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Relaxational dynamics of the Edwards-Anderson model and the mean-field theory of spin-glasses

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Langevin equations for the relaxation of spin fluctuations in a soft-spin version of the Edwards-Anderson model are used as a starting point for the study of the dynamic and static properties of spin-glasses. An exact uniform Lagrangian for the average dynamic correlation and response functions is derived for arbitrary range of random exchange, using a functional-integral method proposed by De Dominicis. The properties of the Lagrangian are studied in the *mean-field* limit which is realized by considering an *infinite-ranged* random exchange. In this limit, the dynamics are represented by a stochastic equation of motion of a *single* spin with self-consistent (bare) propagator and Gaussian noise. The low-frequency and the static properties of this equation are studied both above and below T_c. Approaching T_c from above, spin fluctuations slow down with a relaxation time proportional to $|T-T_c|^{-1}$ whereas *at* T_c the damping function vanishes as $\omega^{1/2}$. We derive a criterion for dynamic stability below T_c. It is shown that a stable solution necessarily violates the fluctuation-dissipation theorem below T_c. Consequently, the spin-glass order parameters are the time-persistent terms which appear in both the spin correlations and the local response. This is shown to invalidate the treatment of the spin-glass order parameters as purely static quantities. Instead, one has to specify the manner in which they relax in a finite system, along time scales which diverge in the thermodynamic limit. We show that the *finite*-time correlations decay algebraically with time as $t^{-\nu}$ at all temperatures below T_c, with a temperature-dependent exponent ν. Near T_c, ν is given (in the Ising case) as $\nu(T) \sim \frac{1}{2} - \pi^{-1}(1-T/T_c) + \sigma(1-T/T_c)^2$. A tentative calculation of ν at $T=0$ K is presented. We briefly discuss the physical origin of the violation of the fluctuation-dissipation theorem.

I. INTRODUCTION

The low-temperature properties of magnetic systems with quenched random exchange (spin-glasses) have attracted considerable attention in recent years.[1–3] Much of the theoretical work concentrated on the spin-glass (SG) model introduced by Edwards and Anderson[4] (EA); see Sec. II below. They proposed that the system may undergo a phase transition into a state, in which the local spins S_i are frozen in random directions, which can be described by the "EA order parameter,"

$$q_{\mathrm{EA}} = [\langle S_i \rangle_T^2]_J \ . \tag{1.1}$$

The symbol $\langle\ \rangle_T$ refers to thermal average in a particular system and $[\ ]_J$ stands for averaging over the random exchange. At present, it is still unclear whether such a phase transition actually occurs in a three-dimensional (3D) EA model.[5–8] Monte Carlo simulations[2] clearly show the appearance of a frozen order at low temperatures in a fashion which is very similar to the observed properties of SG's. However, recent exact results[8] for small samples indicate that the frozen order in 2D or 3D systems is a nonequilibrium phenomenon.[2]

Despite this uncertainty, it is useful to investigate the properties of the SG state in the mean-field limit of the EA model. This limit is realized in the infinite-ranged random-exchange model introduced by Sherrington and Kirkpatrick[9,10] (SK). The SK model has a sharp transition at a finite temperature, but the properties of its low-temperature state are highly nontrivial. The model has been extensively studied by the replica method, which is commonly employed in various problems of quenched disorder. The method consists of calculating the average free energy via the average partition function of n replicated systems, and taking the limit $n \to 0$,

$$[\ln Z]_J = \lim_{n\to 0}([Z^n]_J - 1)/n \ . \tag{1.2}$$

In this framework, the EA order parameter is identified as

$$q_{\mathrm{EA}} = q^{\alpha\beta} = [\langle S_i^\alpha S_i^\beta \rangle_T]_J, \ \ \alpha \neq \beta \tag{1.3}$$

where S_i^α, S_i^β denote (Ising) spins of two different replicas. A straightforward application of this method yields[9,10] a mean-field solution in which the SG phase is characterized by a single order parameter $q = q^{\alpha\beta}$ which obeys a simple self-consistent equation. This solution is unstable, however, below T_c (Refs. 11 and 12) and also yields unphysical negative entropy near $T = 0$.[10] In order to alleviate this instability one needs to break the replica symmetry by introducing order parameters $q^{\alpha\beta}$ which depend on (α,β), but there are infinitely many ways of breaking this symmetry, and the replica theory provides neither a clear criterion of which way is correct, nor a physical insight to the meaning of the various order parameters and the broken symmetry.[13–15] Similar difficulties appear in the replica theory of the short-range EA model below T_c.[16]

An alternative approach has been proposed by Thouless, Anderson, and Palmer[17] (TAP), who derived mean-field equations for the local random magnetizations $\langle S_i \rangle_T$. Although the TAP theory shed further light onto the nature of the SG phase, it has its own difficulties. A straightforward perturbative average of the TAP equations yields Sommer's[18] solution, which has some odd features at low temperatures.[18–20] A nonperturbative method of averaging over the TAP equations gives rise to additional solutions but the physical meaning of these solutions is not clear, and there is still no simple way of finding the solution with the lowest energy.[21,22] Also, an extension of the method to short-range models requires knowledge of the properties of the eigenstates of short-range random matrices, many of which are still not available.[19]

Dynamics have been proposed by Ma and Rudnick[23] and subsequently by other authors[24–27] as an alternative approach to the SG problem. The motivation for this is twofold. Firstly, dynamics provide means for calculating average thermodynamic quantities without using the unphysical $n \to 0$ replica limit. Secondly, one would like to understand not only static properties but also the time-dependent features of the SG state, especially since many of the unique low-temperature properties of real SG's are dynamic in nature.

Although previous dynamic theories correctly predicted the mean-field dynamic behavior above and at T_c, as Monte Carlo simulations of the SK model have shown,[10] they failed below T_c in several crucial aspects. Most importantly, they described the static properties of the SG phase by a single EA order parameter which is defined, in a dynamic framework, as

$$q_{\mathrm{EA}} = \lim_{t\to\infty} [\langle S_i(0) S_i(t) \rangle]_J \ . \tag{1.4}$$

Thus they exhibit the same instability that appears in the SK replica solution.[25] Secondly, they arrive at a mean-field-type solution by performing a low-order perturbation calculation in a *short-range* problem. The inadequacy of these approximations even as a proper mean-field theory of SG's, was clearly demonstrated by the fact that they do not distinguish between the cases of random exchange and a local random temperature.[23,24,28]

In this work we present a theory of SG's based on a dynamic approach to the EA model. A brief summary of our results has been reported elsewhere.[29] We use a soft-spin version of the EA model (on a lattice) and define the dynamics of the system by a Langevin equation describing the relaxation of spin fluctuations similar to the time-dependent Ginzburg-Landau model[30] in uniform-spin systems.

A few years ago, De Dominicis[25] showed that the dynamic functional integral method[31–33] of Martin, Siggia, and Rose[31] can be used to average out quenched disorder without using the replica method. Using this formalism we derive a uniform dynamic Lagrangian which generates all average spin correlation and response functions for a general (e.g., short-range) EA model. Of course the price that is paid is the necessity of extracting the statics via a solution of a full dynamic problem. But this "technical" complexity is, in our opinion, worthwhile especially since it turns out that one cannot separate completely the properties of the SG (in the thermodynamic limit) at thermal equilibrium from the nonequilibrium ones.

A study of the properties of the dynamic theory in a short-range EA model will be discussed elsewhere.[34] In the present paper we study its properties in the infinite-range limit where a mean-field solution is exact. We show that, in this limit, the dynamics can be expressed as a stochastic equation of motion of a *single* spin with self-consistent (bare) propagator and noise. Analyzing the equilibrium solution of this equation, we show that the

SG phase is characterized by the appearance of *time-persistent* terms not only in the spin correlations, Eq. (1.4), but also in the average response function, i.e.,

$$\lim_{\omega\to 0} T\chi(\omega) \sim 1 - q_{\rm EA} + \Delta\delta_{\omega,0} \ . \tag{1.5}$$

The breakdown of the Fischer[35] relation $\chi=(1-q_{\rm EA})/T$ in the SG phase has been previously derived by various static studies[13,15,18–22] of the SK model. Here we show that the appearance of Δ gives rise to a nonunique thermodynamic limit of the static solution, unless one specifies the *time dependence* in a *finite* system of the spin correlations (and response functions) along time scales which become infinitely long in the thermodynamic limit. This invalidates the treatment of the "frozen" spin correlations as a single static order parameter, Eq. (1.4), and gives rise to a multitude of order parameters and consequently to infinitely many possible static solutions. Using this approach together with some physically plausible assumptions about the time dependence of q and Δ, a *static* solution has been recently constructed[36] which agrees with Parisi's replica results[14] and seems to be the correct mean-field solution of the SK model.

