Random symmetry-breaking fields and the XY model

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The two-dimensional classical XY model in a random p-fold symmetry-breaking field is studied using the replica method. For $p > 2\sqrt{2}$ an XY-like phase exists at intermediate values of temperature and weak field. For $p \leq 4$ we are able to describe the transition into the low-temperature glassy continuation of the paramagnetic phase, while for $p > 4$ the results suggest the transition may be first order, driven by the unbinding of vortices. Several new fixed points and lines are found in the replicated Kosterlitz-Thouless-type recursion relations corresponding to these various transitions. The method we use considers n coupled XY models from which we construct a Coulomb gas with $n(n-1)/2$ types of $(n-1)$-dimensional vector charges.

I. INTRODUCTION

The present research investigates the XY model coupled to a random p-fold symmetry-breaking field, a problem closely related to the effect of substrate randomness on layers of adsorbed atoms. The replica method and renormalization-group methods are used in the analysis. This problem has previously been considered by Houghton *et al.*[1] Here we extend their analysis to derive a full set of renormalization-group equations for the theory, permitting an investigation of the nature of the phase transition to the low-temperature disordered phase. According to our analysis, the pure *Gaussian* model with this type of disorder may have a low-temperature "glass" phase.[2]

The model we study is determined by the temperature-reduced Hamiltonian

$$H=\sum_{\langle \vec{R},\vec{R}'\rangle} K\{1-\cos[\theta(\vec{R})-\theta(\vec{R}')]\} + \sum_{\vec{R}} \tilde{h}_p \cos[p\theta(\vec{R})+\phi(\vec{R})] , \tag{1.1}$$

where $\vec{R}$ and $\vec{R}'$ define nearest-neighbor pairs on a square lattice of lattice constant unity, and $\phi(\vec{R})$ is a random field which is uncorrelated between sites:

$$\langle\langle \exp\{ i[\phi(\vec{R})-\phi(\vec{R}')]\}\rangle\rangle = \delta_{\vec{R},\vec{R}'} . \tag{1.2}$$

The double angle brackets indicate an average over the random variables. We take $\phi(\vec{R})$ to have a uniform distribution, but our results are true for any distribution for which $\langle\langle e^{i\phi}\rangle\rangle = 0$.

We would like to analyze the free energy F, where

$$F = k_B T \langle\langle \ln Z \rangle\rangle , \tag{1.3}$$

to describe the effect of quenched impurities. By exploiting the relation[3]

$$\ln(Z) = \lim_{n\to 0} (Z^n - 1)/n , \tag{1.4}$$

n replicas of the partition function Z are constructed, an average over the quenched variables $\phi(\vec{R})$ is performed, and the trace over the thermodynamic variables is taken to find Z^n. Finally, n is set equal to 0, an appropriate prescription if F is an analytic function of n. In this context, the replica method is a bookkeeping device to organize the perturbation expansion, which can also be obtained with more difficulty, without using replicas.

The analysis suggests that in the presence of vortices[4] the phase diagram may be like that shown in Fig. 1 for $p > 2\sqrt{2}$, where regions labeled paramagnetic and glass are not strictly separate phases since one path can connect the two phases without encountering a transition. We are able to analyze the transition into the glassy continuation of the paramagnetic phase in the case $p < 4$ and believe we understand the entire phase diagram for the XY model in a threefold symmetry-breaking field.[5]

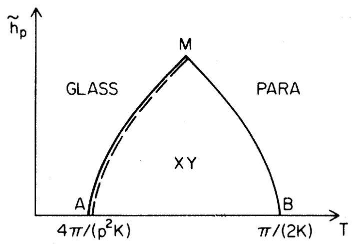

FIG. 1. Phase diagram of the two-dimensional (2D) XY model in a random p-fold symmetry-breaking field of strength $\tilde{h}_p$ is shown as a function of temperature T and $\tilde{h}_p$. For $p>2\sqrt{2}$, there is a region with ordinary XY behavior labeled XY.

When p exceeds 4 the recursion relations do not seem to provide a description of the phase transition, although we believe the phase diagram remains qualitatively as in Fig. 1. There is some indication from the renormalization-group equations that the transition for $p>4$ into the glassy phase may be first order.

An important feature of this research is that it sheds light on the effect of disorder on a model which we believe is understood completely in the absence of disorder.[4–8] Within the replica formulation we are therefore able to make exact predictions which can be checked numerically and perhaps experimentally.

Houghton *et al.*[1] studied the case when the magnitude of $\tilde{h}_p$ as well as the phase was random. By contrast, in this research $\tilde{h}_p$ is fixed while the phase $\phi(\vec{R})$ is uniformly distributed in the interval $0<\phi<2\pi$. The difference with the model of Ref. 1 is expected to be unimportant, but our results significantly extend that analysis.

José has studied the planar model with bond disorder[9] where a fraction x_f of antiferromagnetic bonds were introduced, and x_f was taken to be near 0 or $\frac{1}{2}$. For x_f close to zero, the randomness was found to be unimportant, while for x_f near $\frac{1}{2}$, the randomness seemed to destroy the XY phase. Weak-bond disorder was also considered by Solla and Riedel.[9]

II. EQUIVALENCE TO A VECTOR COULOMB GAS

To proceed, we define the replicated variables $\theta_\alpha(\vec{R})$ and the partition function Z^n by

$$Z^n=\sum_{\{\theta_\alpha(\vec{R})\}}\sum_{\{\phi(\vec{R})\}}\exp\left[-\sum_\alpha[H_\alpha+V_\alpha\{\phi(\vec{R})\}]\right], \tag{2.1}$$

where α is a replica index between 1 and n. The quantities H_α and V_α are simply given by the first and second terms of Eq. (1.1) with a subscript α on $\theta(\vec{R})$. Since V_α is assumed to be small, we can expand $e^{-\sum_\alpha V_\alpha}$ as

$$e^{-\sum_\alpha V_\alpha}=\left[1-\sum_\alpha V+\frac{1}{2}\sum_{\alpha\beta}V_\alpha V_\beta-\cdots\right]. \tag{2.2}$$

