Vacuum states of the Korteweg—De Vries equation

C. Au and P. C. W. Fung
Physics Department, University of Hong Kong, Pokfulam Road, Hong Kong
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In this investigation we present a set of analytical solutions to the Korteweg—De Vries equation via a Bäcklund transformation using a differential geometrical approach. Some of these solutions are new and regular, while others are irregular. We have examined a set of regular solutions and found that a parameter b, which may be called the "vacuum parameter," appears in the expression for the soliton velocity. In previous analyses, b has been set to zero. With the proper choice of b in the one-soliton solutions, we have demonstrated graphically that the velocity can be zero, positive, or negative. By ascribing the soliton to different physical states corresponding to different values of the vacuum parameter b, we are able to give an interpretation to explain some previous seemingly ambiguous results relating velocity, amplitude, and width of a soliton. The physical and mathematical meaning of b is discussed.

I. INTRODUCTION

Using a differential geometrical approach, Loo *et al.*[1] arrived at a system of Bäcklund transformation of the Korteweg—de Vries (KdV) equation

$$u_t+u_{xxx}+12uu_x=0 , \tag{1}$$

$$u^*=b , \tag{2}$$

$$u^*=u(x,t) , \tag{3}$$

$$u^*=-u(x,t)-y^2+\lambda , \tag{4}$$

where b and λ are constants and the function y satisfies

$$y_x=-[2u(x,t)+y^2-\lambda] , \tag{5}$$

$$y_t=4\{ [u(x,t)+\lambda][2u(x,t)+y^2-\lambda] + \tfrac{1}{2}u_{xx}-u_xy \} , \tag{6}$$

with the usual subscript notations to signify partial derivatives. If $u(x,t)$ is a solution to the KdV equation, Eqs. (5) and (6) should be compatible. Using Bäcklund transformations (4), (5), and (6), the solution to the one-soliton case has been obtained in Ref. 1. In this paper we shall continue the work and obtain a set of analytical solutions to the KdV equation and we shall discuss some new features of the solutions.

A. A set of analytical solutions

Starting from the Bäcklund transformation for u^* in (2),

$$u^*=b \tag{7}$$

is a solution to the KdV equation. Substituting (7) in (5) and (6), we obtain

$$y_x=-(y^2+2b-\lambda) , \tag{8}$$

$$y_t=4[(b+\lambda)(y^2+2b-\lambda)] , \tag{9}$$

which give

$$y_t=-4(b+\lambda)y_x . \tag{10}$$

The solution to (10) is obvious:

$$y=y(r) ,$$

where

$$r=x-4(b+\lambda)t-x_0 , \tag{11}$$

indicating a propagating wave. We would note that the propagating velocity is

$$v=4(b+\lambda) . \tag{12}$$

Only if $b=0$, the velocity becomes 4λ.[2] Clearly, from (8) we obtain the Riccati equation

$$\frac{dy}{dr}=-(y^2+2b-\lambda) \tag{13}$$

where y, instead of being a function of x and t, now is only a function of r. Using a process of separation of variables, one has the solutions:

$$y=\pm\sqrt{\lambda-2b}\ , \tag{14a}$$

$$y=\frac{1}{r+C}\ ,\quad \lambda-2b=0\ , \tag{14b}$$

$$y=\sqrt{\lambda-2b}\left[\frac{Ce^{(\lambda-2br)^{1/2}}-e^{-(\lambda-2br)^{1/2}}}{Ce^{(\lambda-2br)^{1/2}}+e^{-(\lambda-2br)^{1/2}}}\right], \tag{14c}$$

where C is a constant.

From our Bäcklund transformation (4), we obtain

$$u^*=b\ , \tag{15a}$$

$$u^*=b-\frac{1}{(x-12bt-x_0)^2}\ ,\quad \lambda=2b\ , \tag{15b}$$

$$u^*=\lambda-b-(\lambda-2b)\times\frac{[Ce^{(\lambda-2br)^{1/2}}-e^{-(\lambda-2br)^{1/2}}]^2}{[Ce^{(\lambda-2br)^{1/2}}+e^{(-\lambda-2br)^{1/2}}]^2}\ . \tag{15c}$$

If we substitute (15) into Eq. (1), we readily find that these solutions satisfy the KdV equation. We would like to point out here that the Ricatti equation (13) may be solved by the transformation

$$y=\frac{\psi_x}{\psi}\ ,$$

where $\psi=\psi(x,t)$. Now (13) reads

$$\psi_{xx}+(2u-\lambda)\psi=0\ .$$

Using the above linearized equation, we can obtain all the solutions presented in (14). We now proceed to study the characteristics of our analytical solutions (15) under different special cases.

