Effective-field model with correlations for Ising systems

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Using the techniques introduced by Honmura and Kaneyoshi, we show that two sets of identities for the two-state Ising model with z nearest neighbors can be used as a basis for various approximation schemes which can systematically include correlation effects. In one dimension the results in zero field are exact for the magnetization, nearest-neighbor correlation function, and critical temperature. By neglecting correlations of more than three sites we introduce a simple approximation in two and three dimensions for the critical temperature. This simple approximation yields results which improve on those of the Bethe approximation. We discuss how the approximation may be improved and extended to other Ising problems.

I. INTRODUCTION

The two-state, or spin-$\frac{1}{2}$ Ising model has been one of the most investigated models of statistical mechanics since its introduction in 1925.[1] This interest is generated by the facts that (i) the model is a simple one relative to other models of cooperative phenomena, (ii) a phase transition is exhibited by the model at finite temperature in two and three dimensions, (iii) exact solutions exist in one[2] and two dimensions,[3] and (iv) the model has a wide range of applicability to real physical systems.

The problem of finding a solution to the three-dimensional Ising model has generated a number of approximation schemes. Also, as new theoretical techniques have been introduced into statistical mechanics, they have been applied to the Ising model in its role as the prototypical statistical system. The work of Domb and others on series expansion techniques[4] and the various cluster approximations[5] typify the former. In the latter category the recent development of the renormalization-group methodology[6] is a prime example. A critical test of any approximation to the three-dimensional Ising model is to be able to reproduce the exact results in one and two dimensions, or to approach these results to an arbitrary degree of accuracy. In three dimensions, approximation schemes have the renormalization-group results and the series expansions as a guide.

The most sensitive test for any of these approximation techniques is the ability to predict the correct transition temperature and critical exponents. Again, in one and two dimensions the exact results are available. However, in three dimensions the results from series expansions or renormalization-group methods must be considered to be the "exact" results. Away from the critical region there is very little to distinguish one approximation scheme from another. Consequently in the present paper we shall be primarily concerned with determining the critical temperature for the various Ising lattices. We will comment only briefly about critical exponents.

In particular, in the present paper we introduce a new and, we believe, potentially very useful scheme for obtaining approximate solutions to the Ising model. The approximation is based on a pair of exact formal identities for the two-state Ising model, and then utilizes the exponential operator technique introduced by Honmura and Kaneyoshi[7] (HK). This procedure yields two sets of equations which are formally the inverses of each other. However, this symmetry is broken under most plausible approximation assumptions. The net result is an approximation scheme which can explicitly and systematically include the effects of correlations.

Since we are primarily interested in comparing the procedure of the present paper with existing approximation schemes, we introduce a simple approximation to the set of equations and solve for the critical temperature. The results of this approximation represent an improvement to the Bethe approximation. In addition it appears that the approximation of the present paper is but the first of many.

In the following section we discuss the set of Ising identities used by HK. In Sec. III we discuss the exponential operator technique used by HK and compare the HK mean-field approximation (MFA) with other approximations. We introduce a second set of Ising identities in Sec. IV and again apply the technique of HK to this set. A simple approximation is made on the two sets of equations in Sec. V and the results are presented for the critical temperatures for various lattices. Finally, in Sec. VI we discuss the limits and usefulness of the procedure.

II. THE SET OF CORRELATION IDENTITIES

During the early 1960's Doman, Tahir-Kheli, and ter Haar[8] noted that the usual commutator Green's-function equations of motion could be formally summed for Ising systems. This resulted in a set of formal identities for the correlation functions for the Ising model of arbitrary spin. Callen[9] pointed out that if anticommutator Green's functions are used the resultant formal identity achieves a much more useful and transparent form, i.e.,

$$\langle \sigma_g \{g\} \rangle = \langle \{g\} \tanh \beta E(g) \rangle \ . \tag{2.1}$$

Here the thermal averages are given as usual by

$$\langle A \rangle = \frac{\mathrm{Tr} e^{-\beta \mathscr{H}} A}{\mathrm{Tr} e^{-\beta \mathscr{H}}} \ , \tag{2.2}$$

and the Hamiltonian is that of the two-state Ising model,

$$\mathscr{H} = -\tfrac{1}{2} \sum_{gp} J_{gp} \sigma_g \sigma_p - h \sum_g \sigma_g \ . \tag{2.3}$$