The random variable can now be averaged over. Upon re-exponentiating, one finds an effective interaction

$$\sum_{\vec{R}}\tilde{V}(\vec{R})=\frac{1}{2}\tilde{h}_p^2\sum_{\vec{R}}\sum_{\alpha\neq\beta}\cos\{p[\theta_\alpha(R)-\theta_\beta(\vec{R})]\}. \tag{2.3}$$

An overall constant in-replica term and higher cumulants have been ignored. The higher cumulants have an eigenvalue $(2-p^2/\pi\tilde{K})$ [in contrast to $\tilde{h}_p^2$, see Eq. (3.7)], and are therefore irrelevant. Note that the model has become equivalent to n XY models whose relative phases are coupled together. By writing Eq. (2.3) as a product $\cos(p\theta_\alpha)\cos(p\theta_\beta)$ minus the corresponding sine term, it is easy to see that the renormalization-group eigenvalue λ_p for $\tilde{h}_p^2$ is given by $2-\eta$, since each cosine provides a power $-\eta/2$ of R in the correlation function $\langle\tilde{V}(\vec{R})\vec{V}(\vec{O})\rangle$.[5] According to well-known results from the theory of the XY model in the Gaussian limit, η is given by

$$\eta=p^2/(2\pi K). \tag{2.4}$$

The condition that an XY-like phase with algebraic decay of correlations can exist when there are vortices present is then given by the condition

$$2/\pi<K<p^2/4\pi. \tag{2.5}$$

For $K<2/\pi$, vortices destroy the algebraic order, and for $K>p^2/4\pi$ the random field induces correlations which do not decay at large length scales. This implies that for $p>2\sqrt{2}$ there is a portion of the Gaussian line which is stable against both randomness and vortices in the renormalization group. For weak randomness, an intermediate XY-like phase where the randomness is irrelevant may exist, in addition to a high-temperature paramagnetic phase and there may be some sort of paramagnetic or "glass" phase at low temperature. In fact, we will see that the glass phase might not exist as a separate phase from the paramagnetic phase in the presence of vortices.

To proceed further, the full set of renormalization-group equations, including temperature and replica-coupling renormalization by both randomness and vortices, must be considered. Using standard methods[5] to convert the symmetry-breaking field present in Eq. (2.3) to a coupled Coulomb gas, one finds quite easily that

$$Z^n=\prod_{\alpha<\beta}\left[\sum_{n_{\alpha\beta}(\vec{R})=-\infty}^{\infty}\right]\left[\prod_{\gamma,\vec{R}}\int d\theta_\gamma(\vec{R})\right]e^{-H}, \tag{2.6a}$$

$$H=\frac{1}{2}\sum_{\vec{R}}\sum_{\alpha,\beta}K_{\alpha\beta}\vec{\nabla}\theta_\alpha(\vec{R})\cdot\vec{\nabla}\theta_\beta(\vec{R}) +ip\sum_{\vec{R}}\sum_{\alpha<\beta}\{n_{\alpha\beta}(\vec{R})[\theta_\alpha(\vec{R})-\theta_\beta(\vec{R})] +\ln(y_p)n^2_{\alpha\beta}(\vec{R})\}, \tag{2.6b}$$

where $y_p=\exp(-\frac{1}{2}\tilde{h}^2_p)$. Thus there are $n(n-1)/2$ charges $n_{\alpha\beta}$ which take on integer values $-\infty<n_{\alpha\beta}<\infty$. In this expression, vortices are ignored; to lowest order in $\tilde{y}_p$ and the vortex fugacity,[5] it is appropriate to consider vortices and charges separately.

The matrix $K_{\alpha\beta}$ is defined by two constants, K and $\tilde{K}$, through

$$K_{\alpha\beta}=K\delta_{\alpha\beta}+(K-\tilde{K})(1-\delta_{\alpha\beta}). \tag{2.7}$$

Initially $K=\tilde{K}$ (the replicas are not coupled), but an off-diagonal coupling will be generated by the renormalization. Integrating over the fields $\theta_\alpha(\vec{R})$ we obtain the Coulomb gas Hamiltonian whose interacting part is

$$H_C=\frac{p^2}{2\pi}\sum_{\vec{R}\neq\vec{R}'}\sum_{\alpha\beta}\sum_{\gamma\delta}n_{\alpha\beta}(\vec{R})(K^{-1})_{\beta\delta} \times n_{\gamma\delta}(\vec{R}')G(\vec{R}-\vec{R}'), \tag{2.8}$$

where $n_{\alpha\beta}=-n_{\beta\alpha}$ for $\alpha>\beta$ and $G(\vec{R}-\vec{R}')=\ln(|\vec{R}-\vec{R}|/a)$. The matrix K^{-1} is explicitly

$$(K^{-1})_{\beta\delta}=\frac{\delta_{\beta\delta}}{\tilde{K}}-\frac{K-\tilde{K}}{[nK-(n-1)\tilde{K}]\tilde{K}}. \tag{2.9}$$

We see that, since $\sum_{\alpha,\beta}n_{\alpha\beta}=0$, the second term of (2.9) does not contribute to (2.8), which simplifies to

$$H_C=\frac{p^2}{2\pi\tilde{K}}\sum_{\alpha\beta\gamma}\sum_{\vec{R}\neq\vec{R}'}n_{\alpha\beta}(\vec{R})n_{\gamma\beta}(\vec{R}')G(\vec{R}-\vec{R}'). \tag{2.10}$$

Exploiting the antisymmetry of the $n_{\alpha\beta}$, this can be written in the more symmetrical form

$$H_C=\frac{p^2}{4\pi\tilde{K}}\sum_{\vec{R}\neq\vec{R}'}\sum_{\alpha<\beta}\sum_{\gamma<\delta}n_{\alpha\beta}(\vec{R}) \times(\delta_{\alpha\gamma}+\delta_{\beta\delta}-\delta_{\alpha\delta}-\delta_{\beta\gamma}) \times n_{\gamma\delta}(\vec{R}')G(\vec{R}-\vec{R}'). \tag{2.11}$$