In this paper we proceed to investigate the dynamic properties of the SG phase for frequencies which are small compared to *microscopic* characteristic frequencies but do not vanish as the size of the system approaches infinity. In this scale, the fluctuation-dissipation theorem holds and we are able to express uniquely the low-frequency properties in terms of the moments of the local frozen magnetization (measured in *finite* time). We derive a criterion for the dynamic stability of the various static solutions. In the case of the SK solution, the criterion for dynamic stability is identical to the Almeida-Thouless[11] replica stability condition. The recently derived mean-field static solution has been shown[36] to satisfy the condition for marginal dynamic stability at all temperatures. This is shown here to lead to a power-law decay in time of spin correlations $\sim t^{-\nu}$ with an exponent ν which acquires a universal value $\frac{1}{2}$ at T_c but *decreases* upon cooling below T_c. In the Ising case ν is given, near T_c, as

$$\nu(T) \sim \tfrac{1}{2} - \pi^{-1}(1-T/T_c) + O(1-T/T_c)^2 \ . \tag{1.6}$$

However, calculation of its value at $T\to 0$ K depends on as yet unknown properties of the static solution at very low T.

The outline of the paper is as follows. In Sec. II we define the relaxational dynamic model and derive the effective uniform dynamic Lagrangian for the EA model. In Sec. III an exact self-consistent local equation of motion is derived for the infinite-ranged case. The dynamic properties of this equation at and above T_c are analyzed in Sec. IV. Section V deals with the equilibrium solutions below T_c. The dynamic properties in the SG phase are studied in Sec. VI. Section VII contains concluding remarks and a brief discussion of the breakdown of the fluctuation-dissipation theorem.

Some of the results of this work, in particular, the identification of the Almeida-Thouless instability as a dynamic instability and the appearance of a zero-frequency singularity in the response function, were also found recently by Hertz *et al.*[37] The averaged dynamic Lagrangian discussed in Sec. III has also been derived very recently by Schuster.[38]

II. THE DYNAMIC MODEL

The EA Hamiltonian is

$$H = -\sum_{\langle ij\rangle} J_{ij} S_i S_j \ , \tag{2.1}$$

where $\langle ij\rangle$ means a sum over nearest-neighbor pairs, and the exchange J_{ij} are random variables with a Gaussian distribution,

$$P(J_{ij}) = (2\pi z/\tilde{J}^2)^{-1/2} \times \exp[-z(J_{ij}-J_0/z)^2/2\tilde{J}^2] \ . \tag{2.2}$$

The spin variables S_i take the values ± 1, and z is the number of nearest neighbors. The disorder is assumed to be quenched so that the average over J_{ij} has to be carried out on physical observables such as the free energy or spin-spin correlations. We consider here a soft-spin version of the EA model defined by

$$\beta H = \tfrac{1}{2}\sum_{\langle ij\rangle} (r_0\delta_{ij} - 2\beta J_{ij})\sigma_i\sigma_j + u\sum_i \sigma_i^4 + \sum_i h_i\sigma_i, \quad \beta = 1/T \ . \tag{2.3}$$

The length of the soft spin σ_i is allowed to vary continously from $-\infty$ to $+\infty$. The parameters r_0 and u are independent of temperature, and h_i is an external magnetic field divided by temperature. The model with fixed spin length is recovered from (2.3) in the limit $r_0\to -\infty$ and $u\to +\infty$, such that their ratio remains finite.

To study the relaxational dynamics of spin glasses, we propose a simple phenomenological Langevin equation,

$$\begin{aligned}\Gamma_0^{-1}\partial_t\sigma_i(t) &= -\frac{\delta(\beta H)}{\delta\sigma_i(t)}+\xi_i(t)\\ &= \sum_j (r_0\delta_{ij}-\beta J_{ij})\sigma_j(t)\\ &\quad +4u\sigma_i^3(t)+h_i(t)+\xi_i(t)\ .\end{aligned} \tag{2.4}$$

The noise ξ_i is a Gaussian random variable with zero mean and variance

$$\langle \xi_i(t)\xi_j(t')\rangle = \frac{2}{\Gamma_0}\delta_{ij}\delta(t-t')\ , \tag{2.5}$$

which ensures the proper equilibrium distribution and also that locally, the *fluctuation-dissipation theorem* (FDT) holds. The physical quantities are products of spin variables averaged over the noise ξ. Of particular interest are the two-spin correlation

$$C_{ij}(t-t') = \langle \sigma_i(t)\sigma_j(t')\rangle \tag{2.6}$$

and the linear-response function

$$G_{ij}(t-t') = \frac{\partial\langle\sigma_i(t)\rangle}{\partial h_j(t')},\quad t>t' \tag{2.7}$$

where $\langle\ \rangle$ means average over ξ.

The Fourier transform

$$G_{ij}(\omega) = \int_0^\infty dt\, e^{i\omega t}G_{ij}(t)\ , \tag{2.8}$$

is an analytic function in the upper-half plane, i.e., $\mathrm{Im}\omega>0$. Its real and imaginary parts obey the Kramers-Kronig relations

$$\begin{aligned}\mathrm{Re}G_{ij}(\omega) &= -\int\frac{d\omega'}{\pi}\frac{\mathrm{Im}G_{ij}(\omega')}{\omega-\omega'}\ ,\\ \mathrm{Im}G_{ij}(\omega) &= \int\frac{d\omega'}{\pi}\frac{\mathrm{Re}G_{ij}(\omega')}{\omega-\omega'}\ ,\end{aligned} \tag{2.9}$$

where the principal value of the integrals has to be taken. The FDT reads in the present context

$$\begin{aligned}C_{ij}(\omega) &= \int_{-\infty}^{+\infty} dt\, e^{i\omega t}C_{ij}(t)\\ &= \frac{2}{\omega}\mathrm{Im}G_{ij}(\omega)\ ,\end{aligned} \tag{2.10}$$

and

$$C_{ij}(t=0) = G_{ij}(\omega=0)\ . \tag{2.11}$$

Using the functional integral formulation of De Dominicis[32] and Janssen *et al.*,[33] we define a generating functional for dynamic correlations and response functions,

$$Z\{J_{ij},l_i,\hat{l}_i\} = \int D\sigma D\hat{\sigma}\exp\left[\int dt\, l_i(t)\sigma_i(t)+i\hat{l}_i(t)\hat{\sigma}_i(t)+L\{\sigma,\hat{\sigma}\}\right]\ , \tag{2.12}$$

where

$$L\{\sigma,\hat{\sigma}\} = \int dt\sum_i i\hat{\sigma}_i(t)\left[-\Gamma_0^{-1}\partial_t\sigma_i(t)-r_0\sigma_i(t)+\beta\sum_j J_{ij}\sigma_j(t)-4u\sigma_i^3(t)-h_i(t)+\Gamma_0^{-1}i\hat{\sigma}_i(t)\right]+V\{\sigma\}\ . \tag{2.13}$$

The term V, which arises from the functional Jacobian, is given by[32,33]

$$\begin{aligned}V &= -\tfrac{1}{2}\int dt\sum_i\frac{\delta^2(\beta H)}{\delta\sigma_i^2}\\ &= -\int dt\sum_i[\tfrac{1}{2}r_0+6u\sigma_i^2(t)]\end{aligned} \tag{2.14}$$

and ensures the proper normalization of Z,

$$Z\{J_{ij},l_i=\hat{l}_i=0\}=1\ . \tag{2.15}$$

As usual, $\int D\sigma D\hat{\sigma}$ means

$$\int_{-\infty}^{+\infty}\prod_{i,t}[d\sigma_i(t)d\hat{\sigma}_i(t)]\ .$$

The cummulants are given as the coefficients in the Taylor expansion of $\ln Z$ in l and $\hat{l}$

$$\left.\frac{\delta^n\delta^m \ln Z}{\delta\hat{l}_1(\hat{t}_1)\cdots\delta l_m(t_m)}\right|_{l_i=\hat{l}_i=0} = \langle i\hat{\sigma}_1(\hat{t}_1)\cdots\sigma_m(t_m)\rangle_c\ . \tag{2.16}$$

The auxiliary field $i\hat{\sigma}_i(t)$, which was introduced by Martin, Siggia, and Rose,[31] acts as a response field $\partial/\partial h_i(t)$. As can be seen from Eqs. (2.12) and (2.13), l_i plays the role of an applied magnetic field, so that all cummulants in (2.16) with $n\geq 1$ are response functions. In particular,

$$\langle i\hat{\sigma}_j(t')\sigma_i(t)\rangle = G_{ij}(t-t')\ .$$

As required by causality, the response functions vanish if any of the $\hat{t}_i$ are larger than *all* t_i. For more details on the general formalism, see Refs. 25, 32, and 33.