The inverse temperature is given by $\beta = (k_B T)^{-1}$ and h is an effective field which we set equal to zero. The Ising variables are $\sigma_g = \pm 1$ and J_{gp} is a translationally-invariant effective interaction between sites g and p which we restrict to the z nearest neighbors. The operator $\{g\}$ represents any function of the Ising variables so long as it is not a function of the site g and the energy operator $E(g)$ is given by

$$E(g) = \sum_p J_{gp} \sigma_p + h \ . \tag{2.4}$$

Callen pointed out in his paper that if $\{g\} = 1$ then the usual MFA, with zero external field, follows from (2.1); i.e.,

$$m \equiv \langle \sigma_g \rangle = \langle \tanh \beta E(g) \rangle \simeq \tanh \beta \langle E(g) \rangle \ , \tag{2.5}$$
$$m = \tanh m \beta z J \ .$$

At the same time, Callen also noted that the identity (2.1) could be generated by the method of restricted trace, as had been earlier used by Fisher[10] and was later generalized by Suzuki.[11] The set of identities so generated, i.e., (2.1), is what we call the *set of correlation identities* (SCI).

The SCI has been used off and on since its introduction as the basis for a number of approximation schemes. One of the first of these was Tahir-Kheli's[12] use of the SCI as the basis for a high-temperature series expansion in order to calculate the correlation functions and the critical temperature for the two-level Ising model in an external field. Later, Anderson[13] generated exact differential difference equations for the correlation functions based on Green's-function techniques. The correlation function identities of Doman, Tahir-Kheli, and ter Haar can be obtained from these equations as special cases, and he was also able to show how the MFA and the Bethe-Peierls-Weiss approximation could be obtained from his equations.

In recent years the SCI has appeared more frequently in the literature in connection with improved approximations to the Ising model and particularly with improved estimates of the critical properties.[14] Much of this interest has been prompted by the work of Frank and Mitran[15] who used an integral representation of the SCI to form the basis for approximations to the critical properties of Ising models. Variations and criticisms of their procedures have been discussed by a number of authors.[16,17]

In addition to these approaches to approximate solutions to the Ising problem, there is the recent paper of Tanaka and Uryû[18] on the two-dimensional Ising antiferromagnet. This is one of the few papers in which the arbitrary operator in (2.1), i.e., $\{g\}$, is replaced by a functional form of the Ising variables so as to facilitate the process of obtaining a solution. On a more formal basis, recent work by McCoy, Perk, and Wu[19] on correlation function identities for the two-dimensional Ising model has led to a set of quadratic difference equations for the n-spin correlations.

The primary motivation for the present work is the technique proposed by HK for use with the SCI. This technique produces a form for the SCI which, we believe, is particularly amenable to solution. In its simplest approximation it yields results which are better than the MFA. The HK technique can also be related to the work of Frank and Mitran, as well as be applied to a large range of Ising systems.

III. THE HONMURA-KANEYOSHI TECHNIQUE

HK[7,20] have introduced an exponential operator technique directly on the SCI with $\{g\}=1$ and $h=0$. This is much in the same spirit as the earlier work of Anderson. The result is that (2.1) becomes

$$\langle \sigma_g \rangle = \left\langle \tanh\beta \sum_p J_{gp}\sigma_p \right\rangle \tag{3.1}$$

or

$$\langle \sigma_g \rangle = \left\langle \exp\left[\beta D \sum_p J_{gp}\sigma_p\right] \right\rangle \tanh x \,|_{x=0} \,, \tag{3.2}$$

where

$$D = \frac{\partial}{\partial x} \,. \tag{3.3}$$

With the exponential operator we can write the summation in (3.2) as a product,

$$\langle \sigma_g \rangle = \left\langle \prod_p \exp(\beta D J_{gp}\sigma_p) \right\rangle \tanh x \,|_{x=0} \,, \tag{3.4}$$

and by using the van der Waerden identity for the two-state Ising model finally write

$$\langle \sigma_g \rangle = \left\langle \prod_p (\cosh D\beta J_{gp} + \sigma_p \sinh D\beta J_{gp}) \right\rangle \times \tanh x \,|_{x=0} \,. \tag{3.5}$$

This expression is particularly amenable to systematic approximations. If we restrict ourselves to the z nearest neighbors having an effective interaction J we obtain (setting the central site $g\equiv 0$)

$$\langle \sigma_0 \rangle = \left\langle \prod_{p=1}^{z} (\cosh D\beta J + \sigma_p \sinh D\beta J) \right\rangle \times \tanh x \,|_{x=0} \,. \tag{3.6}$$

