Wave mechanics of sine-Gordon solitons

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We continue our study of the acceleration of a single sine-Gordon (SG) soliton kink wave by an external field [cf. G. Reinisch and J. C. Fernandez, Phys. Rev. B 24, 835 (1981), hereafter referred to as paper I]. We exhibit, both qualitatively and quantitatively, the basic physical process which prevents the kink dynamics to be *a priori* Newtonian: the self-consistent interaction between the small linear oscillations (phonon waves) excited by the external field about the kink profile, and the kink itself. In order to evaluate the importance of this interaction, and therefore build a kink wave mechanics, we define a kink momentum $P=-(\pi/4)(\partial/\partial t)\int_{-\infty}^{\infty} u -(r/4)(\partial/\partial t)\int_{-\infty}^{\infty} u\,dx$ and check that it measures, in an acceptable sense, the momentum of a particle associated with the kink center, having a mass equal to the energy of the kink. This enables us to recover our previous results (paper I), obtained within the framework of the (rather severe) adiabatic assumption (consisting in retaining only the small-wave-number phonon waves), and correct them by taking into account the whole phonon spectrum. As a matter of fact, we obtain an important correction, of the order of 20%, of the kink dynamics in presence of an external field, and verify that it leads to a better fitting with the numerical results of paper I. We show that the above-mentioned kink wave mechanics is based on the existence of a general force equation of the type $(d/dt)P=\Xi[\psi_k]$, where Ξ is a linear operator and ψ_k is the phonon spectrum. This equation shows that the phonon dressing of SG kinks may lead to first-order (with respect to the perturbation function) dynamical effects concerning the kink, when the phonon-soliton interaction is coherent, i.e., when the characteristic interaction time is comparable with the time of coherence of the excited phonons.

I. INTRODUCTION

In two previous papers,[1,2] we studied the specific dynamics of a single driven sine-Gordon (SG) kink satisfying the (dimensionless) SG equation including a constant field χ:

$$u_{tt}-u_{xx}+\sin u=\chi\ . \tag{1}$$

(Here subscripts mean partial derivations. The variables x and t, respectively, measure the one-dimensional space and the time.) This study, which uses standard perturbation techniques,[3,4] describes the solution $u(x,t)$ of Eq. (1) as

$$u(x,t)=4\tan^{-1}e^{\alpha x}+\psi(x,t)\ . \tag{2}$$

The first term of expression (2) is the well-known kink ($\alpha=+1$) or antikink ($\alpha=-1$) solution of the unperturbed ($\chi=0$) SG equation.[4] The second term is the perturbation function ψ and describes the evolution of the kink profile and its dynamical properties. As is usual in perturbation techniques, the function ψ is always assumed small compared with the kink amplitude:

$$\psi \ll 2\pi\ . \tag{3}$$

Adding a small viscous term $(-\Gamma u_t)$ to the right-hand side (rhs) of (1) was shown to lead to some minor quantitative changes in the soliton dynamics. Therefore we discard such a damping effect and consider the conservative situation described by (1). This equation may be derived from the following Hamiltonian density[5]:

$$H(x,t)=\mu^2[\tfrac{1}{2}u_t^2+\tfrac{1}{2}u_x^2+1-\cos u -\chi(u-u_0)]\ , \tag{4}$$

where

$$u_0(x)=4\tan^{-1}e^{\alpha x}\,, \tag{5}$$

by use of canonical equations relating the generalized variables $\{\mu u;\mu u_t\}$. The parameter μ will be described below. Note that the total mass M_0 of the system is conserved:

$$M_0=\int_{-\infty}^{\infty}H(x,t)dx=\text{const}=8\mu^2\,. \tag{6}$$

The physical interest of this study was mentioned in Ref. 2. There are several extensive reviews of the literature devoted to the subject (see, for instance Refs. 4 and 6–11). Here we only recall that Eq. (1) plays an important role in the propagation of "fluxons" accelerated in long Josephson-junction transmission lines, which are being given considerable attention at the present time for the application of this device for transmission, storage, and processing of information. We note with historical interest that the idea of using the relationship between the solitary waves and moving domain walls in order to build a memory device was suggested as early as 25 years ago.[12]

Although the following is not directly related to the physical situation described by Eqs. (1)–(4), we also wish to mention recent theoretical developments in the statistical mechanics of the SG chain[13–15] and in the long-term behavior of its equilibrium.[16] These works focus on the statistical mechanics of dynamical properties of the SG chain, and therefore seem the natural continuation to SG-kink gas theory of the single-kink dynamics.

In Refs. 1 and 2, we showed that, as long as inequality (3) is satisfied, the dynamics of the SG kink is not Newtonian, i.e., the kink acceleration is not proportional to the field amplitude χ.[17] Here we show that this departure from Newtonian dynamics is all the more important as χ becomes "large," i.e., typically[1,2,18]

$$0.1\lesssim\chi<1\,. \tag{7}$$

The specific kink dynamics can be understood in terms of coherent and/or incoherent coupling processes between the solitons and the phonon modes [which build the continuum spectrum of the Schrödinger-type operator, $\mathscr{O}=-\partial^2/\partial x^2+1-2\,\text{sech}^2x$, obtained by linearizing the driven SG equation (1) about the exact kink solution[4,19] (5)]. Comparing the characteristic time τ (proportional to χ^{-1}) of the soliton acceleration with the time of coherence of the excited phonon waves seems a better physical approach than asking if the deviation of the kink dynamics from Newton's law is—or is not—an artifact of the initial conditions.[20]

As was shown in Ref. 2, an efficient coupling process between the so-called translation (or Goldstone) eigenmode of operator $\mathscr{O}$

$$f_b(x)\propto\frac{\partial u_0}{\partial x}\sim\text{sech}x$$

and the phonon modes

$$f_k(x)\propto\exp[i(kx+\theta_k)]$$

may occur when the SG kink comes under the influence of the external field χ [we assume $\psi(x,0)\equiv 0$]. As a consequence, it starts moving as χt^4, instead of χt^2 (Newton's law).

This coupling is all the more important as the phonon-soliton interaction significantly develops within the time of coherence of the excited phonon waves. In situations described by (7), we have

$$\omega_k^{-1}\lesssim\tau\ll\Delta\omega_k^{-1}\,, \tag{8}$$

where

$$\omega_k^2=1+k^2 \tag{9}$$

is the phonon wave dispersion relation and $\Delta\omega_k$ is the spectral width of the phonon spectrum. Hence, in terms of standard nonlinear optics, the interaction is strongly coherent. Important phase effects then explain the non-Newtonian kink dynamics.

