Phase transition in the $\pm J$ Ising-spin-glass model

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The two-dimensional $\pm J$ model (antiferromagnetic and ferromagnetic bonds at random) is studied by transfer-matrix and Monte Carlo calculations. Large clusters of spins pointing in one direction are found for $0.12 < x \leq 0.16$ (x being the concentration of negative bonds). Evidence against a phase transition at finite temperature is found by investigating the correlation function $\langle S_0 S_R \rangle^2$.

I. INTRODUCTION

The thermodynamics of two-dimensional Ising lattices with random exchange interactions have been studied extensively in the context of the spin-glass problem.[1–3] The question of a phase transition has been heavily debated. At the present time there exists a lot of (mostly numerical) evidence against a phase transition in all short-range interaction models considered so far.[4–6] On the other hand, experimental results are still ambiguous concerning this question.

In the most abstracted case, the $\pm J$ model (antiferromagnetic and ferromagnetic bonds at random). Transfer-matrix calculations (TM) showed for the correlation function $\{\langle S_0 S_R \rangle^2\}$ (two spins in distance R) a power law decay with R at temperature $T=0$ for a concentration $x=0.5$ of negative bonds. For finite temperature an exponential decay is seen. At lower concentrations of negative bonds not all aspects of the system are investigated so far. Monte Carlo (MC) calculations[4,7] found a breakdown of the ferromagnetic order at $x=0.12 \pm 0.1$. This effect was also seen by TM at the same concentration[6] considering the ferromagnetic susceptibility

$$\frac{k_B T \chi_f}{N} = \frac{1}{N} \sum_{ij} \langle S_i S_j \rangle^2 \quad . \tag{1}$$

Several analytical calculations showed qualitatively similar results.[9] But the question of an equilibrium phase transition slightly above the breakdown concentration of the ferromagnetic order was not considered. In a recent publication, Maynard *et al.*[10] mentioned the possibility of a "random antiphase" for $0.10 \leq x \leq 0.15$ characterized by the existence of long-range order and zero magnetization, the latter already seen by MC and TM. The present work now applies the most powerful tool so far in questions of phase transitions in spin-glasses, the TM, in this case considering the decay of the correlation function $\{\langle S_0 S_R \rangle^2\}_{\text{av}}$ at zero temperature.

II. $\pm J$ MODEL AND TRANSFER-MATRIX METHOD

We consider the Ising Hamiltonian with random nearest-neighbor interaction $\{J_{ij}\}$ for an $L \times M$ square lattice

$$-\beta \mathcal{H} = \sum_{\langle i,j \rangle} J_{ij} S_i S_j \quad , \tag{2}$$

where we have for the J_{ij}'s the probability distribution

$$P(J_{ij}) = x \delta(J + J_{ij}) + (1-x) \delta(J - J_{ij}) \tag{3}$$

In the present work we mainly consider the range $0.09 < x \leq 0.16$.

The transfer-matrix method TM computes the free energy recursively, starting by generating the statistical factors for all 2^L states of the spins in the first row. When the *first* spin of the *second* row is added, the trace over the *first* spin of the *first* row is taken, and so on. Adding spins TM has to update the statistical weights at each step, generating recursively the partition function Z, or, in order to keep numbers small $(1/LM) \ln Z$, the free energy. At each step 2^L states have to be kept. Therefore the storage requirement for the computation sets a limit. Thus only small lattices (here mainly 20×12 and 12×16) can be treated. The correlation function $\langle S_0 S_R \rangle^2$ is obtained by numerical differentiation after fixing the spin S_0 on the free boundary (we have cylindric boundary conditions) and applying a small magnetic field on the spin S_R. The quenched average value $\{\cdots\}_{\text{av}}$ is approximated by averaging over only about 50–120 random configurations leading to "small enough" error bars as seen from standard statistical analysis (see Fig. 2).

III. NUMERICAL RESULTS

We introduce cylindric boundary conditions and an even number of spins in both directions in our systems.

A. Transfer-matrix results

Figure 1 shows the behavior of the correlation function $\{\langle S_0 S_R\rangle^2\}_{av}$ extrapolated to temperature $T=0$. TM is only able to treat temperatures down to $k_B T/J=0.1$; but in comparison to $k_B T/J=0.15$, no significant differences in the values for the correlation function are seen, i.e., the ground states are already reached for the small systems.