This result is entirely equivalent to the SCI with $\{g\}$ set to unity and the restriction of nearest-neighbor interactions only. Applying this result to the linear chain ($z=2$) we get

$$m = 2mA_2(1) \,. \tag{3.7}$$

$A_z(n)$ is the coefficient of the n-point correlation function having z nearest neighbors and in this case is given by

$$A_2(1) = \tfrac{1}{2} \tanh 2\beta J \,. \tag{3.8}$$

Observation of (3.7) readily yields the solution

$$2A_2^c(1) = \tanh 2\beta_c J = 1 \tag{3.9}$$

or

$$(2\beta_c J)^{-1} = \frac{k_B T_c}{2J} = 0 \,, \tag{3.10}$$

i.e., the exact result for the one-dimensional Ising model in zero field.

The above result is certainly encouraging and differs appreciably from the usual MFA result of $k_B T_c / J = z$. However, in order to introduce the HK-MFA we will also look at the honeycomb lattice ($z=3$). In this case (3.6) becomes

$$\langle \sigma_0 \rangle = (\langle \sigma_1 \rangle + \langle \sigma_2 \rangle + \langle \sigma_3 \rangle) \times A_3(1) + \langle \sigma_1\sigma_2\sigma_3 \rangle A_3(3) \,. \tag{3.11}$$

where the coefficients $A_z(n)$ are given in Table I. If we assume translational invariance and also statistical independence of sites we obtain the HK-MFA to (3.11), i.e.,

$$m = 3mA_3(1) + m^3 A_3(3) \,. \tag{3.12}$$

This relation yields the magnetization curve for the honeycomb lattice, and if we linearize it as $m \to 0$ we get the critical condition

$$3A_3^c(1) = 1 \tag{3.13a}$$

or

$$\tfrac{3}{4}(\tanh 3\beta_c J + \tanh \beta_c J) = 1 \,. \tag{3.13b}$$

Solution of (3.13) yields

$$\frac{k_B T_c}{J} = 2.1037 \,, \tag{3.14}$$

as compared to the MFA result of 3.0. We note that (3.13) can be viewed as either a transcendental equation in $\beta_c J$ or a polynomial in $\tanh \beta_c J$.

This result for the critical temperature is an improvement on the usual MFA. It is also the same

TABLE I. Coefficients $A_z(n)$ of the n-site correlation functions in Eq. (3.6) for the zero-field Ising model with z nearest neighbors using the HK technique.

$A_2(1)=2^{-1}(\tanh 2\beta J)$
$A_3(1)=2^{-2}(\tanh 3\beta J+\tanh \beta J)$
$A_3(3)=2^{-2}(\tanh 3\beta J-3\tanh \beta J)$
$A_4(1)=2^{-3}(\tanh 4\beta J+2\tanh 2\beta J)$
$A_4(3)=2^{-3}(\tanh 4\beta J-2\tanh 2\beta J)$
$A_6(1)=2^{-5}(\tanh 6\beta J+4\tanh 4\beta J+5\tanh 2\beta J)$
$A_6(3)=2^{-5}(\tanh 6\beta J-3\tanh 2\beta J)$
$A_6(5)=2^{-5}(\tanh 6\beta J-4\tanh 4\beta J+5\tanh 2\beta J)$
$A_8(1)=2^{-7}(\tanh 8\beta J+6\tanh 6\beta J+14\tanh 4\beta J+14\tanh 2\beta J)$
$A_8(3)=2^{-7}(\tanh 8\beta J+2\tanh 6\beta J-2\tanh 4\beta J-6\tanh 2\beta J)$
$A_8(5)=2^{-7}(\tanh 8\beta J-2\tanh 6\beta J-2\tanh 4\beta J+6\tanh 2\beta J)$
$A_8(7)=2^{-7}(\tanh 8\beta J-6\tanh 6\beta J+14\tanh 4\beta J-14\tanh 2\beta J)$
$A_{12}(1)=2^{-11}(\tanh 12\beta J+10\tanh 10\beta J+44\tanh 8\beta J+110\tanh 6\beta J+165\tanh 4\beta J+132\tanh 2\beta J)$
$A_{12}(3)=2^{-11}(\tanh 12\beta J+6\tanh 10\beta J+12\tanh 8\beta J+2\tanh 6\beta J-27\tanh 4\beta J-36\tanh 2\beta J)$
$A_{12}(5)=2^{-11}(\tanh 12\beta J+2\tanh 10\beta J-4\tanh 8\beta J-10\tanh 6\beta J+5\tanh 4\beta J+20\tanh 2\beta J)$
$A_{12}(7)=2^{-11}(\tanh 12\beta J-2\tanh 10\beta J-4\tanh 8\beta J+10\tanh 6\beta J+5\tanh 4\beta J-20\tanh 2\beta J)$
$A_{12}(9)=2^{-11}(\tanh 12\beta J-6\tanh 10\beta J+12\tanh 8\beta J-2\tanh 6\beta J-27\tanh 4\beta J+36\tanh 2\beta J)$
$A_{23}(11)=2^{-11}(\tanh 12\beta J-10\tanh 10\beta J+44\tanh 8\beta J-110\tanh 6\beta J+165\tanh 4\beta J-132\tanh 2\beta J)$