On the other hand, the quasi-Newtonian kink dynamics related to the presence of very weak external fields in the system ($\chi\ll 0.1$) is due to the random-phase aspect of the phonon-soliton interactions, since τ increases now to such an extent that it becomes larger than the time of coherence $\Delta\omega_k^{-1}$ of the phonon waves.

The outline of this paper is as follows. In Sec. II, we define a kink momentum. This is not as simple as it seems, since we show that the soliton response to the field χ is not adiabatic, i.e., the kink does not keep the same profile as it moves under the influence of this field. We adopt the rather natural definition[1,2]

$$V_{\text{sol}}=\frac{d}{dt}\left[\frac{\int_{-\infty}^{\infty}xu_x dx}{\int_{-\infty}^{\infty}u_x dx}\right]\simeq\frac{\alpha}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty}xu_x dx \tag{10}$$

related to the picture of a particle associated to the kink and located at its "center" defined as the point of steepest slope in the kink profile. One must bear in mind that other definitions could also be adopted. In particular the so-called field momentum[15]

$$\Pi(t)=-\int_{-\infty}^{\infty} u_t u_x dx \qquad (11)$$

refers to the velocity of the center of mass $V_{\text{c.m.}}$ of the whole system constituted by both the particle and its cloud of phonons

$$V_{\text{c.m.}}=\frac{1}{8\mu^2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} xH\,dx \ , \qquad (12)$$

since we have

$$\frac{\partial}{\partial t}H(x,t)=\frac{\partial}{\partial x}(u_t u_x) \qquad (13)$$

[cf. Eqs. (3) and (4)]. Its time variation is given by Newton's law[5]:

$$\Pi(t)=-2\pi\mu^2\alpha\chi t \ . \qquad (14)$$

Equation (14) is an exact relation between Π, as defined by (11), and t, and is obtained without any approximation directly from the basic equation (1) once an initial kink profile is assumed. Since it is not relativistically covariant, the field momentum (11) is not a physically acceptable choice of the system momentum. It is possible to show by geometrical arguments based on a "tail effect" related to the term $\chi(u-u_0)$ in (3) that both velocities (8) and (10) are related in such a way that (10) still defines a relativistic (particle) velocity, while this is not the case for (12) (see Appendix B). Hence we obtain an additional argument favoring the particlelike definition (10).

We show that the kink velocity (10) is simply related to the generalized impulse

$$P=-\tfrac{1}{4}\alpha\pi\int_{-\infty}^{\infty}\psi_t dx \ . \qquad (15)$$

We calculate this momentum in terms of the phonon spectrum $\psi_k(t)$ defined, as usual,[4,19] by the expansion of the perturbation function $\psi(x,t)$:

$$\psi(x,t)=\psi_b(t)f_b(x)+\int_{-\infty}^{\infty}dk\,\psi_k(t)f_k(x) \ . \qquad (16)$$

Owing to the coupling between the translation mode and the phonon modes, described by the completeness relation

$$\int_{-\infty}^{\infty}dk\,f_k^*(x_1)f_k(x_2)=\delta(x_1-x_2)-f_b(x_1)f_b(x_2) \ , \qquad (17)$$

we obtain a formula of the type

$$P_t=\int_{-\infty}^{\infty}dk\,\Xi[\psi_k]+\text{const} \ , \qquad (18)$$

where Ξ is a linear operator, depending on the wave number k. Formula (18) shows that the translation (Goldstone) mode $f_b(x)$ disappears from the dynamical description of the (particlelike) soliton motion, and that this motion is controlled entirely by the soliton wave functions $f_k(x)$ through the generalized forces $\Xi[\psi_k]$.

In Sec. III, we correct our previous results obtained in Refs. 1 and 2 by using the momentum (15). We show that the so-called adiabatic velocity obtained in Refs. 1 and 2 is as a matter of fact erroneous by a factor $8/\pi^2$, and we have, instead of $V_{\text{sol}}=-\frac{1}{24}\alpha\pi\chi t^3$,

$$V_{\text{sol}}=\frac{\alpha\chi t^3}{3\pi} \qquad (19)$$

for small values of t. The correction due to the factor $8/\pi^2$ improves the fits to the numerical simulations given in Refs. 1 and 2 (see Fig. 1). We show that this is due to the nonzero-wave-number phonons that were neglected.

As a conclusion, we point out that SG phonon-soliton interactions are not always a second-order process. Indeed, for intermediate field amplitudes [Eq. (7)], the coherent (fixed-phase) interaction leads to first-order corrections with respect to the expected Newtonian dynamics of the kink, as seen from formulas (15)–(19). In contrast, random-phase phonon-soliton coupling due to weaker fields should lead to second-order dynamical effects similar to those described in Ref. 21.

II. PHONON-INDUCED KINK DYNAMICS

A. Inappropriateness of the translation mode for the description of the initial kink dynamics

As already pointed out in Refs. 1 and 2, the translation mode $f_b(x)$ is a somewhat ambiguous tool in order to describe the motion of the accelerated kink. Indeed, consider the addition of a constant function, say $\psi\equiv 1$, to the pure kink profile (see Fig. 2). This leads to the uniform amplification shown in the left-hand side (lhs) of Fig. 2. Now project ψ on the translation mode $f_b(x)$. The result is

$$(\psi,f_b(x))=\int_{-\infty}^{\infty}f_b(x)dx=\frac{1}{\sqrt{2}}\pi \qquad (20)$$