The correlations generated by Eq. (2.16) still depend on the random variables J_{ij}. We are, however, interested in averaged quantities. As noted by De Dominicis,[25] since the generating functional is normalized [see Eq. (2.15)], the quenched average is done directly on Z. This is particularly convenient in cases such as ours in which the Lagrangian is linear in the Gaussian random variable (in our case J_{ij}). A straightforward integration then yields

$$[Z]_J \equiv \int \prod dJ_{ij} P(J_{ij}) Z\{J_{ij}\} = \int D\sigma D\hat{\sigma} \exp \Bigg[L_0\{\sigma,\hat{\sigma}\} + \frac{\beta J_0}{z} \sum_{\langle ij \rangle} \int dt\, i\hat{\sigma}_i(t)\sigma_j(t) + 2\frac{\beta^2 \tilde{J}^2}{z} \sum_{\langle ij \rangle} \int dt\, dt' [i\hat{\sigma}_i(t)\sigma_j(t') i\hat{\sigma}_i(t')\sigma_j(t) + i\hat{\sigma}_i(t)\sigma_j(t) i\hat{\sigma}_j(t')\sigma_i(t')] \Bigg] , \tag{2.17}$$

where L_0 is the purely local part of L, i.e.,

$$L_0\{\sigma,\hat{\sigma}\} = \int dt \sum_i [i\hat{\sigma}_i(-\Gamma_0^{-1}\partial_t\sigma_i - r_0\sigma_i - 4u\sigma_i^3 - h_i + i\Gamma_0^{-1}\hat{\sigma}_i) + V\{\sigma\} + i\hat{l}_i\hat{\sigma}_i + l_i\sigma_i] . \tag{2.18}$$

[Note that in deriving (2.17) we use the property $J_{ij} = J_{ji}$.] Thus, integrating out the random exchange, we have introduced a four-spin coupling which is nonlocal in time. This is analogous to the result of averaging of the free energy by the replica method, which generates a four-spin interaction between different replicated systems. The result can be set in a more convenient form (at least for $J_0 = 0$) by using a Gaussian transformation to decouple the four-spin interactions in (2.17) and introducing four auxiliary fields $Q_\alpha^i(t,t')$, $\alpha = 1, \ldots, 4$ which are local in space but not in time. This leads to

$$[Z]_J = \int \prod_{\alpha}^{4} DQ_\alpha^i(t,t') \exp \Bigg[-\frac{z}{\beta^2 \tilde{J}^2} \int dt\, dt' \sum_{i,j} (K^{-1})_{ij} [Q_1^i(t,t')Q_2^j(t,t') + Q_3^i(t,t')Q_4^j(t,t')] + \ln \int D\sigma D\hat{\sigma} \exp L\{\sigma,\hat{\sigma},Q_\alpha\} \Bigg] , \tag{2.19}$$

where K is the short-range matrix ($K_{ij} = 1$ if i,j are nearest neighbors and zero otherwise), and

$$L\{\sigma,\hat{\sigma},Q_\alpha\} = L_0\{\sigma,\hat{\sigma}\} + \tfrac{1}{2} \int dt\, dt' \sum_i [Q_1^i(t,t') i\hat{\sigma}_i(t) i\hat{\sigma}_i(t') + Q_2^i(t,t')\sigma_i(t)\sigma_i(t') + Q_3^i(t,t') i\hat{\sigma}_i(t)\sigma_i(t') + Q_4^i(t,t') i\hat{\sigma}_i(t')\sigma_i(t)] . \tag{2.20}$$

(We have assumed $J_0 = 0$.)

The results (2.17) and (2.18) can serve as a useful starting point for studying the critical properties of the dynamic and the static properties of short-range spin glasses. Furthermore, the same procedure can be used to derive an averaged generating functional for more complicated dynamic processes. Here, we proceed to study the properties of (2.17) in the mean-field limit. In the following, we will assume for simplicity that $J_0 = 0$. Generalizing the results to the $J_0 \neq 0$ case is straightforward.

III. EQUATIONS OF MOTION IN THE MEAN-FIELD LIMIT

The mean-field limit is achieved by considering an infinite-ranged J_{ij}, namely, by taking the number of nearest neighbors, z, equal to the number of spins in the systems, N, similar to the static SK model.[9,10] In the infinite-ranged case, Eq. (2.17) can be reduced (in the $N \to \infty$ limit) to a *local* mean-field equation. To do this we write the quartic interaction in Eq. (2.17) as squares of sums of local quantities,

$$N^{-2} \sum_{i \neq j} i\hat{\sigma}_i(t) i\hat{\sigma}_i(t') \sigma_j(t) \sigma_j(t') = \tfrac{1}{4} N^{-2} \left[\sum_i i\hat{\sigma}_i(t) i\hat{\sigma}_i(t') + \sigma_i(t)\sigma_i(t') \right]^2 - \tfrac{1}{4} N^{-2} \left[\sum_i i\hat{\sigma}_i(t) i\hat{\sigma}_i(t') - \sigma_i(t)\sigma_i(t') \right]^2 + O(1/N) ,$$

and similarly for the second quartic term in Eq. (2.17). Thus, decoupling these squares by a Gaussian transformation yields

$$[Z]_J=\int\prod_{\alpha=1}^{4}DQ_\alpha(t,t')\exp\left[-\frac{N}{\beta^2\tilde{J}^2}\int dt\,dt'[Q_1(t,t')Q_2(t,t')+Q_3(t,t')Q_4(t,t')]\right.$$
$$\left.+\ln\int D\sigma D\hat{\sigma}\exp L\{\sigma,\hat{\sigma},Q_\alpha\}\right], \tag{3.1}$$

where

$$L\{\sigma,\hat{\sigma},Q_\alpha\}=L_0\{\sigma,\hat{\sigma}\}+\tfrac{1}{2}\int dt\,dt'\sum_i[Q_1(t,t')i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')+Q_2(t,t')\sigma_i(t)\sigma_i(t')$$
$$+Q_3(t,t')i\hat{\sigma}_i(t)\sigma_i(t')+Q_4(t,t')i\hat{\sigma}_i(t')\sigma_i(t)]+O(1)\;. \tag{3.2}$$

In the limit $N\to\infty$ the integration over Q_α can be performed using the method of steepest descent, which amounts to substituting for Q_α, their stationary point values,

$$Q_1^0(t,t')=\frac{\beta^2\tilde{J}^2}{2N}\sum_i\langle\sigma_i(t)\sigma_i(t')\rangle\;, \tag{3.3a}$$

$$Q_2^0(t,t')=\frac{\beta^2\tilde{J}^2}{2N}\sum_i\langle i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')\rangle\;, \tag{3.3b}$$

$$Q_3^0(t,t')=\frac{\beta^2\tilde{J}^2}{2N}\sum_i\langle i\hat{\sigma}_i(t')\sigma_i(t)\rangle\;, \tag{3.3c}$$

$$Q_4^0(t,t')=\frac{\beta^2\tilde{J}^2}{2N}\sum_i\langle i\hat{\sigma}_i(t)\sigma_i(t')\rangle\;. \tag{3.3d}$$

The averages at the right-hand side (rhs) of Eqs. (3.3) are calculated with $L\{\sigma,\hat{\sigma},Q_\alpha^0\}$ of Eq. (3.2), leading to self-consistent equations for Q_α^0.

First we discuss Eq. (3.3b). The correlation $\langle\hat{\sigma}\hat{\sigma}\rangle$ has no simple physical meaning and is indeed identically zero in dynamics of ordinary spin systems.[32,33] It can be seen from the structure of Eqs. (3.1) and (3.2) that

$$Q_2^0=\langle\hat{\sigma}\hat{\sigma}\rangle=0 \tag{3.4}$$

is a self-consistent solution in our case as well, at all temperatures. This solution has the property that it preserves the normalization $[Z]_J=1$. On the other hand, the appearance of a term $Q_2^0\sigma\sigma$ in Eq. (3.2) will generate closed loops which will result in $[Z]_J\sim\exp(N\alpha)$, $\alpha\neq0$. Formally, this can happen as a spontaneous symmetry breaking which violates the original normalization in the limit $N\to\infty$ below T_c. Such a solution will probably invalidate *a posteriori* our averaging procedure, (2.17), which relied crucially on the normalization (2.15). Instead, Z will act as a partition function, and one will have to average over $\ln Z$. This situation is very similar and probably closely related to the analysis by De Dominicis *et al.*[22] and Bray and Moore[23] of the mean-field equations of Thouless, Anderson, and Palmer[18] (TAP). Denoting the average number of the solutions of the TAP equations by $[N_s]_J$ they find that while at $T>T_c$, $[N_s]_J=1$, below T_c there are self-consistent solutions which have the property that $[N_s]_J=\exp(N\alpha)$ with nonzero α. These authors interpret $[N_s]_J$ as the number of metastable states and proceed to calculate averages over these states by a replica method. It is also noted that the TAP solutions with $\alpha\neq0$ have been shown to be related to a replica symmetry breaking solution which violates the normalization $\lim_{n\to0}[Z^n]_J=1$, Z being here the static partition function of the SK model.