value obtained recently by Mattis,[21] and previously by Mamada and Takano,[22] who introduce a more sophisticated MFA by utilizing a distribution of molecular fields at a given site. The connection between Mattis's work and that of HK and Frank and Mitran has been discussed in the literature.[23] Interestingly this same result was obtained by Zernike[24] in 1940.

We have listed the values for the critical temperatures obtained using the HK-MFA in Table II for a number of lattices. By way of comparison we have also listed the results of the Bethe approximation, i.e.,

$$\frac{k_B T_c}{J}=\frac{2}{\ln[z/(z-2)]}\,, \tag{3.15}$$

and the exact or high-temperature-series results. In all cases the HK-MFA is an improvement on the usual MFA, but it does not improve on the Bethe approximation.

In concluding this section on the HK technique it should be pointed out that HK have only used one of the SCI. For z nearest neighbors we have the complete SCI:

$$\langle \{g\}\sigma_0\rangle=\left\langle \{g\}\prod_{p=1}^{z}(\cosh D\beta J+\sigma_0\sinh D\beta J)\right\rangle \times \tanh x\,|_{x=0}\,. \tag{3.16}$$

If we let $\{g\}$ be simply Ising variables (as opposed to functions of these variables) we can generate a

TABLE II. Values of the critical temperature $k_B T_c/J$ for the zero-field Ising model with z nearest neighbors. The exact values are taken from Ref. 4.

z	MFA	HK-MFA	Bethe	Here	Exact
2	2.0	0.0	0.0	0.0	0.0
3	3.0	2.1037	1.8205	1.8725	1.5186
4	4.0	3.0898	2.8854	2.6797	2.2692
6	6.0	5.0733	4.9326	4.8896	4.5108
8	8.0	7.0606	6.9521	6.9143	6.3533
12	12.0	11.0445	10.9696	10.6509	9.7952

closed set of correlation identities for clusters having z nearest neighbors.

IV. THE INVERSE SET OF IDENTITIES

When we introduced the SCI in Sec. II we commented that this particular form for the identities was obtained by using anticommutator Green's functions in the equations of motion. The original Ising identities of Doman, Tahir-Kheli, and ter Haar were generated using commutator Green's functions, and the correlation identity corresponding to the two-state Ising model using this approach is

$$\langle \{g\} \rangle = \langle \sigma_g \{g\} \coth \beta E(g) \rangle \ . \tag{4.1}$$

As Callen has pointed out, this expression is the formal inverse of (2.1). Substitution of one into the other leads to a trivial identity. Consequently we call the set of equations (4.1) *the inverse set of correlation identities* (ISCI).

The question to be raised here is whether the ISCI can be used as a basis for approximation using the HK technique. And if so, is the information content the same as the SCI or different? As would be expected if we treat the two sets exactly we do indeed get trivial identities. However, as soon as an approximation is made we break this symmetry, and the two sets of equations can be used to obtain solutions to the Ising model.