(the bold parentheses indicate the scalar product of

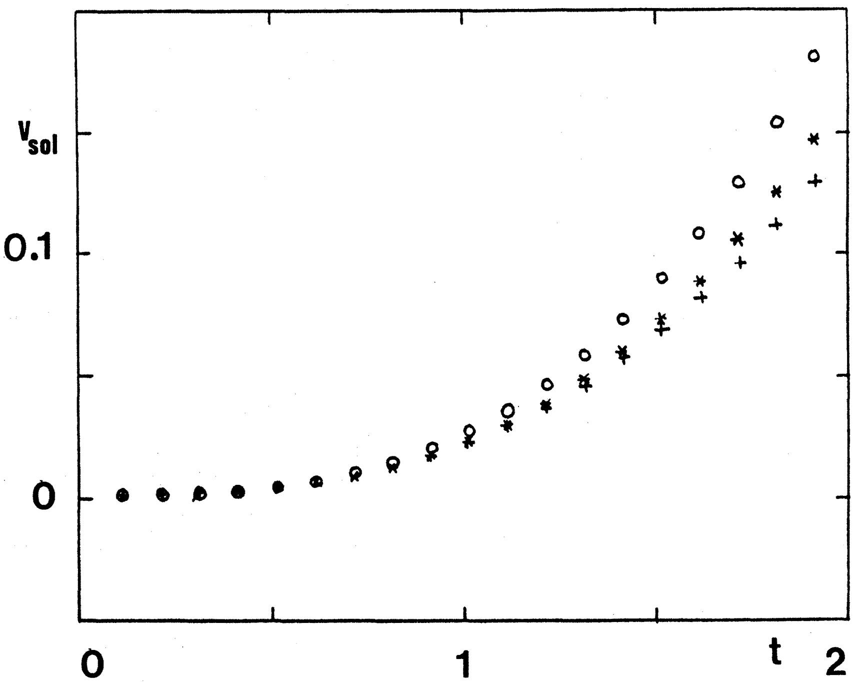

FIG. 1. Theoretical kink velocity vs time in the adiabatic approximation [formula (53): ○] and when taking the whole phonon spectrum into account [formula (55) ∗]. The numerical simulation obtained in Ref. 2 leads to the lowest line: $+$.

ψ and f_b).

According to Refs. 3 and 4, this value should measure the displacement of the soliton due to the addition of $\psi\equiv 1$ to $u_0(x)$ (see the rhs of Fig. 2). This is clearly an erroneous statement. Now note that this misleading dynamical interpretation is due to the neglect of the dynamical effect of the phonon spectrum which clearly appears in the fol-

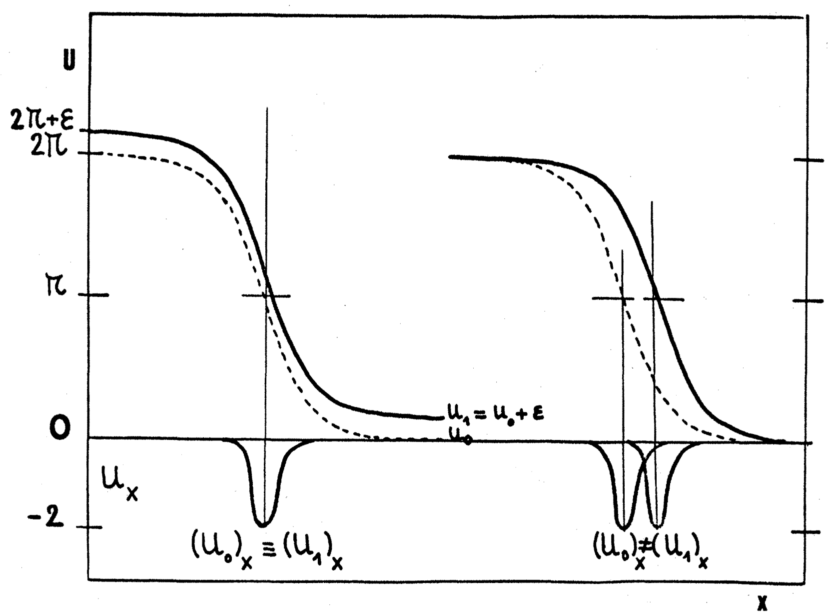

FIG. 2. Illustration of the danger of considering always the projection of the perturbation function ψ on the translation mode f_b as a measure of the kink motion. Indeed, if $\psi=\epsilon=0.5$, we do *not* obtain a shift of the kink over a distance $(1/2\sqrt{2})\int_{-\infty}^{\infty}\epsilon f_b(x)dx=\pi/4$, as seen when looking at the derivatives $(u_0)_x$ and $(u_1)_x$.

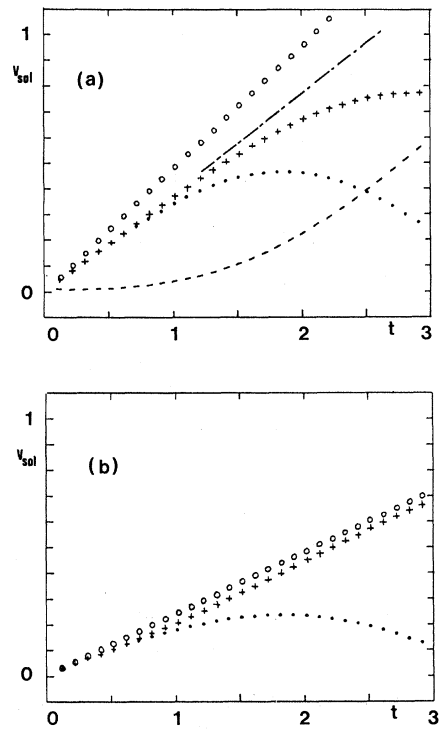

FIG. 3. (a) "Screw effect" for $a=0.6$. ○: theoretical adiabatic linear dynamics $V_{sol}=\frac{1}{4}\pi at$. — · — · —: theoretical nonadiabatic linear dynamics $V_{sol}=2at/\pi$. + +: numerical results in the case $\chi=0.3$. · · · numerical results in the case $\chi=0$. — — —: numerical results of Ref. 2 for the case $\chi=0.3$, $a=0$, given here for comparison. The important point is that the addition of constant a to the kink profile destroys the non-Newtonian start of the motion. (b) Remarkable resonant effect (Ref. 25) in the case $a=\chi=0.3$ (the dotted line still means $a=0.3$, $\chi=0$ for comparison). We note an excellent agreement with the theoretical linear dynamics ("Newton's law"), even for large time values.

lowing equation:

$$\int_{-\infty}^{\infty} dk\, \psi_k(t) f_k(x) = \int_{-\infty}^{\infty} dk (\psi, f_k) f_k(x)$$
$$= 1 - \frac{\pi}{\sqrt{2}} f_b(x) . \qquad (21)$$

The second term on the rhs of (21) exactly cancels the projection of ψ on the translation mode $f_b(x)$.

If we were able, by an artifact, to perturb the balance between these terms, the coefficient of $f_b(x)$ then would not vanish and we would recover a translation motion described by this coefficient. This happens when, for instance we add, at $t=0$, a constant function $\psi(x,0)\equiv a$ to the kink profile. We obtain a Newtonian soliton dynamics characterized by $V_{sol} \sim \frac{1}{4}\pi at$ (see Fig. 3). Here the constant a plays the role of a force. To recover the linear dynamics still requires the account of the continuum phonon spectrum (see Appendix A).