The charges of this Coulomb gas define a lattice generated by the elementary lattice vectors $\hat{e}_{\alpha\beta}$ $(\alpha<\beta)$. These vectors are not independent, however. Since

$$\hat{e}_{\alpha\beta}+\hat{e}_{\beta\gamma}=\begin{cases}\hat{e}_{\alpha\gamma} & (\alpha<\gamma)\\ -\hat{e}_{\alpha\gamma} & (\alpha>\gamma)\end{cases} \tag{2.12}$$

the vectors $\hat{e}_{\alpha\beta}$ define an $(n-1)$-dimensional lattice. A general charge can be thought of as a lattice vector,

$$\vec{n}=\sum_{\alpha<\beta}n_{\alpha\beta}\hat{e}_{\alpha\beta}. \tag{2.13}$$

The form of (2.11) suggests that we define an inner product

$$\hat{e}_{\alpha\beta}\cdot\hat{e}_{\gamma\delta}=\frac{1}{2}(\delta_{\alpha\gamma}+\delta_{\beta\delta}-\delta_{\alpha\delta}-\delta_{\beta\gamma}). \tag{2.14}$$

Note that this is consistent with the addition rule (2.12). The Coulomb gas Hamiltonian becomes

$$H_C=\frac{p^2}{2\pi\tilde{K}}\sum_{\vec{R}\neq\vec{R}'}\vec{n}(\vec{R})\cdot\vec{n}(\vec{R}')G(\vec{R}-\vec{R}') +\ln y_p\sum_{\vec{R}}\vec{n}(\vec{R})\cdot\vec{n}(\vec{R}). \tag{2.15}$$

The last term is correct if we restrict ourselves to values of $\vec{n}^2=0$ or 1. Higher values are irrelevant.

The advantage of this formulation is that (2.15) is analogous to the vector Coulomb gas which appears in the theory of two-dimensional melting.[6,7]

In fact, for $n=3$ the vectors $\hat{e}_{12}$, $\hat{e}_{23}$, $\hat{e}_{31}$ are equivalent to the elementary Burgers vectors for a triangular lattice.

III. SCALING EQUATIONS

These observations enable us to work out the renormalization-group recursion relations (see Appendixes A and B) in a manner similar to that used by Young[7] for the melting of a triangular solid. For $n=2$, the results should agree with those for two coupled planar models, considered by Halperin and Nelson in connection with two-dimensional (2D) liquid-crystal phases.[10] These special cases provide a convenient check on our formulas.

When vortices are included, the recursion relations to lowest order in the vortex fugacity y and the periodic charge fugacity $\tilde{y}=p\sqrt{\pi}y_p$ are

$$\frac{dK}{dl}=(n-1)\tilde{y}^2-[K^2+(n-1)(K-\tilde{K})^2]y^2\ , \tag{3.1}$$

$$\frac{d\tilde{K}}{dl}=n\tilde{y}^2-\tilde{K}^2y^2\ , \tag{3.2}$$

$$\frac{d\tilde{y}}{dl}=\left[2-\frac{p^2}{2\pi\tilde{K}}\right]\tilde{y}+\frac{2(n-2)}{p\sqrt{\pi}}\tilde{y}^2\ , \tag{3.3}$$

$$\frac{dy}{dl}=(2-\pi K)y\ . \tag{3.4}$$

A factor of $4\pi^3$ has been absorbed into y^2.

For $n\geq 2$ no new physics emerges. There is a high-temperature paramagnetic phase, and a low-temperature phase where the replicas are locked together. For $p>2\sqrt{2}$, an intermediate XY-like phase can exist.

When $n=0$, the recursion relations reduce to

$$\frac{dK}{dl}=-\tilde{y}^2-\tilde{K}(2K-\tilde{K})y^2\ , \tag{3.5}$$

$$\frac{d\tilde{K}}{dl}=-\tilde{K}^2y^2\ , \tag{3.6}$$

$$\frac{d\tilde{y}}{dl}=\left[2-\frac{p^2}{2\pi\tilde{K}}\right]\tilde{y}-\frac{4}{p\sqrt{\pi}}\tilde{y}^2\ , \tag{3.7}$$

$$\frac{dy}{dl}=(2-\pi K)y\ . \tag{3.8}$$

Note that $d\tilde{K}/dl$ is independent of $\tilde{y}$. We argue in Appendix A that this is true to all orders in $\tilde{y}$.

It is perhaps worthwhile to discuss the relation of these equations to those derived by Houghton *et al.* in Ref. 1. In that work, a double expansion in y and K was made about $y=0$, $2\pi K=4$. Within the framework of that expansion, it was shown that an infinitesimal h_p does not introduce additional divergences at the point $y=0$, $2=\pi K$. Our equations are consistent with those of Ref. 1 close to this point. However, the effect of a *finite* impurity concentration on the renormalization-group equations was never considered in that paper. In fact, the existence of the entire plane of fixed points introduced by $K\neq\tilde{K}$ was overlooked. This is crucial in the analysis if *finite* impurity concentration is to be considered.

We conclude that the analysis of the transition by Houghton *et al.* is only applicable at the point B in Fig. 1, since for finite h_p, K becomes different from $\tilde{K}$ before y and $\tilde{y}$ renormalize to zero. Indeed, near the temperature corresponding to $2\pi K=\frac{1}{2}p^2$, K renormalizes very differently from $\tilde{K}$ since here y is irrelevant, whereas $\tilde{y}$ is marginal. The analysis of Houghton *et al.* can therefore not conclude any properties of this transition (along the line AM in Fig. 1) nor can the multicritical point M be analyzed without considering $K\neq\tilde{K}$.

There are several physical consequences of the fact that $K\neq\tilde{K}$. The first is the nonuniversal value of the exponent η which determines the power-law decay of spin-spin correlations at the transition. Further, it can be shown that the four-point correlation function $C_4(\vec{R}_1,\vec{R}_2,\vec{R}_3,\vec{R}_4)$ given by

$$C_4(\vec{R}_1,\vec{R}_2,\vec{R}_3,\vec{R}_4)=\langle\psi(\vec{R}_1)\psi^*(\vec{R}_2)\psi(\vec{R}_3)\psi^*(\vec{R}_4)\rangle\ ,$$

where

$$\psi(\vec{R})=e^{i\theta(\vec{R})}$$

does not factorize into a product of two-point correlations as it would in the XY phase of a system without impurities.