In particular if we follow the HK procedure we get

$$\langle \{g\} \rangle = \left\langle \sigma_g \{g\} \prod_p (\cosh D\beta J_{gp} + \sigma_p \sinh D\beta J_{gp}) \right\rangle \times \coth x \,|_{x=0} \ . \tag{4.2}$$

From a mathematical point of view we must be careful of the singularity in $\coth x$ ($x \to 0$). However, this problem can be readily avoided. Thus, if we restrict ourselves to the nearest-neighbor approximation and set $\{g\}=1$ we obtain

$$1 = \left\langle \sigma_0 \prod_p^z (\cosh D\beta J + \sigma_p \sinh D\beta J) \right\rangle \times \coth x \,|_{x=0} \ . \tag{4.3}$$

Applying this to the linear chain we get

$$1 = \langle \sigma_0\sigma_1 + \sigma_0\sigma_2 \rangle \widetilde{A}_2(1) \ , \tag{4.4}$$

where $\widetilde{A}_z(n)$ is the coefficient of the $(n+1)$ point correlation function for z nearest neighbors. This function is obtained from $A_z(n)$ by replacing $\tanh x$ in Table I by $\coth x$. We can immediately solve for the correlation function ρ,

$$1 = 2\rho \widetilde{A}_2(1) \equiv 2\langle \sigma_0\sigma_1 \rangle \widetilde{A}_2(1) \tag{4.5a}$$

or

$$\rho = \tanh 2\beta J \ , \tag{4.5b}$$

which is the exact solution for the linear chain. This result coupled with (3.9) then yields the solution to the one-dimensional Ising model. Thus, by using the first of each of the SCI and the ISCI we are able to obtain information about the system.

As with the SCI, the ISCI also gives a closed set of correlation identities for clusters of z nearest neighbors if $\{g\}$ is replaced by Ising variables. For the linear chain these two sets of identities yield the following relationships:

$$\langle \sigma_0 \rangle = \langle \sigma_0\sigma_1\sigma_2 \rangle \ , \tag{4.6}$$

$$\langle \sigma_0\sigma_1 \rangle = \langle \sigma_0\sigma_2 \rangle \ , \tag{4.7}$$

$$\langle \sigma_1 \rangle = \langle \sigma_2 \rangle \ , \tag{4.8}$$

$$\langle \sigma_1\sigma_2 \rangle = 1 \ , \tag{4.9}$$

$$\langle \sigma_0 \rangle = \langle \sigma_1 \rangle \tanh 2\beta J \ , \tag{4.10}$$

and

$$\langle \sigma_0\sigma_1 \rangle = \tanh 2\beta J \ . \tag{4.11}$$

Besides the correlation identities (4.6)–(4.9), there are two exact equations of particular interest explicitly involving the temperature, i.e., (4.10) and (4.11). One can then conclude that the ISCI can be profitably used in conjunction with the SCI to obtain approximate solutions for the two-state Ising model with $z > 2$. We consider such an approximation for the critical temperature in the next section.

V. THE AVERAGE TRIPLET APPROXIMATION

The average triplet approximation (ATA) proposed here is an indication of one procedure to utilize both sets of correlation identities. We will determine the critical temperature for the various crystal lattices and not be concerned with the correlation functions themselves at this point. Nor will we attempt to determine relationships among the correlation functions for the various lattices. We will use the honeycomb lattice as an example.

In Sec. III we briefly discussed the honeycomb lattice utilizing the HK-MFA. Assuming again translational invariance for the single-site averages the first of the SCI becomes, i.e., (3.12),

$$m = 3mA_3(1) + \tau A_3(3) , \tag{5.1}$$

where we have retained the triplet correlation

$$\tau \equiv \langle \sigma_1\sigma_2\sigma_3 \rangle . \tag{5.2}$$

In the HK-MFA, $\tau = m^3$, and when (5.1) is linearized the critical condition (3.13) is obtained, i.e.,

$$3A_3^c(1) = 1 . \tag{5.3}$$

If we introduce the ISCI for the honeycomb lattice we get

$$1 = 3\langle \sigma_0\sigma_1 \rangle \tilde{A}_3(1) + \langle \sigma_0\sigma_1\sigma_2\sigma_3 \rangle \tilde{A}_3(3) . \tag{5.4}$$

The ATA is obtained by using the next iteration of (5.4) to get a relationship between the single site and triplet correlation functions, but only after we have averaged over (5.4) such that all n site correlations are equivalent. Thus (5.4) would be rewritten as,

$$1 = 3\rho \tilde{A}_3(1) + q\tilde{A}_3(3) , \tag{5.5}$$

where ρ is the pair correlation function and q is the four-site correlation function. In the next iteration we set

$$m = 3\tau \tilde{A}_3(1) + \omega \tilde{A}_3(3) , \tag{5.6}$$

where ω is the five-site correlation.