However, from the dynamic point of view it is clear that the only physical solution is indeed Eq. (3.4). The reason for that is that the introduction of a vertex $Q_2^0(t,t')\sigma(t)\sigma(t')$ in Eq. (3.2) will lead to *violation of causality,* namely, will yield nonzero contributions to $\langle i\hat{\sigma}(t)\sigma(t')\rangle$ with $t>t'$.[39] Thus we adopt here the self-consistent solution (3.4) which in turn ensures the normalization (2.15) and causality. Thereby we are left with a dynamic local Lagrangian,

$$L\{\sigma_i\hat{\sigma}_i\}=L_0\{\sigma_i,\hat{\sigma}_i\}+\frac{\beta^2\tilde{J}^2}{2}\int dt\,dt'[C(t-t')i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')+2G(t-t')i\hat{\sigma}_i(t)\sigma_i(t')]\;, \tag{3.5}$$

where C and G are the average local correlation and response functions,

$$C(t-t')\equiv[\langle\sigma_i(t)\sigma_i(t')\rangle]_J\ ,$$
$$G(t-t')\equiv[\langle i\hat{\sigma}_i(t')\sigma_i(t)\rangle]_J\ , \quad (3.6)$$

which have to be calculated self-consistently with L. Examining the structure of L_0, one notices that the effect of the spin-glass interaction in the mean-field limit is to modify the inverse bare propagator (via the $\beta^2\tilde{J}^2G\hat{\sigma}\sigma$ vertex) as well as the width of the Gaussian noise (via the $\beta^2\tilde{J}^2C\hat{\sigma}\hat{\sigma}$ vertex). Indeed, carrying out the integration over $\hat{\sigma}$ in

$$\int D\sigma_i D\hat{\sigma}_i \exp(L\{\sigma_i\hat{\sigma}_i\})\ ,$$

the generating functional can be again expressed by an equation of motion for σ_i, which is, in Fourier representation,

$$\sigma_i(\omega)=G_0(\omega)[\phi_i(\omega)+h_i(\omega)]-4uG_0(\omega)\int d\omega_1 d\omega_2\sigma_i(\omega_1)\sigma_i(\omega_2)\times\sigma_i(\omega-\omega_1-\omega_2)\ . \quad (3.7)$$

The new effective bare propagator is

$$G_0^{-1}(\omega)=r_0-i\omega\Gamma_0^{-1}-\beta^2\tilde{J}^2G(\omega)\ , \quad (3.8)$$

and the effective noise ϕ is a Gaussian random variable with width

$$\langle\phi_i(\omega)\phi_i(\omega')\rangle=[2\Gamma_0^{-1}+\beta^2\tilde{J}^2C(\omega)]\delta(\omega+\omega')\ . \quad (3.9)$$

The functions $C(\omega)$ and $G(\omega)$ are the Fourier transforms of the autocorrelations and response, (3.6), and must be calculated self-consistently through Eq. (3.7). Note that the effective local equation of motion is non-Markovian: The noise ϕ is not instantaneous and the bare propagator is nonlocal in time.

IV. DYNAMICS FOR $T \geq T_c$

In this section we investigate the low-frequency behavior of response and correlation functions in the paramagnetic phase. There are no time-persistent correlations and the FDT [Eqs. (2.10), (2.11)] between the full response and correlations is expected to hold. Low-frequency spin fluctuations are conveniently characterized by a generalized damping function $\Gamma(\omega)$, defined as

$$\Gamma^{-1}(\omega)=i\frac{\partial G^{-1}(\omega)}{\partial\omega}\ . \quad (4.1)$$

The dynamic-response function obeys a Dyson equation

$$G^{-1}(\omega)=G_0^{-1}(\omega)+\Sigma(\omega)\ , \quad (4.2)$$

with a frequency-dependent self-energy $\Sigma(\omega)$. Above T_c, Eq. (4.2) implies

$$\Gamma^{-1}(\omega)=\frac{\Gamma_0^{-1}+i\dfrac{\partial\Sigma}{\partial\omega}}{1-\beta^2\tilde{J}^2G^2(\omega)}\ . \quad (4.3)$$

The denominator represents the renormalization of the damping function due to the random exchange while $i\partial\Sigma/\partial\omega$ is the further renormalization due to the nonlinear coupling u.

We shall first assume that $\partial\Sigma/\partial\omega$ ($\omega=0$) is finite and discuss the resulting low-frequency behavior. Subsequently we shall show that our assumption is indeed correct. If the low-frequency expansion of the self-energy

$$\Sigma(\omega)\simeq\mathrm{Re}\,\Sigma(0)+i\omega\frac{\partial}{\partial\omega}\mathrm{Im}\,\Sigma(0)\ , \quad (4.4)$$

where

$$\frac{\partial}{\partial\omega}\mathrm{Im}\,\Sigma(0)\equiv\left.\frac{\partial}{\partial\omega}\mathrm{Im}\,\Sigma\right|_{\omega=0}\ ,$$

is inserted into Eq. (4.2), we find

$$\mathrm{Re}\,G(\omega)\simeq\frac{r_0+\mathrm{Re}\,\Sigma(0)}{1+\beta^2\tilde{J}^2|G(\omega)|^2}|G(\omega)|^2\ , \quad (4.5)$$

$$\mathrm{Im}\,G(\omega)\simeq\frac{\Gamma_0^{-1}-\dfrac{\partial}{\partial\omega}\mathrm{Im}\,\Sigma(0)}{1-\beta^2\tilde{J}^2|G(\omega)|^2}|G(\omega)|^2\ . \quad (4.6)$$

In the limit $\omega\to 0$, Eq. (4.6) implies a singularity in the imaginary part of the response function at the transition temperature

$$T_c=\tilde{J}G(0)\ . \quad (4.7)$$

The real part of the response function, i.e., the static susceptibility, remains finite, as is expected at the spin-glass transition. For $T>T_c$ the kinetic coefficient Γ has a finite limiting value as $\omega\to 0$:

$$\Gamma^{-1}(\omega=0)=\frac{\Gamma_0^{-1}+\mathrm{Im}\,\dfrac{\partial\Sigma}{\partial\omega}(0)}{1-\beta^2\tilde{J}^2G^2(0)}\ . \quad (4.8)$$

As T approaches T_c, Γ^{-1} ($\omega=0$) shows critical slowing down,

$$\Gamma^{-1}(0)\propto|\tau|^{-1},\quad \tau\equiv 1-T/T_c \quad (4.9)$$

and spin fluctuations will therefore decay at a rate

$\sim \Gamma G^{-1} \sim |\tau|$. At T_c, we substitute $G^{-1}(\omega) \simeq G^{-1}(0) - i\omega\Gamma^{-1}(\omega)$ in Eq. (4.3), yielding

$$\Gamma(\omega) \sim \omega^{1/2} . \tag{4.10}$$

In order to show that $\partial\Sigma/\partial\omega$ is finite, we construct a dynamic perturbation expansion for $\Sigma(\omega)$ similar to a usual time-dependent Ginzburg-Landau model,[29] with the simplification due to the absence of nonlocal interactions. In renormalized perturbation theory the lowest-order contribution is explicitly given by

$$\frac{\partial\Sigma(0)}{\partial\omega} = 2(12u)^2 \int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi} C(\omega_1)C(\omega_1-\omega_2) \times \frac{\partial}{\partial\omega_2} \mathrm{Im}\, G(\omega_2) . \tag{4.11}$$

Above T_c the frequency integrals are obviously finite. At T_c, $G(\omega) \sim \omega^{1/2}$ and $C(\omega) \sim \omega^{-1/2}$ so that again there is no divergence due to the two integrations over internal frequencies. Since higher order-diagrams contain *at least* two internal integrations of the type of (4.11), they cannot give rise to a divergent $\partial\Sigma/\partial\omega$.

The results (4.9) and (4.10), were found previously by various mean-field approximations to a short-range SG.[23–25] They have also been derived for the SK model using linearized Glauber dynamics. Here we have shown that the nonlinearities do not alter the critical behavior above or at T_c. They do, however, give rise to a finite, temperature-dependent increase of the relaxation rate Γ^{-1} [see Eq. (4.8)]. This is in accordance with the trend seen in the comparison between the Monte Carlo data and the results of the linearized approximation of Ref. 10.

V. STATICS BELOW T_c

The dynamic definition of the EA order parameter is

$$q = \lim_{t\to\infty} [\langle \sigma_i(0)\sigma_i(t)\rangle]_J = \lim_{t\to\infty} C(t) . \tag{5.1}$$

It is convenient to define the *finite-time* part of $C(t)$ as

$$\tilde{C}(t) \equiv C(t) - q . \tag{5.2}$$

According to (3.9), the noise has now a static component which acts as a random static field to generate time-persistent autocorrelations. We then write the noise ϕ, (3.9), as a sum of two Gaussian noises $\phi = f + z$ where f is defined by the finite-time part of the correlations (3.9); i.e.,

$$\langle f(\omega)f(\omega')\rangle = [2\Gamma_0^{-1} + \beta^2\tilde{J}^2\tilde{C}(\omega)]\,\delta(\omega+\omega') , \tag{5.3}$$

and z by the time-persistent part of (3.9),

$$[z(\omega)z(\omega')] = \beta^2\tilde{J}^2 q\delta(\omega)\delta(\omega+\omega') . \tag{5.4}$$

Substituting this in the equations of motion (3.7) it is readily seen that the self-consistent equations for q is

$$q\delta(\omega) = [\langle\sigma(\omega)\rangle^2] , \tag{5.5}$$

where $\langle\ \rangle$ means (here and in the following) average with respect to f keeping z fixed, and [ ] means averaging over the remaining time-persistent noise z.