Case (A): $b=0$. We have

$$u^*=0\ , \tag{16a}$$

$$u^*=-\frac{1}{(x-x_0)^2}\ , \tag{16b}$$

where $\lambda=0$,

$$u^*=\frac{4C\lambda}{(Ce^{\sqrt{\lambda}(x-4\lambda t-x_0)}+e^{-\sqrt{\lambda}(x-4\lambda t-x_0)})^2}\ , \tag{16c}$$

where $\lambda\neq0$. We shall only concentrate on solution (16c).

Case (i): $\lambda>0$, propagation along positive x direction. The situation for which $C=0$ is obvious. If $C=1$,

$$u^*=\lambda\,\mathrm{sech}^2[\sqrt{\lambda}(x-4\lambda t-x_0)]\ , \tag{17}$$

which is the well-known one-soliton solution. If $C=-1$,

$$u^*=-\lambda\,\mathrm{csch}^2[\sqrt{\lambda}(x-4\lambda t-x_0)]\ . \tag{18}$$

This solution is singular at

$$r=x-4\lambda t-x_0=0\ .$$

The form of the solution for the range $C>0$ ($C<0$) is the same as Eq. (17) [Eq. (18)].

Case (ii): $\lambda<0$, propagating along negative x direction. If $C=0$, $u^*=0$. If $C=\pm1$,

$$u^*=-k^2\sec^2[k(x+4k^2t-x_0)]\ , \tag{19}$$

where $k^2=-\lambda$. Clearly, u^* has a "period" π and $u^*_{\max}=-k^2$.

Case (B): $b\neq0$. We now consider the cases in which $\lambda-2b>0$ and $\lambda-2b<0$.

Case (i): $\lambda-2b>0$. If $C=0$, $u^*=b$. If $C=1$,

$$u^*=b+(\lambda-2b)\,\mathrm{sech}^2\{\sqrt{\lambda-2b}\,[x-4(b+\lambda)t-x_0]\}\ . \tag{20}$$

If $C=-1$,

$$u^*=b-(\lambda-2b)\,\mathrm{csch}^2\{\sqrt{\lambda-2b}\,[x-4(b+\lambda)t-x_0]\}\ . \tag{21}$$

This is singular at $r=0$.

We would like to note that under case B i, for $\lambda+b>0$, as $\lambda-2b>0$, we must have $b>0$, $\lambda>2b$, or $b<0$ and $\lambda>-b$. Under such a situation, the soliton propagates in the positive x direction. However, for $\lambda+b<0$, namely $b<0$ and $2b<\lambda<-b$, the solition propagates in the negative x direction.

Case (ii): $\lambda-2b<0$. We let $\lambda-2b=-k^2$, and

$$u^*=(\lambda-b)-(\lambda-2b)\frac{(Ce^{ikr}-e^{-ikr})^2}{(Ce^{ikr}+e^{-ikr})^2}\ . \tag{22}$$

If $C=0$, $u^*=b$, an obvious case. If $C=1$,

$$u^*=b-k^2\sec^2k[x+4(k^2-3b)t+x_0]\ . \tag{23}$$

If $C=-1$,

$$u^*=b-k^2\csc^2k[x+4(k^2-3b)t+x_0]\ . \tag{24}$$

Solutions (23) and (24) are periodic (with "periodicity" π) and possess a series of singularities.

From (12) it is easy to see that for $\lambda+b>0$, the wave represented by (24) propagates in the positive x direction. For $\lambda+b<0$, the wave propagates in the negative x direction. The new forms of solution in case B ii may be applicable to nonlinear physical phenomena and will be published elsewhere. In this paper we shall concentrate on a detailed study of the meaning of solution $u^*=b$, which on first sight looks trivial. It will be shown in the next section that in fact b represents the vacuum state of the soliton and is an important parameter to the solutions of the KdV equation.

II. INTERPRETATION OF THE VACUUM STATE OF A SOLITON

In this section we shall study the special features of the sets of solutions (15c) obtainable through a Bäcklund transformation. In particular we take the set of solutions [Eq. (20)] for a one-soliton as a demonstration of the interesting characteristics we have discovered.

For convenience in discussion we write the KdV equation in the form

$$u_t+\alpha u u_x+\beta u_{xxx}=0 , \tag{25}$$

leaving α and β as yet unspecified parameters. Dropping the superscript * for convenience, Eq. (20) may be written as

$$u=b+\lambda'\operatorname{sech}^2\left[\frac{\alpha\lambda'}{12\beta}\right]^{1/2}\left[x-\frac{\alpha}{3}(\lambda'+3b)t\right] , \tag{26}$$

which is the well-known one-soliton solution and $\lambda'=\lambda-2b$. We shall proceed to discuss special features of the solutions one by one, including our interpretations.