B. Definition of the kink momentum

Consider the generalized impulse associated with the Hamiltonian density (4):

$$P = \mu \int_{-\infty}^{\infty} \psi_t dx , \qquad (22)$$

and assume that it should at least describe the correct (particlelike) kink momentum when the kink is simply shifted over a distance $\Delta x = V_{sol} t$. Taylor expanding to first order the new solution $u(x,t) = u_0(x - V_{sol} t)$, we obtain [cf. Eq. (2)]:

$$\psi_t = -V_{sol} \frac{\partial u_0}{\partial x} , \qquad (23)$$

which shows that the Goldstone mode $f_b(x) \propto \partial u_0/\partial x$ must be a solution of the system (1) and (2) with $\chi=0$. We obtain from (22):

$$P = -2\pi\alpha V_{sol}\, \mu \qquad (24)$$

($\alpha=+1$ for kinks, $\alpha=-1$ for antikinks). Equating this expression to the particlelike kink momentum

$$M_0 V_{sol} = 8\mu^2 V_{sol} , \qquad (25)$$

where M_0 is the kink mass (6), leads to

$$\mu = -\tfrac{1}{4}\alpha\pi , \qquad (26)$$

and therefore

$$P = -\tfrac{1}{4}\alpha\pi \int_{-\infty}^{\infty} \psi_t dx . \qquad (27)$$

We must now relate this generalized impulse P to the particlelike kink momentum derived from (10) and (25):

$$P_{sol} = M_0 V_{sol} = \tfrac{1}{4}\alpha\pi \frac{d}{dt} \int_{-\infty}^{\infty} x u_x dx . \qquad (28)$$

Integrating the rhs of (28) by parts and using definition (2), we obtain

$$P_{sol} - \tfrac{1}{4}\alpha\pi \left[\lim_{x\to+\infty} (x\psi_t) - \lim_{x\to-\infty} (x\psi_t) \right] = P . \qquad (29)$$

Therefore we have a simple relation between P and

$P_{\rm sol}$. In particular, when the boundary effects described in the large parentheses of (29) may be neglected or vanish, the generalized impulse P defined in (27) is indeed the particlelike kink momentum (28).

Consider now the usual expansion (16) of the kink wave function ψ in the eigenfunctions $f_b(x)$ and $f_k(x)$ defined as[3,4]

$$f_b(x)=\frac{1}{\sqrt{2}}\operatorname{sech}x\ ,\qquad f_k(x)=\frac{e^{ikx}}{\sqrt{2\pi}\omega_k}(k+i\tanh x)\ . \tag{30}$$

We let $y=kx$ and obtain

$$\lim_{x\to(\pm)\infty}(x\psi_t)=\lim_{x\to(\pm)\infty}\frac{1}{\sqrt{2\pi}}\int_{-(\pm)\infty}^{+(\pm)\infty}dy\,e^{iy}\frac{(\pm)i+y/x}{(1+y^2/x^2)^{1/2}}\tilde{\psi}_t(y,x,t)\ , \tag{31}$$

where

$$\tilde{\psi}_t(y,x,t)=\psi_t(k,t)\ . \tag{32}$$

Assuming that we may now proceed to the limit $x\to(\pm)\infty$ inside the integral, (31) leads to

$$\lim_{x\to(\pm)\infty}(x\psi_t)=\frac{(\pm)i}{\sqrt{2\pi}}\int_{-(\pm)\infty}^{+(\pm)\infty}dy\,\tilde{\psi}_t(y,\infty,t)e^{iy}\ . \tag{33}$$

The projections $\psi_b(t)$ and $\psi_k(t)$ defined in (9) are[2]

$$\psi_b(t)=\frac{1}{2\sqrt{2}}\pi\chi t^2+\psi_{bt}(0)t+\psi_b(0)\ , \tag{34}$$

$$\psi_k(t)=\psi_k(0)\cos\omega_k t+\psi_{kt}(0)\frac{\sin\omega_k t}{\omega_k}+\chi\frac{1-\cos\omega_k t}{\omega_k^2}\int_{-\infty}^{\infty}f_k^*(x)dx\ , \tag{35}$$

where

$$\int_{-\infty}^{\infty}f_k^*(x)dx=\int_{-\infty}^{\infty}f_k(x)dx=\frac{1}{\sqrt{2\pi}\omega_k}\left[2\pi k\delta(k)-\frac{\pi}{\sinh\frac{1}{2}\pi k}\right]\ . \tag{36}$$

With the hypothesis that the initial phonon spectrum $\psi_k(0)$ and its time derivative $\psi_{kt}(0)$ are regular for vanishing wave numbers $k\to 0$, formula (33) becomes

$$\lim_{x\to(\pm)\infty}(x\psi_t)=-\frac{i\,|x\,|\chi}{\pi}\sin t\int_{-(\pm)\infty}^{+(\pm)\infty}\frac{e^{iy}}{y}dy \tag{37}$$

and we obtain

$$\lim_{x\to+\infty}(x\psi_t)-\lim_{x\to-\infty}(x\psi_t)=\lim_{|x|\to\infty}2\,|x\,|\chi\sin t=L\chi\sin t\ , \tag{38}$$

where L is the length of the system.

Note that we recover our previous result $\lim_{x\to\pm\infty}\psi=\chi(1-\cos t)$ obtained by assuming that the u_{xx} term in (1) was negligible for $x\to\pm\infty$.[1] And indeed, limit (33) results in selecting the zero-wave-number phonons. As a consequence, when we consider (say) a phonon wave packet $[\psi_k]_{k\sim k_0}$ (in the case where $\chi=0$), there is no term related to boundary effects in (29) and

$$P_{\rm sol}=P\quad(\chi=0)\ . \tag{39}$$

In the picture of a chain of weakly coupled pendulums modeling the driven SG equation (1),[22] the "boundary term" (38) describes the rotational degree of freedom of the chain considered as a rigid body, while P_{sol} describes the translational degree of freedom of the soliton wave.