In order to solve Eq. (5.5), we must determine the relation between $G(\omega)$ and $C(\omega)$. In ordinary phase transitions the *full* response function is related to the finite-time part of C by the FDT which, instead of (2.10), now reads

$$\tilde{C}(\omega) = \frac{2}{\omega}\mathrm{Im}\, G(\omega) . \tag{5.6}$$

The static limit of Eq. (5.6) is the Fischer[35] relation $G(0) = [\langle\sigma^2\rangle] - q$. Equation (5.6) can be self-consistently satisfied in our case also. Assuming that (5.6) holds, using Eqs. (3.8) and (5.4) one obtains

$$\langle f(\omega)f(\omega')\rangle = -2\,\mathrm{Im}\, G_0^{-1}(\omega)/\omega ,$$

which means that the relation (5.6) is indeed obeyed by the bare ($u=0$) correlation and response functions. One then uses the usual diagrammatic expansion to show that the nonlinearity in Eq. (3.7) does not invalidate this relation. Equation (5.6) ensures that the static limit of the solution of (3.7) for $\langle\sigma\rangle$ is exactly the *magnetization* induced in *thermal equilibrium* by a static Gaussian field z. Thus, Eq. (5.5) reads

$$q = \int_{-\infty}^{\infty} \frac{dz}{(2\pi q\beta^2\tilde{J}^2)^{1/2}} \exp(-\tfrac{1}{2}z^2) m^2(z) , \tag{5.7}$$

where

$$m(z) = \frac{\int_{-\infty}^{\infty} d\sigma\, \sigma \exp[-\frac{1}{2}G_0^{-1}(0)\sigma^2 - u\sigma^4 + (z+h)\sigma]}{\int d\sigma \exp[-\frac{1}{2}G_0^{-1}(0)\sigma^2 - u\sigma^4 + (z+h)\sigma]} , \tag{5.8}$$

and in the Ising limit,

$$m(z)=\tanh(z+h)\ . \tag{5.9}$$

This solution, which is identical to the SK solution,[9,10] is, however, unstable below T_c, as will be shown in the next section.

Thus, a stable solution of the mean-field equations necessarily violates the FDT below T_c. This conclusion has been previously reached by various static approaches.[13,15,18–22] As first suggested by Bray and Moore,[13] the violation of the FDT is presumably the consequence of the high degeneracy of the spin-glass free-energy ground states: The FDT describes the response to an external field due to transitions of the system into new ground states in the vicinity of the original one. However, the actual response consists also of transitions among states which are separated by energy barriers which become infinitely high in the $N\to\infty$ limit. Thus, the full dynamic response consists of two parts: A finite-frequency part which describes the response which is "local" in phase space and obeys the FDT and a part which appears only at $\omega=0$ namely only on the infinitely long time scales which characterize the crossing of the energy barriers between the ground states. Accordingly, we write

$$G(\omega)=\tilde{G}(\omega)+\Delta(\omega)\ , \tag{5.10}$$

where $\tilde{G}(\omega)$ is the finite-frequency response which is related to $\tilde{C}(\omega)$ by

$$\tilde{C}(\omega)=\frac{2}{\omega}\operatorname{Im}\tilde{G}(\omega)\ , \tag{5.11}$$

and

$$\Delta(\omega)=\Delta\delta_{\omega,0}\ . \tag{5.12}$$

$\delta_{\omega,0}$ is defined here as an analytic (complex) function of ω whose real part becomes a Kronecker δ in the limit $N\to\infty$. We separate also the bare propagator, Eq. (3.8), into a finite-frequency part

$$\tilde{G}_0^{-1}(\omega)=r_0-i\omega\Gamma_0^{-1}-\beta^2\tilde{J}^2\tilde{G}(\omega)\ , \tag{5.13}$$

and a time-persistent part $G_0^{-1}-\tilde{G}_0^{-1}=-\beta^2\tilde{J}^2\Delta(\omega)$. Substituting this in Eq. (3.7) yields, after straightforward algebra, the following equation of motion:

$$\begin{aligned}\sigma(\omega)=\tilde{G}_0(\omega)&\\ &=\tilde{G}_0(\omega)[f+H+h(\omega)]\\ &\quad-4u\tilde{G}_0(\omega)\int d\omega_1 d\omega_2\sigma(\omega_1)\\ &\qquad\times\sigma(\omega_2)\sigma(\omega-\omega_1-\omega_2)\ ,\end{aligned} \tag{5.14}$$

with

$$H=z(\omega)+\beta^2\tilde{J}^2\Delta\delta_{\omega,0}\sigma(\omega)\ . \tag{5.15}$$

Since, however, the area under the curve $\Delta(\omega)$ is vanishingly small, only the part of $\sigma(\omega)$ which is induced by the static noise z, i.e., $\langle\sigma\rangle$, gives a nonzero contribution to H as can be explicitly checked by inspecting diagrams of $\sigma(\omega)$ generated by (5.14). Thus, H is a static random field which is given, in terms of the Gaussian time-persistent noise z, as

$$H(z)=z+\beta^2\tilde{J}^2\Delta\delta_{\omega,0}\langle\sigma\rangle\ . \tag{5.16a}$$

Finally, we note that, since $\langle ff\rangle=2\operatorname{Im}\tilde{G}_0^{-1}(\omega)/\omega$ [see Eqs. (5.3) and (5.11)], $\langle\sigma\rangle$ is again the magnetization induced in thermal equilibrium by H and h, i.e.,

$$\langle\sigma\rangle=m(z)=\frac{\int d\sigma\,\sigma\exp[-\frac{1}{2}\tilde{G}_0^{-1}(0)\sigma^2-u\sigma^4+(H+h)\sigma]}{\int d\sigma\exp[-\frac{1}{2}\tilde{G}_0^{-1}(0)\sigma^2-u\sigma^4+(H+h)\sigma]}\ , \tag{5.16b}$$

and in the Ising limit,

$$m(z)=\tanh(H+h)\ . \tag{5.16c}$$

With this definition of m, the self-consistent equation for q is still given by (5.7) and that of Δ is

$$1-q+\Delta=\left[\frac{\partial m}{\partial h}\right]\ . \tag{5.17}$$

The presence of Δ modifies via Eq. (5.15) the otherwise Gaussian distribution of the local field H. However, the most important consequence is the nonuniqueness of the solution for H. Since both $\delta_{\omega,0}$ and $\langle\sigma\rangle$ are nonzero only at $\omega=0$, the product $\delta_{\omega,0}\langle\sigma\rangle$ in Eq. (5.16a) is ill defined. Actually, the functions $q\delta(\omega)$ and $\Delta\delta_{\omega,0}$ are limits of functions $q_N(\omega)$ and $\Delta_N(\omega)$ which have a finite width at finite N, and the value of $\delta_{\omega,0}\langle\sigma\rangle$ is determined by the convolution of $\Delta_N(\omega)$ and $q_N(\omega)$. Thus, the static solution in the thermodynamic limit depends on the dynamic properties of the finite system in time scales which approach infinity in the limit $N\to\infty$. For instance, if the frequency width of $\Delta_N(\omega)$ is much larger than that of $q_N(\omega)$, then clearly $\Delta_N(\omega)q_N(\omega)\to\Delta q\delta(\omega)$ and Δ is coupled to the full magnetization m. In such a case, Eqs. (5.16c) and (5.17) read

$$m(z)=\tanh[z+\beta^2\tilde{J}^2\Delta m(z)+h] , \tag{5.18}$$

$$1-q+\Delta=\left[\frac{(1-m^2)}{1-\beta^2\tilde{J}^2\Delta(1-m^2)}\right] , \tag{5.19}$$

which together with Eq. (5.7) completely specify a solution which is identical to Sommer's solution[18,19] of the SK model. On the other hand, if the width of $\Delta_N(\omega)$ is much smaller than that of $q_N(\omega)$, the coupling between $\Delta(\omega)$ and $\langle\sigma\rangle$ is negligible, and the only self-consistent solution is $\Delta=0$ leading back to the SK solution. Physically, however, neither assumption seems to be right. Both the appearance of Δ and the complete decay of the autocorrelations are the results of crossing the barriers between the various ground states; hence it is plausible that at least part of the time scales of $\Delta_N(\omega)$ and $q_N(\omega)$ are of the same order of magnitude. In such a case, they cannot be represented in the thermodynamic limit by a single number. Indeed, adopting this point of view, a static solution has been recently constructed[36] which has many desired properties and seems to be the correct mean-field theory of the SG transition. Here we proceed to analyze the *finite*-time properties below T_c.