IV. RANDOMNESS IN THE GAUSSIAN MODEL

If we set $y=0$ to exclude vortices, the resultant model is exactly a Gaussian model with a random field. Note that $\tilde{K}$ is unrenormalized, a result which we argue in Appendix A is true to all orders, and $K=\tilde{K}$ initially. If $\tilde{K}<p^2/4\pi$, $\tilde{y}$ is driven to zero as the length scale is increased. This phase, therefore, has power-law decay of spin-spin correlations at large distances, given by

$$\langle e^{i\theta_\alpha(\vec{R})} e^{-i\theta_\alpha(\vec{0})} \rangle - \langle e^{i\theta_\alpha(\vec{R})} e^{-i\theta_\beta(\vec{0})} \rangle \tag{4.1}$$

when $\alpha \neq \beta$. The second term vanishes in the Gaussian model, and the first term behaves asymptotically as $|\vec{R}|^{-\eta}$ where, from (2.9),

$$\eta = \frac{1}{2\pi \tilde{K}} \left[1 - \frac{K_\infty - \tilde{K}}{\tilde{K}} \right], \tag{4.2}$$

where K_∞ is the renormalized value of K at infinite length scales. Since $dK/dl < 0$, $K_\infty < \tilde{K}$ so that η is not universal at the transition.

One can also study correlations of the Edwards-Anderson order parameter

$$q_{\alpha\beta}(\vec{R}) = e^{i\theta_\alpha(\vec{R}) - i\theta_\beta(\vec{R})} \quad (\alpha \neq \beta), \tag{4.3}$$

$$\langle q_{\alpha\beta}(\vec{R}) q_{\alpha\beta}(\vec{0}) \rangle \propto |R|^{-\bar{\eta}}, \tag{4.4}$$

where $\bar{\eta} = 1/\pi \tilde{K}$ is universal (equal to $4/p^2$) at the transition. For $\tilde{K} \geq p^2/4\pi$, the renormalization group flows to the stable fixed point

$$\tilde{y} = \tilde{y}^* = (p\sqrt{\pi}/4)(2 - p^2/2\pi\tilde{K}), \quad K = -\infty. \tag{4.5}$$

Equations (3.5)–(3.8) are of course valid only to lowest order in $\tilde{y}$, and we can, therefore, only use (4.5) when $\tilde{K} - p^2/4\pi$ is small. However, this does indicate the existence of a line of fixed points emanating from the Gaussian line, similar to the line of fixed points found when a fourfold nonrandom field is present,[5,8] except that the line is now a line of stable fixed points. The fact that K is negative does not signify an instability in the model without vortices since the eigenvalues of the matrix $K_{\alpha\beta}$ remain positive.

For $(\tilde{K} - p^2/4\pi)$ small, the magnetic eigenvalue at this fixed point should be approximately the same as its value on the Gaussian surface $\tilde{y} = 0$. Thus (4.2) should apply; with, however, $K_\infty \to -\infty$. This means that $\eta \to +\infty$, and there are only short-range correlations in the spins. The random field has destroyed the quasi-long-range order present at higher temperatures. However, K_∞ does not enter into the correlations of $q_{\alpha\beta}(\vec{R})$, and we find a finite $\bar{\eta}$ as defined by (4.5).

This situation is quite different from that which occurs for $n \geq 2$, when $\tilde{y}$ becomes large when $\tilde{K} > p^2/4\pi$. In that case, one can argue that $q_{\alpha\beta}$ orders; that is, the replicas are locked together.

V. RANDOMNESS IN THE PRESENCE OF VORTICES

A. General results

We now consider the effect of vortices. Assume $p > 2\sqrt{2}$, that y and $\tilde{y}$ are small, and that the initial values of K and $\tilde{K}$ obey $2/\pi < K = \tilde{K} < p^2/4\pi$. If y is not too large, y flows to zero compared to $\tilde{y}$ after a few length rescalings. Further rescaling iterates $\tilde{y}$ to zero. If $\tilde{K} > p^2/(4\pi)$, however, the recursion relations take us towards the glass fixed point $\tilde{y} = \tilde{y}^*$, $y = 0$. The coupling K then iterates toward zero. Eventually the vortices will become relevant when $K(l) < 2/\pi$. Thus, one naively expects that the low-temperature phase will be characterized by a gradual unbinding of vortices as the length scale is increased further, and the transition into this phase will be continuous. This is not necessarily so, however, because although it is true that $y(l)$ increases when K has dropped below $2/\pi$, the fugacity y has become exponentially small during the preceding iterations, and it does not necessarily become large again before $\tilde{y}$ causes K to iterate to zero. When K falls below zero, the interaction between vortices becomes *repulsive*, signaling an instability at large length scales. We interpret this as a possible signal of a first-order transition.

To find out if y remains small when K attains zero, assume y remains small relative to $\tilde{y}^*$. Then

$$K(l) = K_0 - (\tilde{y}^*)^2 l \tag{5.1}$$

and

$$y(l) = y_0 e^{(2-\pi K_0)l + (1/2)(\tilde{y}^*)^2 l^2}, \tag{5.2}$$

where K_0 is the value of $K(l)$ when y initially becomes small compared to $\tilde{y}$. The condition $K(l^*) = 0$ implies $l^* = K_0/(y^*)^2$. We therefore find that at this length scale

$$y(l^*) = \exp[K_0(2 - \pi K_0/2)/(y^*)^2]. \tag{5.3}$$

At the transition into the glass phase where K_0 is arbitrarily close to $p^2/4\pi$, for $p < 4$, $y(l^*)$ becomes large before an instability at $K = 0$ develops, while if $p > 4$, $K(l)$ is eventually driven to negative values. Therefore, $p = 4$ represents a special value for p above which the low-temperature transition out of the intermediate phase at weak coupling is expected to be first order. The three cases $p = 3$, $p = 4$, and $p \geq 5$ need to be analyzed separately.