Formulas (16) and (27) lead to

$$P=-\tfrac{1}{4}\alpha\pi\left[\frac{\pi}{\sqrt{2}}\psi_{bt}+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dk\,\psi_{kt}\omega_k^{-1}\left[2\pi k\delta(k)-\frac{\pi}{\sinh\frac{1}{2}\pi k}\right]\right]. \tag{40}$$

Using (34)–(36), this becomes

$$P_t=-\tfrac{1}{4}\alpha\pi\left[\tfrac{1}{2}\pi^2\chi+\left[\frac{\pi}{2}\right]^{1/2}\int_{-\infty}^{\infty}dk\frac{\omega_k}{\sinh\frac{1}{2}\pi k}\psi_k+\chi\int_{-\infty}^{\infty}dk\left[\int_{-\infty}^{\infty}f_k(x)dx\right]^2-\sqrt{2\pi}\lim_{k\to 0}k\psi_k\right]. \tag{41}$$

The third term on the rhs of (41) is written, by use of the completeness relation (17),

$$\begin{aligned}-\tfrac{1}{4}\alpha\pi\chi\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dk\,f_k^*(x)f_k(x')&=-\tfrac{1}{4}\alpha\pi\chi\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'[\delta(x-x')-f_b(x)f_b(x')]\\&=-\tfrac{1}{4}\alpha\pi\chi L+\tfrac{1}{8}\alpha\pi^3\chi\end{aligned} \tag{42}$$

(this is just the Parseval-Planckerel formula in the functional space spanned by the eigenfunctions $\{f_b,f_k\}$). Therefore (41) gives

$$P_t=-\tfrac{1}{2}\pi\alpha\left[\frac{1}{2}\left[\frac{\pi}{2}\right]^{1/2}\int_{-\infty}^{\infty}dk\frac{\omega_k}{\sinh\frac{1}{2}\pi k}\psi_k+\tfrac{1}{2}\chi L-\left[\frac{\pi}{2}\right]^{1/2}\lim_{k\to 0}k\psi_k\right]. \tag{43}$$

We recover (18) in which

$$\Xi(\psi_k)=\frac{-\alpha\pi^{3/2}}{4\sqrt{2}}\left[\frac{\omega_k}{\sinh\frac{1}{2}\pi k}-2k\delta(k)\right](\psi_k), \tag{44}$$

and the constant is $(-\frac{1}{4})\pi\alpha\chi L$.

As already pointed out, the interest of (43) and (44) is mainly theoretical. The kink moves, according to its particlelike trajectory (10) determined by (29), under the influence of a "phonon force" given by (44). The translation mode component $\psi_b(t)f_b(x)$ has disappeared from the kink dynamics,[23] due to its relation to the phonon modes $f_k(x)$ through the completeness relation (42). Formulas (43) and (44) therefore describe the phonon-induced kink dynamics (if we focus our attention on the excited phonon spectrum ψ_k that interacts with the kink) or, equivalently, the kink wave mechanics.

Note that for small time values, the constant $(-\frac{1}{4})\pi\chi\alpha L$ is eliminated from the description of the kink dynamics. Indeed, (29), (38), (43), and (44) give, for $t \gtrsim 0$,

$$\frac{d}{dt}P_{\mathrm{sol}}=\int_{-\infty}^{+\infty}dk\,\Xi[\psi_k], \tag{45}$$

and the above identification of $\Xi(\psi_k)$ with a phonon force acting on the particlelike kink becomes evident.

In formulas (43) and (44), the operator Ξ has a strong singularity for $k\to 0$. It is possible to obtain an equivalent formula in which the kernel of the linear operator dealing with the phonon spectrum ψ_k is regular for all k, but the price we have to pay is the reappearance of the translation mode component $\psi_b(t)f_b(x)$.

Equations (43) and (9) lead to

$$P_{tt}=\frac{-\alpha\pi^{3/2}}{4\sqrt{2}}\int_{-\infty}^{\infty}dk\,\psi_{kt}\left[\frac{k^2}{\omega_k\sinh\frac{1}{2}\pi k}-\frac{1}{\pi\omega_k}\left[2\pi k\delta(k)-\frac{\pi}{\sinh\frac{1}{2}\pi k}\right]\right] \tag{46}$$

and, according to (40), we obtain

$$P_{tt}+P=\frac{-\alpha\pi^{3/2}}{4\sqrt{2}}\left[\sqrt{\pi}\psi_{bt}+\int_{-\infty}^{\infty}dk\,\psi_{kt}\frac{k^2}{\omega_k\sinh\frac{1}{2}\pi k}\right]. \tag{47}$$

There is an alternative proof of (47) given in Appendix C.

III. NONADIABATIC KINK DYNAMICS IN THE PRESENCE OF AN EXTERNAL FIELD

Assume $\chi \neq 0$, and consider the onset of the perturbation: $t \gtrsim 0$. In this section we wish to correct our previous results[1,2] by taking into account the above phonon-induced dynamics described by formula (47).

Starting with an initially static kink, formula (47) becomes, for small time values,

$$P_{tt} = T_1 + T_2 + T_3 , \tag{48}$$

where [by use of (34)–(36)]

$$T_1 = \frac{-\alpha\pi^2}{4\sqrt{2}} \psi_{bt} = -\tfrac{1}{8}\alpha\pi^3 \chi t , \tag{49}$$

$$T_2 + T_3 = \frac{-\alpha\pi^{3/2}}{4\sqrt{2}} \int_{-\infty}^{\infty} dk\, \psi_{kt} \frac{k^2}{\omega_k \sinh\frac{1}{2}\pi k} = -\tfrac{1}{8}\alpha\pi\chi t \int_{-\infty}^{\infty} dk \frac{k^2[2\pi k\delta(k) - \pi \sinh^{-1}\frac{1}{2}\pi k]}{\omega_k^2 \sinh\frac{1}{2}\pi k} . \tag{50}$$

Hence

$$T_2 = -\tfrac{1}{4}\alpha\pi^2\chi t \int_{-\infty}^{\infty} dk \frac{k^3 \delta(k)}{\omega_k^2 \sinh\frac{1}{2}\pi k} = 0 , \tag{51}$$

$$T_3 = \tfrac{1}{8}\alpha\pi^2 \chi t \int_{-\infty}^{\infty} dk \frac{k^2}{(1+k^2)\sinh^2\frac{1}{2}\pi k} = -T_1 \left[1 - \frac{8}{\pi^2} \right] . \tag{52}$$

Formula (48) describes the kink dynamics induced by all phonons whose wave number ranges from $-\infty$ to $+\infty$. If we decide to select a monochromatic wave packet of frequency $\omega_k \equiv 1$, i.e., $k \equiv 0$, as we did in Ref. 2, we recover the adiabatic velocity of Ref. 2:

$$V_{\rm sol}^{\rm adiab} = \frac{-\alpha\pi\chi t^3}{24} . \tag{53}$$

Indeed, the rhs of (48) reduces in this case to $T_1 + T_2$ (since there is no singularity in T_3 for $k=0$), i.e., to T_1 [cf. (51)]. Since [cf. (29) and (38)]

$$P_{tt} \simeq \frac{d^2 P_{\rm sol}}{dt^2} \quad \text{for} \quad t \gtrsim 0 , \tag{54}$$

and since the soliton rest mass is $M_0 = 8\mu^2 = \frac{1}{2}\pi^2$ [cf. (6) and (26)] the equation $p_{tt} \simeq T_1$ leads to (53).