VI. DYNAMICS BELOW T_c

In this section we study the dynamic properties of the system on time scales which are very long compared to the "microscopic" time scale (set by Γ_0^{-1}) but are still finite even in the limit $N\to\infty$. These properties are described by $\tilde{C}(\omega)$ and $\tilde{G}(\omega)$ which by definition vary in a frequency scale which is much larger than that of Δ_N or q_N and have a well-defined zero-frequency limit. It is convenient to introduce the "unaveraged" propagator and correlation functions $\tilde{G}(\omega,z)$ and $\tilde{C}(\omega,z)$ which are derived from Eq. (5.14) before averaging over z, namely,

$$\tilde{G}(\omega,z)=\frac{\partial\langle\sigma(\omega)\rangle}{\partial h(\omega)} , \tag{6.1}$$

$$\tilde{C}(\omega,z)=\langle\delta\sigma(\omega)\delta\sigma(-\omega)\rangle , \tag{6.2}$$

where $\delta\sigma(\omega)\equiv\sigma(\omega)-m(z)$. (Recall that $\langle\ \rangle$ refers to averaging only over the fast noise f.) The averaged quantities are then given as $\tilde{G}(\omega)=[\tilde{G}(\omega,z)]$ and $\tilde{C}(\omega)=[\tilde{C}(\omega,z)]$. The diagrammatic expansion of $\tilde{G}(\omega,z)$ and $\tilde{C}(\omega,z)$ is straightforward. We define a self-energy $\Sigma(\omega,z)$ by

$$\tilde{G}(\omega,z)=1/[\tilde{G}_0^{-1}(\omega)+\Sigma(\omega,z)] . \tag{6.3}$$

Differentiating this equation with respect to ω and using Eq. (5.13) yields for the averaged response $\tilde{G}(\omega)$,

$$\frac{\partial\tilde{G}(\omega)}{\partial\omega}\{1-\beta^2\tilde{J}^2[\tilde{G}^2(\omega,z)]\}=\frac{i}{\Gamma_0}[\tilde{G}^2(\omega,z)]-\left[\tilde{G}^2(\omega,z)\frac{\partial\Sigma(\omega,z)}{\partial\omega}\right] . \tag{6.4}$$

In a dynamically stable system, both $\operatorname{Im}\partial\tilde{G}/\partial\omega$ and $-\operatorname{Im}\partial\Sigma/\partial\omega$ are non-negative, in the limit of low frequency. The imaginary part of (6.4) then implies that

$$T^2/\tilde{J}^2-[\tilde{G}^2(0,z)]=T^2/\tilde{J}^2-[(\langle\sigma^2\rangle-\langle\sigma\rangle^2)^2]\geq 0 , \tag{6.5}$$

where the last equality is a consequence of Eq. (5.11). In the Ising case, Eq. (6.5) reads

$$T^2/\tilde{J}^2\geq 1-2q+[m^4] . \tag{6.6}$$

The inequality (6.6) was first derived by Almeida and Thouless[11] as the stability criterion for the SK solution. They have also shown that Eq. (6.6) *with equality* defines a line in the (h,T) plane which separates the high T region, where the SK solution is stable, from the low T "unstable" region. At $h=0$, the SK solution is unstable for all $T\leq T_c$. In fact, near T_c, $q_{\rm SK}\sim\tau+\frac{1}{3}\tau^2$, $[m^4]\sim 3\tau^2$ which yields

$$T^2/\tilde{J}^2-1+2q_{\rm SK}-[m^4]_{\rm SK}\sim-\tfrac{4}{3}\tau^2 .$$

For small fields, the Almeida-Thouless instability line $T_c(h)$ is given by

$$T_c(h)\sim T_c-(\tfrac{3}{4}h^2)^{1/3},\ \ T_c(h)\lesssim T_c . \tag{6.7}$$

On the other hand, near zero temperature the equation of the line is

$$T_c(h)\sim\frac{4\tilde{J}}{3\sqrt{2\pi}}\exp(-h^2/2\tilde{J}^2),\ \ T_c(h)\simeq 0 . \tag{6.8}$$

It should be noted that Eq. (6.6) was found here to be a criterion of dynamic stability not only for the SK solution but for all possible mean-field solutions.

The above dynamic-stability condition is not necessarily a *sufficient* condition for stability. In fact, Sommer's solution[18,19] satisfies the above stability condition. Near T_c it yields

$$T^2/\tilde{J}^2-1+2q-[m^4]\sim\tfrac{4}{3}\tau^2 ,$$

hence it has a positive kinetic coefficient $\Gamma^{-1}(0)$

which diverges as $T \to T_c^-$, like $\Gamma^{-1}(0) \sim \tau^{-2}$ [compare with Eq. (4.9)]. Nevertheless, based on the discussion at the end of the last section we suspect that that solution is still unstable, as is indicated also by the replica stability analysis of De Dominicis and Garel.[20] Indeed, a complete stability analysis must include the variations of the full dynamic Lagrangian of (3.1) and (3.2) and has not yet been completed. We note however that the recent work[36] on the static mean-field solution shows that the correct solution obeys the condition of *marginal* stability

$$T^2/\tilde{J}^2 = [(\langle\sigma^2\rangle - \langle\sigma\rangle^2)^2] \,, \tag{6.9a}$$

or, in the Ising case,

$$T^2/\tilde{J}^2 = 1 - 2q + [m^4] \,, \tag{6.9b}$$

at all $T < T_c$ which implies according to Eq. (6.4), a divergence of $\Gamma^{-1}(0)$ below T_c. This is supported by the Monte Carlo results[10] for the SK model which exhibited algebraic rather than exponential decay of dynamic correlations.

In order to study the finite-time properties of the marginally stable solution, we make the ansatz

$$\tilde{G}(\omega) = \tilde{G}(\omega=0) + \alpha\,|\,\omega\,|^{\nu} + i\gamma\,|\,\omega\,|^{\nu}\operatorname{sgn}(\omega) \,, \tag{6.10}$$

and solve Eq. (6.4) self-consistently for the exponent ν and the constants α and γ. The ratio of the latter is restricted by the Kramers-Kronig relations for the real and imaginary part of $\tilde{G}(\omega)$. Substituting the ansatz (6.10) into Eq. (2.9) we find

$$\frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} = -\frac{2}{\pi}\int_0^\infty \frac{\omega^{\nu-1}}{1+\omega}d\omega = -\frac{2}{\pi}B(\nu, 1-\nu) \,, \tag{6.11}$$

where $B(x,y)$ is the beta function. This equation allows for two solutions

$$\left[\frac{\alpha}{\gamma}\right]_1 = -\tan\frac{\pi\nu}{2} \,,$$

and

$$\left[\frac{\alpha}{\gamma}\right]_2 = -\cot\frac{\pi\nu}{2} \,, \tag{6.12}$$

but only the second one is the correct solution as is shown explicitly below.

To proceed further, we expand the left-hand side (lhs) of Eq. (6.5) for small frequencies

$$2\beta^2\tilde{J}^2\frac{\partial\tilde{G}(\omega)}{\partial\omega}\left[[\tilde{G}_0^{-1}(\omega) - \tilde{G}_0^{-1}(0)][\tilde{G}^3(0,z)] + \omega\left[\frac{\partial\Sigma(\omega,z)}{\partial\omega}\tilde{G}^3(0,z)\right]\right] = \frac{i}{\Gamma_0}[\tilde{G}^2(0,z)] - \left[\tilde{G}^2(0,z)\frac{\partial\Sigma(\omega,z)}{\partial\omega}\right] . \tag{6.13}$$

and examine the perturbation expansion for $\partial\Sigma/\partial\omega$. We first calculate the leading singularity of $\partial\Sigma/\partial\omega$ to lowest order in the coupling u. Later we will sum the contributions to the leading singularity to all orders in u. Some low-order contributions to $\partial\Sigma/\partial\omega$ are shown in Figs. 1 and 2. To lowest order in u^2, only the two diagrams 1(a) and 1(b) contribute to $\partial\Sigma/\partial\omega$. Using the bare response function the contribution of the first diagram is:

$$\frac{\partial\Sigma^a(z,\omega)}{\partial\omega} = -\frac{2}{\pi}(12u)^2\langle\sigma\rangle^2\int_{-\infty}^{\infty} d\omega'\frac{2}{\omega'}\operatorname{Im}\tilde{G}_0(\omega') \times\frac{\partial}{\partial\omega}\tilde{G}_0(\omega-\omega') \,. \tag{6.14}$$

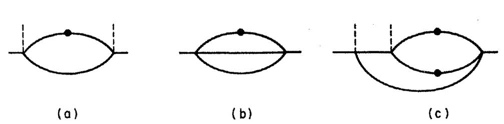

FIG. 1. Some low-order contributions to $\partial\Sigma(\omega,z)/\partial\omega$. A solid line stands for $\tilde{G}_0(\omega)$, a dot for $\langle ff\rangle$, and a dashed line for $\langle\sigma\rangle$.