For all concentrations $x \leq 0.13$ we notice a power law decay analogous to the case $x=0.5$,

$$\{\langle S_0 S_R\rangle_0^2\}_{av} = R^{-P} , \tag{4}$$

when $\langle \cdots \rangle_0$ denotes thermodynamically averaged for $T \to 0$.

The decay of the correlation function is related to the existence of "nontrivial loops" crossing the lattice.[6] A nontrivial loop is a zero-energy surface surrounding a larger part or, in most of our cases, the whole lattice. For one particular lattice the correlation function of two spins on both sides of the nontrivial loop reads

$$\langle S_0 S_R\rangle_0^2 = \left[\frac{Z(+)-Z(-)}{Z(+)+Z(-)}\right]^2 , \tag{5}$$

where $Z(+)$ denotes the number of states connected to the nontrivial loop with $S_0 S_R=+1$ and $Z(-)$ analogous. While the total number of states $Z=Z(+)+Z(-)$ is very large, the difference becomes very small (as also expected from symmetry arguments). Thus the sudden occurrence of a very small number of the correlation function indicates the existence of a nontrivial loop. The area surrounded by the loop leads to a certain energy barrier which is proportional to R. Thus a power law of the correlation function indicates a zero probability for the existence of an infinite barrier height. Due to the finite barriers, we have no phase transition at finite temperature.

At the most interesting concentrations $x=0.13$ and 0.14, we still obtain a power law decay of $\{\langle S_0 S_R\rangle^2\}_{av}$ qualitatively analogous to $x=0.5$. Thus we conclude that a phase transition occurs at these concentrations *only* at $T=0$. The random antiphase is destroyed by shells of zero-energy surface, i.e., nontrivial loops. For concentrations $x \leq 0.12$ we obtain a constant correlation function for larger distances, which is due to the existence of the ferromagnetic phase in this region. On the other hand, the existence of nontrivial loops destroys the ferromagnetic susceptibility[1] as shown in Fig. 2. The existence of a nontrivial loop is caused by the appearance of frustrated plaquettes and not by negative bonds. Thus Fig. 2 shows a natural curve of χ_f only plotted against x_F, the concentration of frustrated plaquettes, where

$$x_F = 4[x(1-x)^3 + x^3(1-x)] . \tag{6}$$

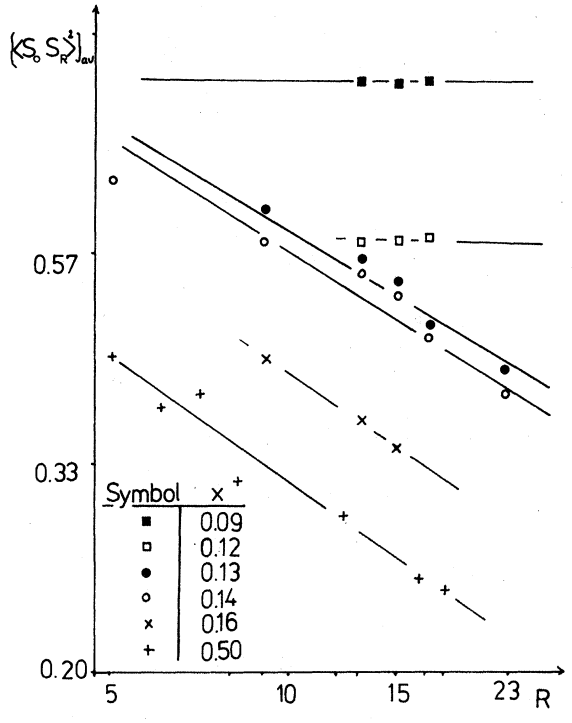

FIG 1. $\{\langle S_0 S_R\rangle^2\}_{av}$ against distance R for various concentrations of negative bonds x at $T \to 0$. Double-logarithmic plot.