Formulas (48)–(52) enable us to calculate the nonadiabatic correction to formula (53) due to the account of all other phonons of wave number $k \neq 0$ interacting with the kink. Since $T_2 = 0$ [cf. (51)], Eqs. (54), (48), (49), and (52) lead to a factor $8/\pi^2$ correcting the adiabatic particlelike kink velocity (53):

$$V_{\rm sol} = \frac{8}{\pi^2} V_{\rm sol}^{\rm adiab} = \frac{-\alpha\chi t^3}{3\pi} . \tag{55}$$

In Appendix D we give an alternative "direct" proof of this formula. We note that it leads to better fits of the numerical results of Ref. 1 (see Fig. 1).

In conclusion, we summarize the three above-mentioned SG kink dynamics in presence of an external field χ [see (8)].

(i) For very weak amplitude χ, leading to random-phase phonon-soliton coupling effects, the kink dynamics is "classical"—i.e., Newtonian—since the phase effects of all phonon modes generated at the onset of the perturbation may be ignored in its description. The equation of motion therefore reads [cf. (40) and see also Ref. 18]

$$P = \frac{-\alpha\pi^2}{4\sqrt{2}} \psi_{bt}$$

or [cf. (34)]

$$V_{\rm sol}^{\rm class} = -\tfrac{1}{4}\alpha\pi\chi t , \tag{56}$$

when the kink is assumed static at $t=0$.[3,4]

(ii) For larger field amplitude, $\chi \gtrsim 0.1$, a rough approximation of the dynamical interaction of the phonon gas with the kink consists in selecting small-wave-number phonons [see formulas (35) and (36) and Ref. 2]. The monochromatic assumption: $k \sim 0$ leads to the so-called adiabatic kink velocity (53).

(iii) The exact procedure consists in taking into account the interaction of *all* phonon waves, generated by the external field $\mathcal{X}$, with the kink. It leads to the rather important correction of about 20% of formula (53).

Note added in proof

There has recently been some controversy about the concept of a Newtonian kink dynamics (Refs. 24–26). Therefore, we here emphasize that the title and the content of Ref. 1 are related to the basic statement developed in Refs. 3 and 4, namely the soliton motion is measured, *for all time* (no matter how short or large), by Eq. (34). This statement is clearly erroneous on short time scales, as shown by Refs. 1 and 2, and by the present paper. Since it first introduced the concept of a Newtonian kink dynamics, we adopted in Ref. 1 the opposite view to it. Now we do recognize that "may" should be more appropriate to our recent knowledge of the problem than "do," in the expression of the title of Ref. 1.

The present paper is an effort to go beyond this discrepancy in the definitions and to present new effects in the frame of an appropriate time scale, comparable to the time scale defining coherent wave-wave coupling effects in nonlinear optics or in plasma physics. On the contrary to an elementary time scale obtained by considering the time t_0 required for a typical signal to cross the width of a soliton (Ref. 26), our time scale takes into account the efficiency of the perturbation in deforming and/or accelerating the soliton during (roughly) a characteristic period of the excited phonon waves. As an example, consider the case of a weak ($\mathcal{X}\sim 1$) periodic force $\mathcal{X}(t)=\mathcal{X}\cos\omega_0 t$ (cf. III in Ref. 2). We numerically checked that the characteristic time in which the soliton deviates from a Newtonian law [i.e., from a *motion* described by (34)] is *much larger* than t_0 (by at least a factor of 10, since t_0 is of order unity), as soon as the forcing frequency ω_0 becomes "great" (i.e., $\omega_0>1$).

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APPENDIX A: THE "SCREW EFFECT"

Consider the following initial condition:

$$\psi(x,0)\equiv\text{const}=a\ll 2\pi\ , \tag{A1}$$

$$\psi_t(x,0)\equiv 0\ . \tag{A2}$$

Then Eq. (16) gives

$$\psi_b(0)=a\int_{-\infty}^{\infty}f_b(x')dx'=\frac{1}{\sqrt{2}}\pi a\ , \tag{A3}$$

$$\psi_k(0)=a\int_{-\infty}^{\infty}f_k^*(x')dx' \tag{A4}$$

and Eqs. (34)–(36) read

$$\psi_b(t)=\frac{\pi}{\sqrt{2}}\left(\tfrac{1}{2}\mathcal{X}t^2+a\right)\ , \tag{A5}$$

$$\psi_k(t)=\left[a\cos\omega_k t+\mathcal{X}\frac{1-\cos\omega_k t}{\omega_k^2}\right]\times\int_{-\infty}^{\infty}f_k^*(x')dx'\ . \tag{A6}$$

Then, for $t\gtrsim 0$ and in the adiabatic approximation $\omega_k\simeq 1$ (cf. Ref. 2):

$$\psi_k(t)\simeq[a+\tfrac{1}{2}t^2(\mathcal{X}-a)]\int_{-\infty}^{\infty}f_k^*(x')dx'\ . \tag{A7}$$

Equation (16) implies

$$\psi(x,t)\underset{t\to 0}{\simeq}\frac{\pi}{\sqrt{2}}\left(\tfrac{1}{2}\mathcal{X}t^2+a\right)f_b(x)+\left[\tfrac{1}{2}(\mathcal{X}-a)t^2+a\right]\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dk f_k(x)f_k^*(x')\ . \tag{A8}$$

By use of the completeness equation (17), we obtain

$$\psi(x,t) \underset{t\to 0}{\simeq} a+\tfrac{1}{2}(\chi-a)t^2+\frac{\pi a t^2}{2\sqrt{2}}f_b(x) \ . \quad \text{(A9)}$$

This result clearly shows a modification of the profile, described by both first terms on the rhs of (A9), and a shift, at least for small times and within the adiabatic approximation, leading to an error of about 20% (cf. Sec. III). The shift of the solitary wave according to the third term of (A9) leads to a trajectory which is indeed of the Newtonian type since the velocity of the wave is

$$V=-\tfrac{1}{4}\pi\alpha a t \ . \quad \text{(A10)}$$

The comparison between the theoretical trajectory and numerical results is shown in Fig. 3. We see that they agree within the 20% error limit due to the adiabatic assumption (A7). Including the $8/\pi^2$ factor due to the nonadiabaticity of the initial kink dynamics [see (55)] leads to $V_{\mathrm{sol}}=2at/\pi$, which fits the numerical results well.

Note that the real force χ disappeared from this kinetic description and has been replaced by the constant a. Hence, even when $\chi=0$, the addition of the perturbation function (A1) and (A2) to the kink profile shifts the kink according to the velocity (A10). When using the very simple and suggestive illustration of the SG kink as a particular configuration of a pendulum chain,[22] result (A10) means that the kink moves ahead when the whole chain is turned according to an angle equal to a. This suggests the mechanism of the screw.