B. The case $p \leq 4$

When $p=3$, the fixed point governing the transition into the glass phase at weak coupling is given by

$$K > \frac{2}{\pi}, \quad \tilde{K} = \frac{p^2}{4\pi}, \quad y=0, \quad \tilde{y} \simeq 0 . \tag{5.4}$$

Therefore, when approaching the glass-transition line AM in Fig. 1 at weak coupling from inside the intermediate phase, the critical behavior is like that for the Gaussian model without vortices since y renormalizes to zero. When approaching the same transition from the low-temperature side from the glass phase, there is a crossover length scale ξ_G defined by

$$\xi_G = A_G e^{\text{const}/\tau^2} , \tag{5.5}$$

where $\tau = |T_c - T| / T_c$ measures the (temperature) deviation from the transition. This result is obtained directly from Eq. (5.3) by demanding that $y(l^*)$ be of order unity and noting that y^* is linear in the deviation of the initial parameters from the values they must have to flow to $y^*=0$.

For length scales $L << \xi_G$, the system looks as if it is at the glass fixed point $\tilde{y}=y^*, y=0$ while for $L >> \xi_G$, vortices are unbound and the system is paramagnetic.

The transition at weak randomness to the paramagnetic phase, along BM in Fig. 1, is an ordinary XY transition governed by the fixed point

$$K = \frac{2}{\pi}, \quad \tilde{K} < \frac{p^2}{4\pi}, \quad \tilde{y}=0, \quad y=0 , \tag{5.6}$$

since here $\tilde{y}$ is irrelevant.

Although $K=\tilde{K}$ initially at small length scales, this equality is no longer valid after renormalization. The intermediate XY phase, as in the $(\tilde{h}_p, T)$ phase diagram of Fig. 1, flows under renormalization to the Gaussian region $y=0, \tilde{y}=0$, and $\tilde{K} < p^2/4$, $K > 2/\pi$, so that to understand critical behavior for stronger random coupling it is necessary to investigate the phase diagram at infinitesimal y and $\tilde{y}$, but arbitrary K and $\tilde{K}$.

By examining Eqs. (3.5)–(3.8) one sees that the renormalization-group flows must cross $K=\tilde{K}$ to flow into the region $K<\tilde{K}$. Since initial conditions correspond to $K=\tilde{K}$, the region of the phase diagram as a function of $\tilde{h}_p$ and T which terminates on the fixed surface CMA in Fig. 2 will be finite and topologically equivalent to this triangle. The boundary of the region AMB in Fig. 1 will flow under renormalization to the boundary of the region AMC in the Gaussian plane in Fig. 2. These conclusions could be modified by higher-order terms in the renormalization-group equations.

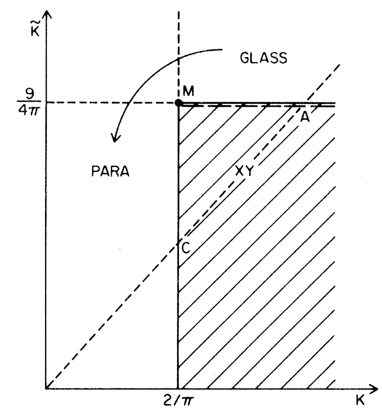

FIG. 2. Infinitesimal fugacity phase diagram as a function of K and $\tilde{K}$, the coupling between vortex and periodic charges is shown for $p \leq 4$. This phase diagram may be topologically equivalent to the $\tilde{h}_p, T$ phase diagram, since the region which is stable against the periodic charges and vortices is the image of the XY phase in the $\tilde{h}_p, T$ phase diagram under renormalization.

Since the quantity $(2K-\tilde{K})$ remains positive along the line CM in Fig. 2, the high-temperature transition line BM in Fig. 1 (which is the renormalization-group pre-image of this line) corresponds to an ordinary XY transition, and likewise the line AM in Fig. 1 which is the pre-image of AM in Fig. 2 has behavior identical to the glass transition described in this section for small $\tilde{h}_p$. The point M in Fig. 1, therefore, maps into the multicritical point

$$y=0, \quad \tilde{y}=0, \quad K = \frac{2}{\pi}, \quad K = \frac{9}{(4\pi)} . \tag{5.7}$$

Although it is possible to analyze the multicritical point M in detail, we have not done so. It suffices to note that the direction of renormalization-group flows projected onto the $K,\tilde{K}$ plane for small y and $\tilde{y}$ are as shown by the arrows in Fig. 2. There is no line of phase transitions emanating from the multicritical point M which would separate what has been labeled the glass and paramagnetic phase in Fig. 1.

When $p=4$, the preceding discussion applies qualitatively. The character of the multicritical point M is changed, however, since the coefficient

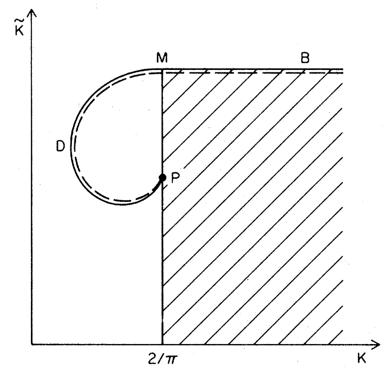

FIG. 3. When $p>4$, the nature of the behavior along the line of xy transition at $K=2/\pi$ changes at point P in the infinitesimal fugacity phase diagram. The separatrix is of imaginary slope from P to M, resulting in a region PDM which flows around PM and maps into the interior of PMB.

$2K-\tilde{K}$ vanishes at this point, but the phase diagram in Figs. 1 and 2 remains applicable.

C. The case $p>4$

When $p>4$, new behavior occurs in the small fugacity phase diagram. As discussed in Sec. V A, the line BM in Fig. 3 to which the parameters flow for a transition out of the glassy phase may be first order. The line AP corresponds to an ordinary XY transition. At the multicritical point P, however, the renormalized coefficient $(2K-\tilde{K})$ vanishes in Eq. (3.5), terminating the line of XY transitions. The line segment PM actually does not correspond to phase transitions. The reason requires some explanation.