(a) Previously, the amplitude of a KdV soliton is taken to be proportional to its velocity,[3] justified by experimental results.[4–6] It has been assumed that the larger the amplitude, the faster the velocity of the soliton will be.

(b) The width of a KdV soliton is taken to be inversely proportional to the square root of its velocity; again this property agrees with some observations.[3–6]

(c) The propagation of a KdV soliton is one-directional, i.e., it cannot have negative velocity.

The theoretical deductions in (a)–(c) have been obtained based implicitly on the assumption of the invariance property of a Galilean transformation,[7] which amounts to setting $b=0$ in Eq. (26). We shall analyze such an assumption.

From (26), the soliton velocity is specified by

$$v\propto\lambda'+3b . \tag{27}$$

We would, however, emphasize that as early as 1965, Zabusky and Kruskal[8] had already shown that the KdV solutions (26) for velocity, width, and amplitude agree numerically with computer experimental simulations only if b takes on values which are not equal to zero. This property has been checked further by Makino *et al.* recently.[9] They fed a sinusoidal wave into the KdV equation. They have found that the number of solitons and the recurrence time agree well with those expected from the KdV equation. However, when they studied the relation between amplitude, width, and propagation velocity of the soliton as time evolves, they found that properties (a)–(c) are violated. We would like to remark that b has been set to zero in their analysis. As will be shown later, setting $b=0$ may be the main cause of such violation. We would note also that in the treatment of Makino *et al.* a process of separation of variables has been carried out for the two-dimensional problem and the methodology of analysis is essentially the same as that used in a one-dimensional problem.

In view of the previous result it appears that a close examination of the physical meaning of "different b states" is essential. Considering that b could take on nonzero values, we may offer the following interpretations of different physical states of a one-soliton system based on the solutions of Eq. (26): The velocity v of a KdV soliton does not depend only on the amplitude λ' but depends also on the parameter b. We would like to name this parameter as the "vacuum parameter." Since $u=b$ [namely Eq. (7)] is also a solution to the KdV equation, $u=b$ may be called the "vacuum state" of the equation. To illustrate the above new idea, we plot in Fig. 1 the temporal evolution for solitons with different values of the vacuum parameter b and the amplitude λ'. Thus Fig. 1(a) represents the well-known situation where $b=0$. When b takes a positive value ($b=2$), as in Fig. 1(b), the solitons still propagate along the positive x axis but with a velocity different from that in Fig. 1(a). In this situation the soliton with a larger amplitude still travels faster. When b takes on a negative value [e.g., $b=-2$, Fig. 1(c)] the solitons can propagate along the negative x axis and the

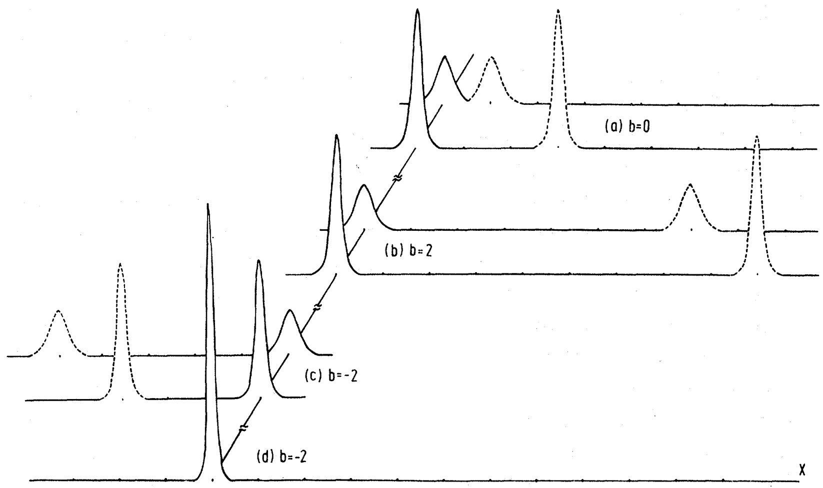

FIG. 1. Time evolution of one soliton at two instants [time $t=0$ (solid line), and $t=\frac{1}{4}$ arbitrary unit (dotted line)] along the x axis with amplitude $\lambda'=\lambda-2b$ for the case where (a) vacuum parameter $b=0$, amplitude $\lambda'=1,3$ ($\lambda=\lambda'$), (b) vacuum parameter $b=2$, amplitude $\lambda'=1,3$ ($\lambda=5,7$), (c) vacuum parameter $b=-2$, amplitude $\lambda=1,3$ ($\lambda=-3,-1$), (d) stationary wave with vacuum parameter $b=-2$ and amplitude $\lambda'=6$ ($\lambda=2$).