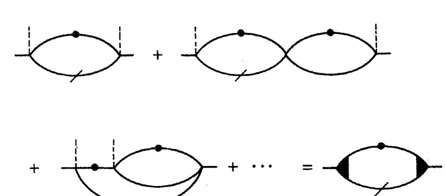

FIG. 2. Most divergent contributions to $\partial\Sigma(\omega,z)/\partial\omega$. A solid line stands for $\tilde{G}(\omega,z)$, line with dot for $C(\omega,z)$, line with slash for $\partial\tilde{G}/\partial\omega$, and dashed line for $\langle\sigma\rangle$. The combinatoric weight of the diagrams is *not* displayed.

For $\nu>\frac{1}{2}$, $\partial\Sigma^a/\partial\omega$ is finite, while for $\nu<\frac{1}{2}$ we find $\partial\Sigma/\partial\omega\sim\omega^{2\nu-1}$, and a logarithmic divergence for $\nu=\frac{1}{2}$. Since $\partial\Sigma/\partial\omega$ diverges for all $\nu\leq\frac{1}{2}$, including $\nu=\frac{1}{2}$, whereas the lhs of Eq. (6.11) is finite for $\nu=\frac{1}{2}$, consistency requires that ν be *smaller* than $\frac{1}{2}$ for all temperatures below T_c. Note that at T_c the prefactor of the divergent integral vanishes. The contribution from the diagram in Fig. 1(b) is finite for $\nu>\frac{1}{3}$ and if it is divergent at all, it is less so than $\partial\Sigma^a/\partial\omega$. For the calculation of the leading singularity of the response function this contribution can be neglected. We then obtain from Eq. (6.14) a self-consistent equation for the exponent ν, correct to lowest order in the coupling constant u. To proceed beyond low-order perturbation we notice that the leading divergence $\omega^{2\nu-1}$ is obtained from diagrams which can be separated into two three-point functions by cutting two internal lines: one a correlation function with "internal" frequency ω' and the other one a propagator with frequency $\omega-\omega'$, as illustrated in Fig. 2. The singularity comes from the frequency dependence of these two lines and, hence the frequencies of all other lines can be set to zero. Thus, the sum of all contributions to the leading singularity of $\partial\Sigma/\partial\omega$ has the same frequency integral as (6.14), but with the renormalized propagators and vertices, i.e.,

$$\frac{\partial\Sigma(\omega,z)}{\partial\omega}\sim-\Gamma_3^2(z)\int\frac{d\omega'}{\pi\omega'}\mathrm{Im}\,\tilde{G}(\omega',z)\times\frac{\partial}{\partial\omega}\tilde{G}(\omega-\omega',z)\,, \tag{6.15}$$

where

$$\Gamma_3(z)=\lim_{\omega_i\to 0}\langle i\hat{\sigma}(\omega_1)i\hat{\sigma}(\omega_2)\delta\sigma(\omega_3)\rangle\tilde{G}^{-1}(\omega_1,z)\times\tilde{G}^{-1}(\omega_2,z)\tilde{G}^{-1}(\omega_3,z)\,. \tag{6.16}$$

All other diagrams are either finite or less divergent. An example is shown in Fig. 1(c). The inserted bubble diverges (after taking the derivative) as $\omega^{2\nu-1}$. However, the full diagram gives a contribution of $O(\omega^{4\nu-1})$ which is finite if $\nu>\frac{1}{4}$ and in any case, less divergent than $\omega^{2\nu-1}$.

Note that the self-energy itself (not its derivative) is less divergent than $\tilde{G}(\omega)$ [or $\tilde{G}_0(\omega)$]; hence, the low-frequency limit of $\tilde{G}(\omega,z)$ is [see Eqs. (3.8) and (6.3)],

$$\tilde{G}(\omega,z)\sim\tilde{G}(0,z)+\beta^2\tilde{J}^2\tilde{G}^2(0,z)\times[\alpha|\omega|^\nu+i\gamma|\omega|^\nu\mathrm{sgn}(\omega)]\,. \tag{6.17}$$

Substituting Eq. (6.17) in Eq. (6.15) we obtain for the leading divergence of $\partial\Sigma/\partial\omega$

$$\begin{aligned}\mathrm{Im}\frac{\partial\Sigma(\omega,z)}{\partial\omega}&=-\Gamma_3^2(z)\gamma^2\nu\int\frac{d\omega'}{\pi}|\omega'|^{\nu-1}|\omega-\omega'|^{\nu-1}[\beta\tilde{J}\tilde{G}(0,z)]^4\\&=-\Gamma_3^2(z)\frac{\gamma^2\nu}{\pi}[B(\nu,\nu)+2B(\nu,1-2\nu)]|\omega|^{2\nu-1}[\beta\tilde{J}\tilde{G}(0,z)]^4\,,\end{aligned} \tag{6.18a}$$

and similarly,

$$\mathrm{Re}\frac{\partial\Sigma}{\partial\omega}=-\Gamma_3^2(z)\frac{\alpha\gamma\nu}{\pi}B(\nu,\nu)|\omega|^{2\nu-1}\times\mathrm{sgn}\,\omega[\beta\tilde{J}\tilde{G}(0,z)]^4\,. \tag{6.18b}$$

To determine the ratio of the two constants α/γ, we consider the *real* part of Eq. (6.13),

$$\frac{\gamma}{\alpha}-\frac{\alpha}{\gamma}=\frac{B(\nu,\nu)}{2\pi}\frac{[\langle\delta\sigma^3\rangle^2]}{[\langle\delta\sigma^2\rangle^2]}\,, \tag{6.19}$$

where again $\omega(\partial\Sigma/\partial\omega)$ has been neglected compared to $\tilde{G}_0^{-1}(\omega)$. Since $B(\nu,\nu)=\Gamma^2(\nu)/\Gamma(2\nu)$, the rhs of Eq. (6.19) is positive definite. Therefore, only the second of the two solutions in Eq. (6.12) is consistent with Eq. (6.1). Inserting this value into (6.19) we obtain an implicit equation for the exponent ν:

$$f(\nu)\equiv4\pi\cot(\pi\nu)/B(\nu,\nu)=\frac{[\langle\delta\sigma^3\rangle^2]}{[\langle\delta\sigma^2\rangle^3]}\,. \tag{6.20}$$

The function $f(\nu)$ increases monotonically with decreasing ν, such that $f(\frac{1}{2})=0$ and $f(0)=2$.

To calculate explicitly the temperature dependence of the exponent ν we specialize to the Ising case ($\sigma^2=1$). In that case, we have

$$\frac{[\langle(\delta\sigma)^3\rangle^2]}{[\langle(\delta\sigma)^2\rangle^3]}=4\frac{[m^2(1-m^2)^2]}{[(1-m^2)^3]}=4\left[\frac{[\tilde{G}^2]}{[\tilde{G}^3]}-1\right]=4\left[\frac{\tilde{T}^2}{\tilde{J}^2[\tilde{G}^3]}-1\right], \tag{6.21}$$

where the last equality is a consequence of Eq. (6.9). Close to T_c $q \sim \tau$, $f(\nu) \sim 2\pi(1-2\nu)$; hence

$$\nu(\tau) \simeq \frac{1}{2} - \frac{\tau}{\pi} + O(\tau^2) . \tag{6.22}$$

Evaluating $\nu(T)$ at low temperatures requires knowledge of the low-temperature expansion of the spin moments in the marginally stable static solution which is not available at the moment. Instead, we outline here a *tentative* result based on the TAP theory.[17,40] The TAP mean-field equations are

$$\tanh^{-1} m_i = \sum_j \beta J_{ij} m_j - \beta^2 \widetilde{J}^2 (1-q) . \tag{6.23}$$

At low temperatures, the deviations of the values of the spin moments from 1 are proportional to T^2 as can be seen from Eq. (6.9). Hence, Eq. (6.23) can be written as

$$\beta \widetilde{h}_i = \alpha m_i + \tanh^{-1} m_i , \quad T \sim 0 \tag{6.24}$$

where α is defined by

$$q = 1 - \alpha (T/\widetilde{J})^2 , \tag{6.25}$$

and $\widetilde{h}_i \equiv \sum J_{ij} m_j$ is the local mean field.