Along this line, the recursion relations have the form

$$\frac{dK}{dl}=\alpha y^2 , \tag{5.8}$$

$$\frac{dy}{dl}=(2-\pi K)y , \tag{5.9}$$

where $\alpha>0$. For an ordinary XY transition, α is negative. In that case, as shown in Fig. 4(a), there is an incident separatrix corresponding to the trajectory which flows to the fixed point $K=2/\pi$, $y=0$. Below the separatrix the trajectory flows onto the Gaussian line, while everywhere else the trajectory is expected to flow to large coupling.

When α is positive, the separatrix is of imaginary slope, and the flows circle the point $y=0$,

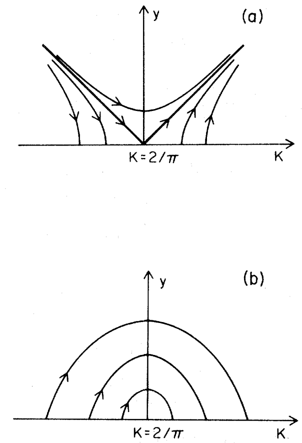

FIG. 4. (a) Flow structure of the ordinary Kosterlitz-Thouless renormalization-group equations is shown as a function of vortex fugacity y and T. Below the separatrix DE the flows eventually terminate on the Gaussian line. (b) When the coefficient of the coupling constant "temperature" renormalization changes sign, the separatrix has imaginary slope and the flows circle the terminus of the vortex-stable region of the Gaussian axis.

$K=2/\pi$ as shown in Fig. 4(b). There is therefore a region PDM in Fig. 3 which flows around the line PM and maps into the triangle PBM. The line PDM maps into the transitions along MB. These considerations lead to a phase diagram as a function of $\tilde{h}_p$ and T as shown in Fig. 5, where the point P in Fig. 3 is the image under renormalization of the point P in Fig. 5.

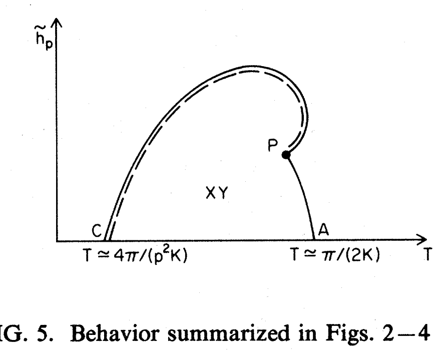

FIG. 5. Behavior summarized in Figs. 2–4 and discussed in more detail in the text results in a phase diagram of the type shown above for $p>4$. The line of phase transitions CP is believed to be first order, while AP is ordinary XY-like. Although the topology of the phase diagram is similar to that shown for $p\leq4$, the character of the point P is different from that of the point M in Fig. 1.

VI. RELATION TO OTHER THEORETICAL PROBLEMS

This section attempts to put the present research into context with other problems in random systems. Of particular interest is the behavior of q-state models in random fields. These models may be obtained by adding uniform fields of the form

$$h_q \sum_{\vec{R}} \cos[q\theta(\vec{R})] \tag{6.1}$$

to the Hamiltonian, and allowing $h_q \to \infty$. Our analysis is performed in the vicinity of the Gaussian model, onto which the q-state model renormalizes only if $q \geq 5$.[5] Thus we expect, for sufficiently large q, this model in p-fold random fields to have a phase diagram of the type shown in Fig. 1 for $p \geq 3$. However, our analysis does not allow us to make any statement about the physically more interesting cases of $q=2$ (Ising model) or $q=3$ (three-state Potts model), and $p=1$ (random field)[11] or $p=2$ (random anisotropy).[12]

From a practical point of view, the XY model is most directly related to superfluid layers, where the angular variable corresponds to the local phase of the condensate wave function. However, in this case, the fields coupling to this phase can have no physical interpretation.

Adsorption experiments provide a more suitable system to which the present model applies. As discussed in Refs. 6 and 13, the XY model in a p-fold symmetry field corresponds to an adsorbed overlayer forming a $1\times p$ unit cell relative to the substrate which is "accidentally commensurate" even with free-boundary conditions. More generally, a two-component generalization of the XY model is needed,[6] but many questions of principle can still be discussed within the present model.

A problem in surface-adsorption experiments is to understand the effect of steps in the surface. If the typical step is high compared to the characteristic length of the adsorbate interactions, the problem requires a finite-size analysis and the present research is not directly relevant. If, however, the height is small compared to the characteristic length, the adsorbed overlayer can be thought of as being stretched out over a bumpy surface without the breaking at the step edges. The net effect of the edge can be modeled by adding a random displacement field corresponding to the necessary displacement vector involved in the stretch across the edge. Of course, this is described by a random variable which is highly correlated out to some characteristic length ξ_R corresponding to the distance across the perfect sections of surface. We may qualitatively understand the effect of extended correlations in the symmetry-breaking field by using the recursion relations for a p-fold nonrandom field up to the length scale ξ_R, thereafter using those for the random field.

At the length scale ξ_R, the renormalized field is

$$\tilde{h}_p(\xi_R)=\tilde{h}_p(0)\xi_R^{(2-p^2/4\pi\tilde{K})}\ , \tag{6.2}$$

where we have assumed that vortices can be neglected, i.e., $\tilde{K} > 2/\pi$. If $p=3$ the exponent will be positive, and if ξ_R is large enough, $\tilde{h}_p(\xi_R)$ will be outside the XY phase of Fig. 1. Thus, a large correlation length of the random field will destroy the "floating solid" phase for $p=3$. If $p\geq 4$, the effect is reversed for $\tilde{K} < p^2/8\pi$, and a large ξ_R will act to extend the floating solid phase boundary to larger values of $\tilde{h}_p$ in the $(T,\tilde{h}_p)$ phase diagram. For $\tilde{K} > p^2/8\pi$, the floating phase will shrink. Also, the smallness of $\tilde{h}_p(\xi_R)$ at the transition will weaken the first-order transition. For large enough ξ_R, the phase diagram will be indistinguishable from that for a nonrandom field.