soliton with a *smaller* amplitude can travel *faster.* In Fig. 1(d) we can choose a set of values for λ' and b such that the propagation velocity is zero (e.g., $\lambda'=6$, $b=-2$)—we have a stationary soliton. In the simple case where $u=b$, b simply modulates the velocity of the soliton(s).

We would like to remark also that for a fixed coordinate system, for a system of "almost individual" solitons[8] which have almost identical shape to that of a one-soliton solution (26), each one-soliton may be associated to its own vacuum parameter b if the same coordinate system is chosen for the system. Under this situation, one cannot in general find a Galilean transformation such that all the b's can be taken to be zero. In fact, in order to explain the relationship between the amplitude, velocity, and width of a one-soliton system obtained from computer simulation, different values of b (namely various values of u_∞) were introduced by Zabusky.[8]

The KdV equation is Galilean invariant.[7] Solutions obtained in this investigation through a Bäcklund transformation are identical to that obtained earlier using the Galilean-invariant property of the KdV equation; in fact the above two methods of analysis lead to the same family of one-parameter solutions. However, previous[7] workers have taken the view that the Galilean invariance of the KdV equation amounts to constraining b to zero value, implying that b has no physical meaning. Guided by the result of our analysis, we are inclined to believe that due to the nonlinearity of the KdV equation, the parameter b is a physical observable. This feature is different from the result obtained from physical phenomena in the linearized regime. It is a pity that such a nonlinear property in physics has not been pursued. Perhaps people have paid too much attention to the linear world.

Using this vacuum-state concept we may provide an explanation to some apparently ambiguous results in the study of KdV solitons. For example, Scott[3] in his review writes: "... Figure 4 shows the collision of two such solitons that are traveling in opposite directions. However, in this case, the ion-acoustic wave solitons should perhaps be described by some other wave equation like the Boussinesq equation which allows waves propagating in both directions rather than the KdV equation which allows only one-directional propagation" (see also the work of Ikezi *et al.*[4]). As shown in our investigation, the KdV equation does allow solitons traveling in opposite directions.

The recurrence property and the identity property[8] are two well-known nonlinear characteristics of solitons. In the example given in this investigation, it appears that different vacuum states of that nonlinear physical process have different effects on the observable physical state. Is this property a general feature of a nonlinear process? A more thorough analysis has to be carried out in order to obtain a decisive answer to the above question. If the answer turns out to be positive, one can then add the third general property to a soliton.

III. CONCLUSION

In this investigation we have obtained a set of analytical solutions to the KdV equations. Some of these represent propagating one-soliton regular solutions along both the positive and negative x directions. Some other solutions are irregular, indicating singularity property. We think that solutions under case B ii are new.

To analyze these solutions, we have confined ourselves in this paper to a close reexamination of the one-parameter family of regular KdV solutions. When we fix a certain Galilean coordinate system for the soliton, different values of the vacuum parameter b would lead to different propagation velocity values. In fact the propagation velocity v of a KdV soliton is a function of both the amplitude λ' and the parameter b: $v \propto \lambda' + 3b$. Previously, the parameter b has been assigned zero value even under different physical situations. We would emphasize that we think the vacuum parameter b represents the particular physical state of the soliton(s) and can take on nonzero values. With proper choice of b the soliton can be stationary or can propagate to positive or negative directions, as illustrated in Fig. 1. As b can take on nonzero values, the sign of the solution to u as given in (26) may be positive or negative. The width D of one-soliton is, however, not an explicit function of b: $D \propto 1/\sqrt{\lambda'}$.

From the mathematical point of view, under Galilean transformations we would obtain a one-parameter family of solutions to the KdV equation. The parameter b has a definite meaning and cannot be set arbitrarily to any particular value, such as zero.

It appears then that with our interpretation of the vacuum parameter and hence the vacuum state, we may provide one logical explanation to some seemingly ambiguous results relating velocity, amplitude, and width of a soliton. Our result suggests that b may be a physical observable. Such a property is basically contributed by the nonlinear term of the KdV equation. It might be worthwhile to explore further whether it is generally true that different vacuum states of a nonlinear physical process may have different effects on the observable physical state.

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