Thouless *et al.*[17] and others[10,41] argue that the probability distribution of $\widetilde{h} = |\widetilde{h}_i|$ behaves in the following manner:

$$P(\widetilde{h}) \sim \frac{\widetilde{h}}{H_0^2} , \quad T \sim 0 \tag{6.26}$$

for small $\widetilde{h}$. This determines the values of the various spin moments near zero temperature, via the equation,

$$\begin{aligned} 1 - [m^{2s}]_J &= \int_0^\infty \{1 - [m(\widetilde{h})]^{2s}\} P(\widetilde{h}) d\widetilde{h} \\ &= H_0^{-2} \int_0^1 (1 - m^{2s}) \widetilde{h}(m) \frac{d\widetilde{h}}{dm} dm \\ &= H_0^2 T^2 \int_0^1 (1 - m^{2s})(\alpha m + \tanh^{-1} m) \\ &\quad \times \left[\alpha + \frac{1}{1-m^2} \right] dm , \end{aligned} \tag{6.27}$$

which together with Eqs. (6.25) and (6.9) yield $H_0/\widetilde{J} \simeq 1.28$ and $\alpha \simeq 1.81$ (see Ref. 40). Equation (6.27) also yields

$$\begin{aligned} [\widetilde{G}^3]_J &= [(1-m^2)^3]_J \\ &= T^2 \left[-\frac{\alpha}{70} + \frac{16}{35} \alpha \ln 2 + \frac{8}{15} \ln 2 \right. \\ &\quad \left. - \frac{11}{60} + \frac{\alpha^2}{6} \right] , \end{aligned} \tag{6.28}$$

which implies that

$$\nu(T \to 0) = 0.25 . \tag{6.29}$$

We emphasize again that the result (6.29) is only *tentative* since we have not yet proved the validity of Eqs. (6.24) and (6.26) within the dynamic framework.

Finally, we calculate the exponent ν along the Almeida-Thouless line, where the SK solution is marginally stable. Near T_c, $q_{\rm SK}(h) \sim (\frac{3}{4} h^2)^{1/3}$; hence

$$\nu(h) \simeq \frac{1}{2} - \frac{1}{\pi} \left[\frac{3}{4} h^2 \right]^{1/3} , \quad T \lesssim T_c , \quad \beta h \ll 1 . \tag{6.30}$$

On the other hand, near zero temperature one obtains

$$\begin{aligned} [\widetilde{G}^3] &= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{sech}^6 \beta(z+h) , \\ [\widetilde{G}^3] &\simeq 16T/(15\sqrt{2\pi}J) \exp(-\tfrac{1}{2} h^2/\widetilde{J}^2) . \end{aligned} \tag{6.31}$$

Substitution of Eq. (6.31) in Eqs. (6.20) and (6.21) yields $f(\nu)=1$, or

$$\nu(T \to 0) = 0.395 . \tag{6.32}$$

Recently Parisi *et al.*[42] presented a projection hypothesis according to which $q(T,h)=q(T)$ below the Almeida-Thouless line. If this is correct, then according to Eq. (6.9), $[m^4]_J$ also must be independent of h. If we make the further assumption that higher moments and in particular $[m^6]$ are also independent of h, then the result (6.32) should hold also for $T \to 0$, $h=0$, in contradiction with the result (6.29) obtained from the analysis based on the TAP theory.

The result that ν is a function of T below T_c is new. Previous dynamic treatments[23,24] predicted a mean-field value of $\frac{1}{2}$ at all $T \le T_c$. These treatments however were based on a low-order perturbation (in u) and neglected the important singularity which appears in the self-energy below T_c. The result for ν has been derived here from a Langevin equation for a soft-spin version of the SK model. This raises the interesting question whether our results hold also for a Glauber dynamics of an Ising SK model. The Glauber equations of motion have been solved[10] previously in the linearized approximation which is inadequate below T_c. So far we are not aware of a completely satisfactory method of extracting the low-frequency properties from the

Glauber equations below T_c. This issue has been recently addressed by Shastry.[43] He derived from the Glauber equations, mean-field dynamic equations for the time-dependent local magnetizations $m_i(t)$ which reduce in the static limit to the TAP equations. From these equations he concludes that $\nu=\frac{1}{2}$ at all $T \le T_c$. If this is indeed so, one is inevitably led to the surprising conclusion that the critical behavior of a time-dependent Ginzburg-Landau model in the Ising limit is different in the SG case from that of a Glauber dynamics. However, there are still in our opinion important unanswered questions regarding the correct analytic treatment of the Glauber dynamics of the SK model below T_c.

It should also be mentioned that Monte Carlo simulations of the Glauber dynamics of the SK model yielded[10] $\nu \sim 0.5$ at a temperature range $0.5T_c \le T \le T_c$ but the available data is not sufficiently accurate to check the temperature dependence predicted above.

VII. CONCLUDING REMARKS

Our principal result for the relaxational dynamics of SG's in the mean-field limit is the self-consistent local stochastic equation of motion, Eqs. (3.7)–(3.9). As discussed in Sec. III, this solution is a result of the ansatz $Q_2^0=\langle \hat{\sigma}\hat{\sigma} \rangle=0$. Stationary-point solutions with $Q_2^0 \neq 0$ will not in general be reducible to a simple equation of motion. We have argued that solutions with $Q_2^0 \neq 0$ are physically unacceptable, thus justifying our choice. An additional support to this choice stems from the fact that it is consistent with a *static* solution[36] which is probably the correct lowest-energy state. We have also pointed out the analogy between the solution with $Q_2^0 \neq 0$ and the solutions of the TAP equations by Bray and Moore[21] and De Dominicis *et al.*[22] These solutions are described by order parameters other than q and Δ and were associated with a number of metastable states, the logarithm of which is proportional to N. If the relation between the case $Q_2^0 \neq 0$ and these solutions are correct, then it means that the average properties of the TAP equations can be described by q and Δ only with, however, a "careful" treatment of the products of these order parameters. However, in order to prove the relationship between these solutions, one would have to investigate further the properties of Eqs. (3.1)–(3.3) with $Q_2^0 \neq 0$. Also, it should be pointed out that ultimately the neglect of the solution $Q_2^0 \neq 0$ should be justified by a stability analysis of Eq. (3.1), which has not yet been completed by us.

We have shown that the instability of the SK solution below T_c necessarily means that the FDT is violated by the response at $\omega=0$. Thus both correlation and response functions acquire below T_c time-persistent terms denoted by $q\delta(\omega)$ and $\Delta\delta_{\omega,0}$, Eqs. (5.1) and (5.10). As a result, the static local response is

$$G(0)=1-q+\Delta\ , \tag{7.1}$$

with $0<\Delta<q$. Note that the local susceptibility is $G(0)$, since we defined h_i to be an applied field divided by T. The notion of violation of the FDT deserves further comments. Obviously, in a finite system, the FDT is valid, and in particular the equation

$$\frac{1}{N}\sum_i \left[\frac{\partial\langle\sigma_i\rangle_T}{\partial h_i}\right]_J = 1-\frac{1}{N}\sum_i [\langle\sigma_i\rangle_T^2]_J\ , \tag{7.2}$$

must hold for finite N in any configurations of fields. Note that the *apparent* violation of the FDT cannot be attributed simply to nonzero contributions of off-diagonal spin-spin correlations $\langle\sigma_i\sigma_j\rangle_T$, since they do not contribute directly to a true *local* response. One apparent way of interpreting Eq. (7.1) is to conclude that

$$\lim_{N\to\infty}[\langle\sigma_i\rangle_T^2]_J = q-\Delta\ , \tag{7.3}$$

which would then imply that Eq. (7.1) holds only in the presence of some symmetry-breaking fields which are set to zero only after taking the limit $N\to\infty$. This interpretation is in contradiction with the recent static mean-field solution[36] which predicts that, in *thermal equilibrium,*

$$\lim_{N\to\infty}[\langle\sigma_i\rangle_T^2]_J = 0 \tag{7.4}$$

at all temperatures while at the same time $G(0)$ is smaller (below T_c) than the Curie value 1. Thus, contrary to the common conception, the anomaly is *not* in the *appearance* of Δ which is simply a consequence of the relaxation of q, but rather in the fact that even in thermal equilibrium $q-\Delta>0$.

A possible way[44] of reconciling Eqs. (7.1), (7.2), and (7.4) is that the average local response that enters the mean-field theory is *not* identical to the lhs of Eq. (7.2) but rather corresponds to

$$G(0)=\lim_{N\to\infty}\frac{1}{N}\sum_i\left[\frac{\delta\langle\sigma_i\rangle_T}{\delta h_i}\right]_J\ , \tag{7.5}$$

where δh_i, in a finite N, are very small but not infinitesimal. As first suggested by Bray and Moore,

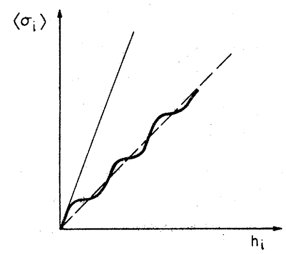

FIG. 3. Schematic plot of the magnetization vs magnetic field demonstrating the possible origin of the apparent violation of the FDT. The slope of the solid straight line is the zero field susceptibility which is equal to 1. The slope of the dashed line is the response $G(0)$, Eq. (7.5), which is equal to $1-q+\Delta$.

it is plausible that the "magnetization versus field" curve contains many steps which correspond to overturning of large clusters of spins. These steps occur as the field changes by an amount which is proportional to some inverse power of N and hence is assumed to be smaller than δh. The actual slope at the origin is 1, but $G(0)$, Eq. (7.5), refers to the slope of the "envelope" of these steps which is smaller than 1; see Fig. 3. This picture has some support from a recent exact solution of the SK model in small samples[45] but, in order to prove it, it is probably necessary to investigate the finite size corrections to the mean-field solution.

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