After this work was completed, papers by Kenway,[14] and Dotsenko and Feigelman[15] have appeared. Neither of these include the off-diagonal coupling or the effect of the vortices at low temperatures, which we have shown to be important. There is also a difference between our renormalization-group equations (with $K=\tilde{K}$, $y=0$) and those of Ref. 15. Aharony[16] has considered the effect of p-fold anisotropies for dimensionless $d>2$, to conclude that there is a low-temperature phase with algebraically decaying correlations, as for the case $p=2$.[17]

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APPENDIX A

In this appendix the derivation of the renormalization-group equations for the Hamiltonian in Eq. (2.10) is sketched in the absence of vortices. The exposition assumes that the reader is familiar with the derivation of the scaling equations for the XY model as derived originally by Kosterlitz. Additional details can be obtained from Refs. 4, 7, or 10. The generalization of the scaling equations of the vector Coulomb gas as derived by Young for the melting of a triangular lattice is easily extended to the present case, and we follow his derivation most closely.

We investigate the partition function

$$Z=\sum_{\{N_\alpha\}}{}' \prod_\alpha \left[\frac{y_p^{N_\alpha}}{N_\alpha!}\right]\left[\prod_{\beta=1}^{n(n-1)}\prod_{n_\beta=1}^{N_\beta}\int d^2r_{n_\beta}^{\beta}\right]e^{-H_C}\,. \tag{A1}$$

The $\sum'$ denotes that the sum is only over those configurations which obey charge neutrality

$$\sum_R \vec{n}(\vec{R})=0 \tag{A2}$$

and where H_C is given by Eq. (2.10), and Greek indices label types of vector charges. The configurational sum places unit charges $\hat{e}_\beta$ at the sites $\vec{r}^{\,\beta}_{n_\beta}$. Consider a pair of oppositely charged unit charges $\vec{n}_\alpha$ and $\vec{n}_\beta a$ located at $\vec{R}_\alpha$ and $\vec{R}_\beta$, and let $\vec{R}_{\alpha\beta}$ be defined by

$$\vec{R}_{\alpha\beta}=\vec{R}_\alpha-\vec{R}_\beta\,. \tag{A3}$$

Define the shell to be integrated over in the renormalization by

$$a_0<r<a_0e^{\delta}\,. \tag{A4}$$

Isolating the part $H_{\alpha\beta}$ in H involving this pair of charges we find

$$H_{\alpha\beta}=\frac{p^2}{2\pi\widetilde{K}}\sum_\gamma(\vec{n}_\gamma\cdot\vec{n}_\alpha)\times[G(\vec{R}^{\,\gamma\alpha})-G(\vec{R}^{\,\gamma\beta})]\,, \tag{A5}$$

where $\vec{R}^{\,\gamma\alpha}=\vec{R}^{\,\gamma}-\vec{R}^{\,\alpha}$, etc. Next consider $\vec{R}_\gamma$ far from $\vec{R}_\alpha$ and $\vec{R}_\beta$ compared to a_0 so that it is permissible to expand the argument in the exponential in gradients,

$$H_{\alpha\beta}=\frac{p^2}{2\pi\widetilde{K}}\sum_\gamma \vec{n}_\alpha\cdot\vec{n}_\gamma\left[\vec{a}\cdot\vec{\nabla}G(\vec{R}_{\alpha\gamma})+\tfrac{1}{2}a_ia_j\frac{\partial}{\partial R_i}\frac{\partial}{\partial R_j}G(\vec{R}_{\alpha\gamma})\right], \tag{A6}$$

where $\vec{a}=\vec{R}^{\,\alpha\beta}$. (A summation over repeated indices is implied.)

Upon expanding the exponential to second order in $\vec{a}$, one finds that the only nonzero term is

$$e^{H_{\alpha\beta}}\cong 1+\frac{p^4}{8\pi^2\widetilde{K}^2}\sum_{\gamma\tau}[\vec{a}\cdot\vec{\nabla}G(\vec{R}_{\gamma\alpha})][\vec{a}\cdot\vec{\nabla}G(\vec{R}_{\tau\alpha})](\vec{n}_\gamma\cdot\vec{n}_\alpha)(\vec{n}_\tau\cdot\vec{n}_\alpha)\,. \tag{A7}$$

In order to sum over α, we note that

$$\sum_\alpha(\vec{n}_\gamma\cdot\vec{n}_\alpha)(\vec{n}_\tau\cdot\vec{n}_\alpha)=\tfrac{1}{2}\sum_{m,n}(\delta_{mr}+\delta_{ns}-\delta_{ms}-\delta_{nr})(\delta_{mt}+\delta_{nu}-\delta_{mu}-\delta_{nt}) \tag{A8a}$$

$$=n(\vec{n}_\gamma\cdot\vec{n}_\tau)\,, \tag{A8b}$$

where $\vec{n}_\gamma=\hat{e}_{rs}$, $\vec{n}_\tau=\hat{e}_{tu}$, and $\vec{n}_\alpha=\hat{e}_{mn}$. The summation is over the indices $1\le m,n\le n(n-1)$. We now integrate in the shell and sum over the vectors α to find

$$\sum_\alpha\int_{\delta_\alpha}e^{H_{\alpha\beta}}=2\pi\delta n a_0\left[1+\frac{p^4}{8\pi^2\widetilde{K}^2}\sum_{\gamma\tau}\vec{\nabla}G(\vec{R}_{\alpha\gamma})\cdot\vec{\nabla}G(\vec{R}_{\alpha\nu})(\vec{n}_\gamma\cdot\vec{n}_\tau)\right]. \tag{A9}$$