At this point we neglect the five-site correlation relative to the triplet correlation. This assumption truncates our system of equations and (5.6) becomes

$$m = 3\tau \tilde{A}_3(1)$$

or

$$\tau = \tfrac{1}{3}[\tilde{A}_3(1)]^{-1} m . \tag{5.7}$$

This approximation is also equivalent to assuming that $\tau = m\rho$ in (5.1) and neglecting the second term in (5.5). Substitution of (5.7) back into (5.1) yields

$$m\{1 - 3A_3(1) - \tfrac{1}{3}[\tilde{A}_3(1)]^{-1} A_3(3)\} = 0 ,$$

or the critical condition for $m \to 0$,

$$3A_3^c(1) + \frac{1}{3}\frac{A_3(3)}{\tilde{A}_3(1)} = 1 . \tag{5.8}$$

This result should be compared to that of the HK-MFA, (5.3). The solution of this transcendental equation, or polynomial, yields

$$\frac{k_B T_c}{J} = 1.8725 ,$$

as compared to values of 2.1037, 1.8205, and 1.5186 for the HK-MFA, Bethe approximation, and exact result.

We have followed this procedure for lattices with four, six, eight, and twelve nearest neighbors and the results are listed in Table II. For $z \geq 4$ our values of the critical temperature improve on those of the Bethe approximation. This is in spite of the fact that the ATA is a first-order approximation. We will discuss the approximation in more detail in the next section.

VI. DISCUSSION

From the exact results presented here for the linear chain and from the approximate values obtained for the critical temperature using the ATA, it appears that the two sets of equations can be used as a basis for obtaining approximate solutions to Ising systems. The use of the ISCI at about the same level of approximation as the HK-MFA has led to a marked improvement over that approximation. In fact, Kaneyoshi *et al.*,[25] have recently shown that the first two identities of the SCI in a correlated-effective-field approximation reproduce the Bethe approximation.

One direction to take in improving the approximation of this paper is to take into account the distinct correlations of the same order, e.g., $\langle \sigma_0\sigma_1 \rangle$ and $\langle \sigma_1\sigma_2 \rangle$. If this is done rigorously using the SCI and the ISCI, one finds a closed set of equations which lead to trivial identities. Thus, the problem comes down to finding the maximum approximation for the particular lattice under consideration. The ATA, as with most effective-field models, makes no distinction between the diamond and simple quadratic lattices for instance, but this difference should be able to be obtained from the identities. Matsudaira[14] used the SCI with $\{g\} = 1$ to obtain estimates for $k_B T_c/J$ for the plane triangle and simple cubic lattices. From Table II one can see that the critical temperature obtained for $z = 4$ using the ATA approaches the simple quadratic lattice value of $k_B T_c/J = 2.2692$ and not the diamond lattice value of $k_B T_c/J = 2.7044$. Taggart[26] has recently introduced an approximation to the SCI and ISCI which explicitly considers the influence of the triplet correlations. Values of $k_B T_c/J$ obtained in this approximation are 2.5284 and 2.9796 for the simple quadratic and diamond lattices, respectively.

As we have mentioned, there has been much work done by Frank and Mitran[15] based on an integral representation of the SCI with $\{g\}=1$. This work has led to various other approximations which can be related back to the HK-MFA. Of some interest is the work of Zhang[17] who has applied various correlation decoupling schemes near the critical temperature. One of these schemes leads to the same values for the critical temperature as found by Tahir-Kheli[27] using a second-order random-phase approximation to the Ising model. Zhang's approximations, however, predict a value of zero for the transition temperature in both one and two dimensions. In addition Frank[28] has used the integral representation of the SCI to determine critical exponents for the Ising model in three dimensions.

The formalism of HK has been applied to spin glasses,[29] dilute ferromagnets,[14,30] amorphous ferromagnets,[31] and systems with competing interactions.[32] Consequently, the methods and equations of this paper should be able to be applied to these systems. Also the correlation identities are not restricted to the spin-$\frac{1}{2}$ Ising model. They can be generalized to many spin interactions[33] and to spins greater than one-half.[34] Thus, it should be possible, for instance, to look at the Blume-Capel[35] model ($S=1$) or Ising systems of triplet interactions.