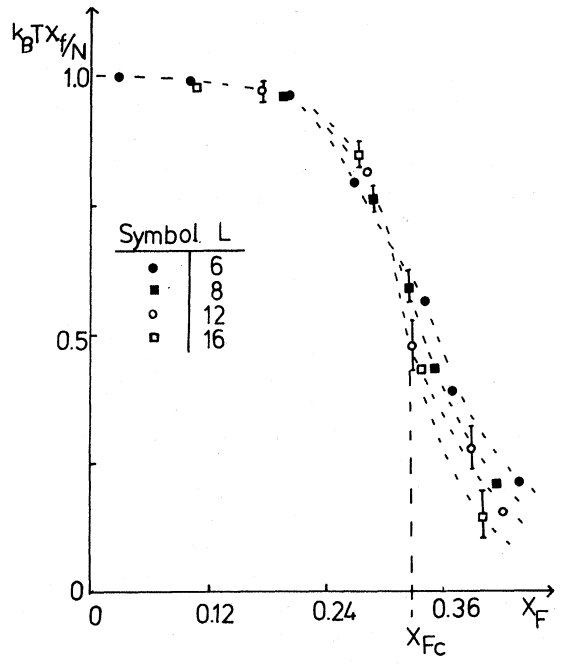

FIG. 2. Ferromagnetic susceptibility $k_B T\chi_f/N$ against concentration of frustrated plaquettes x_F. Various lattice sized ($L \times L$).

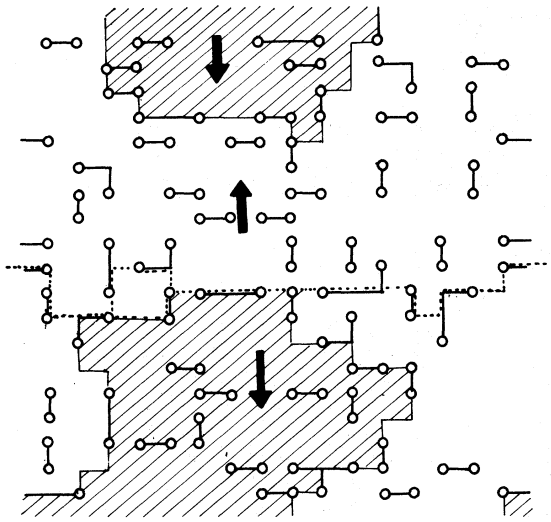

FIG. 3. Ground-state structure of a 20×16 lattice for concentration $x = 0.14$. Open circles denote frustrated plaquettes; thick lines between are strings. Dotted lines indicate nontrivial loops. Grey shaped region surrounded by thinner lines and strings denotes cluster with spins mainly pointing down. Cylindric boundary conditions.

Plotted against the concentrations of negative bonds, the curve looks "unnatural" in contrast to the feeling one has from "ordinary" phase transitions. For $x \leq 0.12$ we have ferromagnetic ordering. Figure 1 shows, therefore, a constant correlation function for larger R. This constant increases for smaller concentrations as the number of "defects" decreases.

B. Monte Carlo calculations

Particular ground states are obtained by slow cooling down of MC calculations. For the given lattice the ground-state energy is known from TM and we are really *sure* to get a ground state. Figure 3 shows a typical example of a 20×16 lattice for concentration $x = 0.14$. A nontrivial loop is crossing the lattice. Notice that only one loop is necessary to obtain different shells with zero-energy surfaces because of the *free* boundary in vertical direction. Periodic boundaries, also in this direction, would destroy the breakdown effect of $\langle S_0 S_R \rangle^2$ in our small lattices,[8] as at least two nontrivial loops are necessary. This would lead to a too small probability to find them in our case.

On the other hand, an *odd* number of spins would also destroy the described effect, as no balance of positive and negative "energy bonds" is possible. It is clear that the thermodynamic limit is well approximated only by an even number of spins, since the energy difference between odd and even infinite loops will vanish.

IV. MAGNETIC WALLS AND SPIN PACKETS

Inside the remaining zero-energy shell we find large areas of spins pointing in one direction. In Fig. 3 the grey shaped regions are down spins in contrast to the not shaped ones. These regions are divided by magnetic walls consisting of a combination of strings and negative bonds leading to a large gain of energy compared to the ferromagnetic case. It has now been accepted that MC as a dynamic process is able to simulate the real world of spin-glasses.[1,4,11] For the considered concentration $x = 0.14$ we get the same freezing effect as at $x = 0.5$, but with large up and down clusters and the related magnetic walls. The existence of nontrivial loops is neglected.

For larger lattices we expect a network of shells surrounded by nontrivial loops. This network is seen in larger lattices,[3,7] where it is not possible to check the ground state by TM. On the other hand, 16×16 lattices with free boundaries also show typically two loops, one in each direction.

V. CONCLUSION

The present work shows no evidence for the existence of a random antiphase. Shells with zero-energy surfaces surround large clusters of up and down spins divided by magnetic walls. As for the case $x = 0.5$ we have a network of zero-energy shells which should be turned over on a large time scale.

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