Using $\nabla^2G=2\pi\delta(\vec{r})$ we find, after integrating over $\vec{R}_\alpha$,

$$y_p^2\sum_\alpha\int_\delta d^2R_{\alpha\beta}\int_\Omega d^2R_\alpha e^{H_{\alpha\beta}}=2\pi\delta y_p^2 n a_0\left[\Omega-\frac{p^4}{4\pi\widetilde{K}^2}\sum_{\gamma\nu}(\vec{n}_\gamma\cdot\vec{n}_\nu)G(R_{\gamma\nu})\right], \tag{A10}$$

where Ω is the system area. The last term is precisely of the form to renormalize the coupling constant $\widetilde{K}$, so that after some rearrangement,

$$\delta(\widetilde{K})=ny_p^2p^2\pi\delta a_0\,, \tag{A11}$$

where $\delta(\widetilde{K})$ denotes the change in the effective value of $\widetilde{K}$. At this point it is worth noting that the renormalization of $\widetilde{K}$ vanishes when $n=0$. *A priori*, it is therefore necessary to calculate the renormalization of $\widetilde{K}$ to higher order in y_p. But we shall argue that the renormalization of $\widetilde{K}$ must vanish at $n=0$ to all orders.

The matrix $K_{\alpha\beta}$ in Eqs. (2.6) and (2.7) has a singly degenerate eigenvalue $\lambda_1 = nK-(n-1)\widetilde{K}$ corresponding to the eigenvector $\hat{\theta}^+$ defined by $\theta_i=\theta_j$ for all i and j, $1 \le i, j \le n$. The eigenvalue $\widetilde{K}$ is $(n-1)$-fold degenerate corresponding to the space of linear combinations of θ_i whose coefficients sum to zero. The eigenvector $\hat{\theta}^+$ does not couple to V, hence λ_1 must remain unrenormalized since this angle field can, in principle, be integrated out of the problem exactly. Doing this integration explicitly complicates the algebra so we have not proceeded by this route. The consequence of this decoupling is that only the renormalization of $\widetilde{K}$ appears explicitly in the equations.

The fact that λ_1 is unrenormalized implies that

$$\frac{dK}{dl}=\frac{(n-1)}{n}\frac{d\widetilde{K}}{dl} . \qquad \text{(A12)}$$

Therefore, if dK/dl did not vanish at $n=0$, we would have an infinite renormalization of K which does not make sense. Our hypothesis that $d\widetilde{K}/dl \le O(n)$ has been checked to the next higher order in $\widetilde{y}$. Evaluation of the coupling constant renormalization to higher than cubic order is rather complicated and we have not checked the result to quartic order. The calculation to cubic order will not be explained in detail here since the results will not be needed in the exposition. It is straightforward to derive the renormalization of the periodic charge fugacity y_p. Upon rescaling the length it is easy to see that to leading order

$$\frac{dy_p}{dl}=\left[2-\frac{p^2}{2\pi\widetilde{K}}\right]y_p , \qquad \text{(A13)}$$

while observing that there are $(n-2)$ ways of forming a given vector charge by summing two others, Eq. (3.3) results after y_p is replaced by the variable $\widetilde{y}$.

APPENDIX B

This appendix discusses the derivation of the renormalization-group equations in the replica representation with vortices present. By considering vortices separately from the random potential it is possible to derive the lowest-order recursion relations as an expansion in the fugacities y and $\widetilde{y}$.

A direct generalization of the method in Ref. 10 yields a Coulomb gas H'_C by considering vortices in the presence of the first term in Eq. (2.6). This Coulomb gas takes the form

$$H'_C=\frac{1}{2}\sum_{\alpha\beta=1}^{n}\sum_{R\neq R'}K_{\alpha\beta}m_\alpha(\vec{R})m_\beta(\vec{R}')G(\vec{R}-\vec{R}') , \qquad \text{(B1)}$$

where $K_{\alpha\beta}$ is given by Eq. (2.7), and the m_α are integer vortex charges.

Instead of repeating the formal derivation in Appendix A and Ref. 10 to obtain recursion relations, it is convenient and illustrative to sketch the derivation in a less rigorous fashion.

The renormalization of the matrix $K_{\alpha\beta}$ occurs due to the screening of distant vortex pairs by tightly bound oppositely charged pairs. Consider a pair of vortices $\pm\alpha$ a large distance apart. This pair is screened by tightly bound α pairs and $(n-1)$ types of different tightly bound γ pairs where $\gamma\neq\alpha$. The interaction α-α is of strength $K\ln(R/a)$, where R is the distance between one of the charges in the tightly bound pair and one of the distant charges. The interaction α-γ is of strength $(K-\widetilde{K})\ln(R/a)$; with these considerations it is easily checked that the renormalization of the diagonal element in $K_{\alpha\beta}$ takes the form

$$\frac{dK}{dl}=-[K^2+(n-1)(K-\widetilde{K})^2]y_p^2 . \qquad \text{(B2)}$$

The renormalization of the interaction between a distant α-β pair comes from the screening by tightly bound α pairs and β pairs and from γ pairs where $\alpha\neq\gamma$ and $\beta\neq\gamma$. The tightly bound α and β pairs together give a factor of $2K(K-\widetilde{K})$ in the renormalization of $K-\widetilde{K}$, while α-γ and β-γ interactions give a factor of $(K-\widetilde{K})^2$ for each γ, resulting in a factor $(n-2)(K-\widetilde{K})^2$ renormalizing $(K-\widetilde{K})$.

Therefore,

$$\frac{d(K-\widetilde{K})}{dl}=-[2K(K-\widetilde{K})+(n-2)(K-\widetilde{K})^2]y^2 \qquad \text{(B3)}$$

and Eq. (3.3) follows.

The derivation of the vortex fugacity renormalization is straightforward since the interactions between α pairs separated by a distance R is equal to $K\ln(R/a)$ so that the Eq. (3.2) follows when $\widetilde{y}=0$.

When both vortices and periodic charges are included, one notes each of these charges must be integrated out in pairs to preserve overall charge neutrality in both vortex and vector charges. Therefore, no terms of the type $y\tilde{y}$ can occur in the renormalization group and, to lowest order, Eqs. (3.1)–(3.4) are the only possibilities for equations which reduce correctly in the limit where either y or $\tilde{y}$ is zero.

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