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[1]S. G. Brush, Rev. Mod. Phys. 39, 883 (1967).

[2]C. J. Thompson, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1972), Vol. 1.

[3]H. N. V. Temperley, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1972), Vol. 1.

[4]C. Domb, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1974), Vol. 3.

[5]D. M. Burley, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1972), Vol. 2.

[6]Th. Niemeijer and J. M. J. van Leeuwen, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M S. Green (Academic, London, 1976), Vol. 6.

[7]R. Honmura and T. Kaneyoshi, Prog. Theor. Phys. (Kyoto) 60, 635 (1978).

[8]B. G. S. Doman and D. ter Haar, Phys. Lett. 2, 15 (1962); R. A. Tahir-Kheli, B. G. S. Doman, and D. ter Haar, *ibid.* 4, 5 (1963); B. G. S. Doman, *ibid.* 4, 156 (1963).

[9]H. B. Callen, Phys. Lett. 4, 161 (1963).

[10]M. E. Fisher, Phys. Rev. 113, 969 (1959).

[11]M. Suzuki, Phys. Lett. 19, 267 (1965).

[12]R. A. Tahir-Kheli, Phys. Rev. 188, 1142 (1969); G. B. Taggart and R. A. Tahir-Kheli, Physica 68, 93 (1973).

[13]F. B. Anderson, Phys. Rev. B 3, 3950 (1971).

[14]N. Matsudaira, J. Phys. Soc. Jpn. 35, 1493 (1973).

[15]B. Frank and O. Mitran, J. Phys. C 10, 2641 (1977); 11, 2087 (1978); O. Mitran, *ibid.* 13, L201 (1980).

[16]S. M. Girvin, J. Phys. C 11, L427 (1978); R. Honmura and T. Kaneyoshi, *ibid.* 11, L973 (1978).

[17]H. I. Zhang and A. K. Rajagopal, J. Phys. C 12, L227 (1979); H. I Zhang, *ibid.* 14, 57 (1981); H. I. Zhang, and B. I. Min, *ibid.* 14, 1769 (1981); B. I. Min and H. I. Zhang, *ibid.* 14, 1779 (1981).

[18]Y. Tanaka and N. Uryû, Phys. Rev. B 21, 1994 (1980).

[19]B. M. McCoy, J. H. H. Perk, and T. T. Wu, Phys. Rev. Lett. 46, 757 (1981).

[20]R. Honmura and T. Kaneyoshi, J. Phys. C 12, 3679 (1979).

[21]D. C. Mattis, Phys. Rev. B 19, 4737 (1979).

[22]H. Mamada and F. Takano, J. Phys. Soc. Jpn. 25, 675 (1968).

[23]T. Kaneyoshi, Phys. Lett. 76A, 67 (1980); H. I. Zhang and D. Kim, Phys. Rev. B 21, 4173 (1980).

[24]F. Zernike, Physica 7, 565 (1940).

[25]T. Kaneyoshi, I. P. Fittipaldi, R. Honmura, and T. Manabe, Phys. Rev. B 24, 481 (1981).

[26]G. B. Taggart, J. Appl. Phys. (in press).

[27]R. A. Tahir-Kheli, Prog. Theor. Phys. (Kyoto) 40, 1312 (1968).

[28]B. Frank, J. Phys. C 11, 2095 (1978).

[29]T. Kaneyoshi, J. Mag. Mag. Mater. 15-18, 119 (1980); Phys. Rev. B 24, 2693 (1981).

[30]T. Kaneyoshi, I. P. Fittipaldi, and H. Beyer, Phys. Status Solidi, B 102, 393 (1981); B. Frank and N. Jan, J. Phys. C 12, 3779 (1979).

[31]T. Kaneyoshi and I. P. Fittipaldi, Phys. Status Solidi B 105, 629 (1981).

[32]J. R. L. de Almeida, I. P. Fittipaldi, and F. C. Sá Bar-

reto, J. Phys. C 14, L403 (1981).
[33]G. B Taggart and R. A. Tahir-Kheli, Prog. Theor. Phys. (Kyoto) 46, 1690 (1971).
[34]R. A. Tahir-Kheli, Phys. Rev. 169, 517 (1968); R. A. Tahir-Kheli and G. B. Taggart, Prog. Theor. Phys. (Kyoto) 42, 1265 (1969).
[35]G. B. Taggart, Phys. Rev. B 20, 3386 (1979).