In the case where $\chi\neq 0$ and $a=\chi$, Eq. (A9) shows that the profile remains time independent and the velocity (A10) is Newtonian. This remarkable resonance between the effect of the field χ and the effect of the initial perturbation function $\psi(x,0)\equiv\chi$ has been described in Ref. 25 and explained as being due to the presence of the far wings of the solitary wave in the fundamental energy level $\psi(x,0)=\sin^{-1}\chi\simeq\chi$ (for small χ). The new object—the SG solitary wave defined by (A1) and (A2) in which $a=\chi$—does then follow a Newtonian dynamics, but it is no longer a soliton [see Fig. 3(b)].

Therefore, within the framework of the linear perturbation theory,[2–4] we obtain the following complementary results.

(i) An exact SG soliton accelerated by a constant field does not obey Newtonian dynamics in the coherent phonon-soliton coupling regime.

(ii) A uniformly amplified SG solitary wave does obey Newtonian dynamics in which the force appears to be the amplification constant

$$\psi(x,0)\equiv a \ .$$

Both of these results are a direct consequence of the coherent interaction between the translation mode $f_b(x)$ and the phonon modes $f_k(x)$, described by the completeness relation (17).

APPENDIX B: INAPPROPRIATENESS OF THE FIELD MOMENTUM (11) FOR THE DESCRIPTION OF THE PARTICLELIKE KINK DYNAMICS

Consider

$$u_{tt}-u_{xx}+\sin u=\chi \ , \quad \text{(B1)}$$

$$u(x,0)\equiv u_0=4\tan^{-1}(e^x) \ , \quad \text{(B2)}$$

$$u_t(x,0)=0 \ . \quad \text{(B3)}$$

Define a Hamiltonian density (see Fig. 4)

$$H=\tfrac{1}{2}u_x^2+\tfrac{1}{2}u_t^2+(1-\cos u)-\chi(u-u_0) \ , \quad \text{(B4)}$$

$$\int_{-\infty}^{\infty} H\,dx=8 \ . \quad \text{(B5)}$$

Define

$$\int_{a}^{\infty} H\,dx=-E \ . \quad \text{(B6)}$$

Then

$$\int_{-\infty}^{a} H\,dx=8+E \ . \quad \text{(B7)}$$

Now look for the center-of-mass motion for this solution, related to the field momentum (11):

$$\bar{X}\equiv\tfrac{1}{8}\int_{-\infty}^{\infty} xH\,dx \ . \quad \text{(B8)}$$

It can be shown [cf. (11)–(14) and Ref. 5]:

$$\frac{d^2\bar{X}}{dt^2}=-\tfrac{1}{4}\pi\chi \ . \quad \text{(B9)}$$

However, this is *not* the motion of the cross-hatched soliton part (see Fig. 4). Define

$$\bar{X}_S\equiv\frac{1}{8+E}\int_{-\infty}^{a} xH\,dx \quad \text{(B10)}$$

for the soliton and

$$\bar{X}_T=\frac{1}{-E}\int_{a}^{\infty} xH\,dx \quad \text{(B11)}$$

for the tail. Then

$$\bar{X}=\tfrac{1}{8}[(-E)\bar{X}_T+(8+E)\bar{X}_S] \ . \quad \text{(B12)}$$

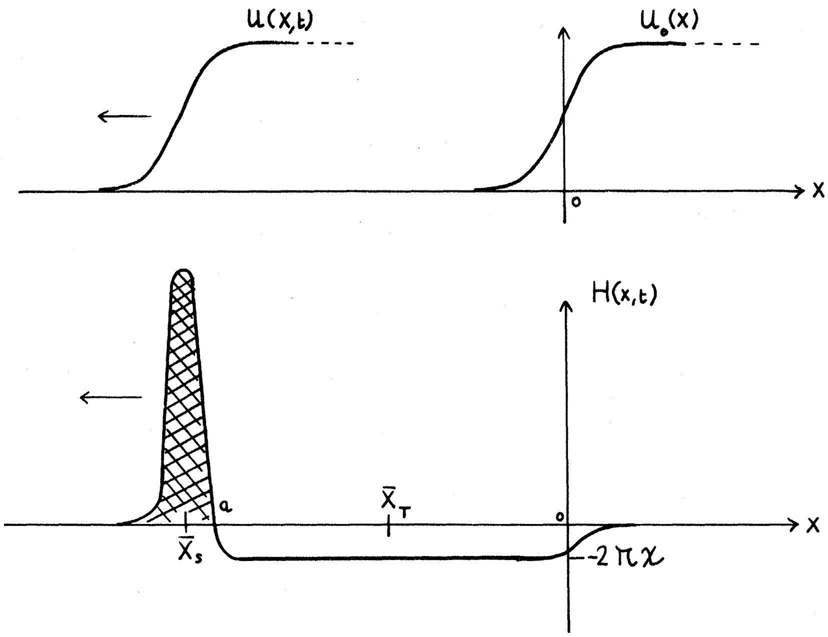

FIG. 4. Hamiltonian density (B4) for a kink at time t.

Now *approximate*

$$\bar{X}_T=\tfrac{1}{2}\bar{X}_S\ , \tag{B13}$$

so

$$\bar{X}=\tfrac{1}{8}(\tfrac{1}{2}E+8)\bar{X}_S\ . \tag{B14}$$

Now

$$E=-2\pi\chi\bar{X}_S\ , \tag{B15}$$

so

$$\bar{X}=(1-\tfrac{1}{8}\pi\chi\bar{X}_S)\bar{X}_S\ , \tag{B16}$$

and

$$\frac{d^2\bar{X}}{dt^2}=\frac{d^2\bar{X}_S}{dt^2}(1-\tfrac{1}{4}\pi\chi\bar{X}_S)-\tfrac{1}{4}\pi\chi\left[\frac{d\bar{X}_S}{dt}\right]^2 . \tag{B17}$$

Thus

$$\frac{d^2\bar{X}_S}{dt^2}(1-\tfrac{1}{4}\pi\chi\bar{X}_S)-\tfrac{1}{4}\pi\chi\left[\frac{d\bar{X}_S}{dt}\right]^2=-\tfrac{1}{4}\pi\chi\ . \tag{B18}$$

This can now be compared with the relativistic equation

$$\frac{d^2\bar{X}_S}{dt^2}=-\tfrac{1}{4}\pi\chi\left[1-\left[\frac{d\bar{X}_S}{dt}\right]^2\right]^{3/2} . \tag{B19}$$

