INCLUSION OF TWO-FLUID EFFECTS

The full problem in Eq. (56) can now be studied by expanding in powers of the small parameters ϵ_i . The first-order correction to Eq. (56) is given by

$$L^{(0)} U^{(1)} = -L^{(1)} U^{(0)}, \quad (69)$$

where

$$\begin{aligned} (L^{(1)})_{11} &= (L^{(1)})_{22} = \epsilon_1 \partial, \\ (L^{(1)})_{33} &= -R^{(1)} (\partial^2 - q^2) (R^{(0)})^{-2}, \\ (L^{(1)})_{42} &= -\epsilon_2 \end{aligned}$$

and the other elements vanish. The condition that the inhomogeneous term $-L^{(1)} U^{(0)}$ be orthogonal to the solution $U^{(0)}$ yields

$$R^{(1)} = 0, \quad (70)$$

because the remaining contributions vanish by symmetry. Thus there is no first-order shift in the critical Rayleigh number from the classical value, and it is therefore necessary to proceed to second order in the expansion.

The first step is to solve the coupled equations (69) for the first-order corrections. Since $\vec{\nabla} \cdot \vec{\nabla}_n^{(1)}$ is nonzero, it is no longer possible to use a vector potential. Nevertheless, the function $v_{nz}^{(1)}$ can be used in an equivalent way. In particular, $v_{nx}^{(1)}$, $T^{(1)}$, and $P^{(1)}$ are all expressible in terms of $v_{nz}^{(1)}$ and its derivatives, and simple manipulations lead to the single inhomogeneous equation

$$\begin{aligned} (\partial^2 - q^2)^2 v_{nz}^{(1)} - q^2 R^{(0)} \int dz' G(z, z') v_{nz}^{(1)}(z') \\ = -iq(\epsilon_1 - \epsilon_2) (\partial^2 - q^2) \partial\psi^{(0)}(z). \end{aligned} \quad (71)$$

This can be solved with an eigenfunction expansion of the form

$$v_{nz}^{(1)}(z) = \sum_l B_l S_l(z), \quad (72)$$

where S_l is the odd counterpart of C_l . Substitution into Eq. (71) and use of the orthogonality of the eigenfunctions lead to a set of linear algebraic equations for B_l in terms of the now known quantities A_l . Note that $v_{nz}^{(1)}$ is odd in z , in contrast to the zero-order contribution $v_{nz}^{(0)}$. Thus the actual convective flow has no definite parity. This specific two-fluid effect is likely to influence the nonlinear behavior beyond threshold.

The second-order contribution to Eq. (56) is

$$L^{(0)}U^{(2)} = -L^{(1)}U^{(1)} - L^{(2)}U^{(0)}, \quad (73)$$

where

$$(L^{(2)})_{22} = \epsilon_2(\epsilon_1\rho/\rho_s - \epsilon_3),$$

$$(L^{(2)})_{33} = -R^{(2)}(\partial^2 - q^2)(R^{(0)})^{-2},$$

and the other elements vanish. The condition that $U^{(0)}$ be orthogonal to the right-hand side now explicitly determines the leading correction $R^{(2)}$. The final expression is a ratio of two quadratic forms in the coefficients A_l and B_l . When evaluated with the approximate two-term solutions for A_1 , A_2 , B_1 , and B_2 , it yields

$$R^{(2)} \approx 24.6\epsilon_1\epsilon_2 + 10.2(\epsilon_1 - \epsilon_2)^2 - 19.9\epsilon_2(\epsilon_1\rho/\rho_s - \epsilon_3) \quad (74)$$

for the shift in the critical Rayleigh number arising from the two-fluid contributions in the hydrodynamic equations. For comparison, the simpler approximation of retaining only A_1 and B_1 gives coefficients 25.5, 11.1, and 19.9, respectively, implying a reasonable rate of convergence. For $T=0.8$ K, $x=0.0047$, and $d=1$ cm, the observations of Ref. 4 imply $\epsilon_1 \approx 6.5 \times 10^{-2}$, $\epsilon_2 \approx 6.4 \times 10^{-2}$, and $\epsilon_3 \approx 2 \times 10^{-3}$ because $(\xi_1 - \rho\xi_3)/\nu_n \approx 1$ in this case.¹¹ As a result, $R^{(2)}$ is ~ 0.02 , far too small to be detectable. Since the observed R should be near 1708, it is clear from Eq. (50) that $\Delta T/T$ varies like d^{-3} . Thus a reduc-

tion to $d \approx 0.5$ cm should increase ϵ_1 and ϵ_2 each by a factor of 8, and the corresponding $R^{(2)} \approx 1.4$ might just be observable in careful experiments.

The small magnitude of ϵ_3 indicates that superfluid dissipation does not play an important role in the experiments of Ref. 4, performed near 1 K. For higher temperatures, however, Steinberg¹¹ has remarked that $(\xi_1 - \rho\xi_3)/\nu_n$ becomes large, leading to qualitatively different behavior, similar to that for a classical two-component fluid. An experimental study of this question would be valuable.

At present, the only systematic study of convection in dilute superfluid ^3He - ^4He mixtures⁴ has used a cylinder with unit aspect ratio ($\Gamma \equiv R/d = 1$), which differs considerably from an unbounded slab. Thus conclusions based on the present study can only be suggestive. Nevertheless, two features of the experiments deserve mention. First, the onset of convection invariably occurred for $R_c \approx 1700$, whereas the classical prediction¹⁷ for a cylinder with insulating sidewalls and $\Gamma = 1$ is $R_c \approx 2262$. Second, in such a cylinder, the classical one-component fluid undergoes a transition to axisymmetric toroidal flow with the same R_c for the two patterns (center falling and center rising) differing only in the sense of motion.¹⁷ In contrast, the experiments⁴ found a clear preference for the center-falling pattern. It might initially be thought that the nonzero \vec{v}_n in the preconvective heat-conducting state could provide a preferred direction, thus splitting these two degenerate flow patterns. More careful analysis for a cylindrical container (presented in the following paper) now suggests that the onset of these two states remains degenerate. Thus the explanation of the observed preference for flow with center falling probably requires the inclusion of the nonlinear terms omitted from the present work. This interesting question requires further study.

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