Integrate (B19):

$$\frac{d\bar{X}_S}{dt}\equiv w\ , \tag{B20}$$

$$\frac{dw}{dt}=-\tfrac{1}{4}\pi\chi(1-w^2)^{3/2}\ , \tag{B21}$$

$$\bar{X}_S=0,\quad \frac{d\bar{X}_S}{dt}=w=0\ \text{ at } t=0\ . \tag{B22}$$

Therefore

$$\tfrac{1}{4}\pi\chi\bar{X}_S=1-1\Big/\left[1-\left[\frac{d\bar{X}_S}{dt}\right]^2\right]^{1/2} . \tag{B23}$$

Integrate (B18):

$$\frac{d\bar{X}_S}{dt}=w\ , \tag{B24}$$

$$\frac{dw}{dt}=\left[-\frac{\pi\chi}{4}\right]\frac{1-w^2}{1-\tfrac{1}{4}\pi\chi\bar{X}_S}\ , \tag{B25}$$

then

$$-\left[\frac{\pi\chi}{4}\right]\frac{d\bar{X}_S}{1-\tfrac{1}{4}\pi\chi\bar{X}_S}=\frac{w\,dw}{1-w^2}\ , \tag{B26}$$

$$\ln(1-\tfrac{1}{4}\pi\chi\bar{X}_S)=-\tfrac{1}{2}\ln(1-w^2)+c\ , \tag{B27}$$

$$X_S=0,\quad w=0 \quad \text{at } t=0\ , \tag{B28}$$

therefore

$$c=0\ ,$$

and

$$\tfrac{1}{4}\pi\chi\bar{X}_S=1-1\Big/\left[1-\left[\frac{d\bar{X}_S}{dt}\right]^2\right]^{1/2} . \tag{B29}$$

We recover formula (B23), which describes the trajectory of a relativistic particle. Therefore the particlelike definition (B10), similar to definition (10) of the present paper since it deals with the picture of a particle associated to the kink and located at its center, leads indeed to a relativistic dynamics, while this is obviously not the case for the dynamics related to the center of mass (B8), (B9) related to the field momentum (11).

APPENDIX C: ALTERNATIVE DIRECT CALCULATION OF THE KINK EQUATION OF MOTION (47)

The linearized driven SG equation $\psi_{tt}+\mathscr{O}(\psi)=\chi$ (cf. Introduction), together with definition (15), implies

$$P_{tt}+P=-\tfrac{1}{4}\alpha\pi[\psi_{xt}]_{x=-\infty}^{x=+\infty} -\tfrac{1}{2}\alpha\pi\int_{-\infty}^{\infty}\psi_t \mathrm{sech}^2x\,dx\ . \tag{C1}$$

The first term on the rhs of (C1) can be treated as in Eq. (31) and following. We obtain

$$\lim_{x\to(\pm)\infty}\psi_{xt}\sim(\pm)\frac{2\chi\,\sin t\,\delta(1)}{|x|}=0\ , \tag{C2}$$

where $\delta(y)$ is the Dirac function. Now we use both of the following identities:

$$I_1=\int_{-\infty}^{\infty}\mathrm{sech}^3x\,dx=\frac{\pi}{2}\ , \tag{C3}$$

$$\begin{aligned} I_2&=\int_{-\infty}^{\infty}e^{ikx}\mathrm{sech}^2x\,dx\\ &=\int_{-\infty}^{\infty}\cos kx\ \mathrm{sech}^2x\,dx\\ &=\frac{\pi k}{\sinh\frac{1}{2}\pi k}\ . \end{aligned} \tag{C4}$$

Then we write, by use of the definition (30) of the phonon modes $f_k(x)$,

$$\int_{-\infty}^{\infty}\mathrm{sech}^2x f_k(x)dx=\frac{k}{\sqrt{2\pi}\omega_k}\left[I_2-\frac{i}{2k}\int_{-\infty}^{\infty}e^{ikx}\frac{d}{dx}(\mathrm{sech}^2x)dx\right] . \tag{C5}$$

Integrating by parts,

$$\int_{-\infty}^{\infty}\mathrm{sech}^2x f_k(x)dx=\frac{k}{2\sqrt{2\pi}\omega_k}I_2\ . \tag{C6}$$

Using now Eqs. (16) and (C1)–(C6), we obtain

$$P_{tt}+P=\frac{-\alpha\pi^{3/2}}{4\sqrt{2}}\left[\sqrt{\pi}\psi_{bt}+\int_{-\infty}^{\infty}dk\ \psi_{kt}\frac{k^2}{\omega_k\sinh\frac{1}{2}\pi k}\right] \tag{C7}$$

and we recover Eq. (47).

APPENDIX D: NONADIABATIC KINK VELOCITY AT THE ONSET OF THE INTERACTION

The equation $\psi_{tt}+\mathscr{O}(\psi)=\chi$ gives for $t\gtrsim 0$ the second-order solution with respect to the small parameter t^2:

$$\psi^{(2)}(x,t)=\chi\left[\frac{t^2}{2}-\frac{t^4}{24}\right]+\tfrac{1}{12}\chi t^4\mathrm{sech}^2x\ . \tag{D1}$$

This equation shows that, as already pointed out in Sec. III, the dynamical response of the kink to the external field χ is not adiabatic. Indeed, such an adiabatic reaction would require a second term on the rhs of (D1) proportional to $\partial u_0/\partial x\sim\mathrm{sech}x$, instead of sech^2x. Hence, the nonzero wave number phonons, which are responsible for this nonadiabaticity, as already explained in Ref. 2, will play an important role.

This effect is illustrated in Eq. (55). In order to derive it from (D1), we first calculate the generalized momentum P defined by (27) and associated with the solution (D1). It becomes

$$P=-\tfrac{1}{4}\chi\pi\alpha L\left[t-\frac{t^3}{6}\right]-\tfrac{1}{6}\chi\pi\alpha t^3\,. \qquad \text{(D2)}$$

Identifying the first term on the rhs of (D2) with (38), Eq. (29) shows that $P_{\rm sol}=-\chi\pi\alpha t^3/6$, which leads to formula (55) when the mass deduced from (6) and (26) is taken into account.

Note that definition (10) directly leads to formula (55) when (D1) is used. We have indeed

$$V_{\rm sol}=\frac{\alpha\chi t^3}{6\pi}\int_{-\infty}^{\infty}x\frac{d}{dx}(\operatorname{sech}^2 x)dx =\frac{-\alpha\chi t^3}{3\pi}\,. \qquad \text{(D3)}$$

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