# Monte Carlo renormalization-group study of the two-dimensional Glauber model

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A recent Monte Carlo renormalization-group study of the two-dimensional kinetic Ising model is extended to obtain renormalized time correlation functions from initial lattices of $16\times16$ and $32\times32$ spins. Different ways to estimate the dynamical exponent $z$ are discussed, with our "best" estimate of $z \simeq 2.23$ being consistent with the original study.

An important, open problem in critical dynamics involves the development of a valid real-space renormalization-group theory.[1,2] The Monte Carlo renormalization-group method[3,4] (MCRG) seems to provide a promising new approach which can be systematically improved. In this paper we extend the original work of Tobochnik, Sarker, and Cordery[3] (TSC) on the two-dimensional Glauber model[5] to obtain more data for the time correlation functions and to explore the effects of larger lattices. We study the renormalized auto and "nearest-neighbor" time correlation functions for kinetic Ising models whose initial lattices contain $16\times16$ and $32\times32$ spins, respectively, as compared to the original work[3] performed on $8\times8$ and $16\times16$ lattices. Although there are discrepancies between our results and the original results for the $16\times16$ lattice, our estimate that the dynamical exponent $z \simeq 2.23$ is consistent with the original study. However, as we discuss below there seems to be no unique way to analyze our data and hence we regard our estimate of $z$ as somewhat inconclusive. Nevertheless, we believe our work to be the most systematic real-space renormalization-group study of the Glauber model performed so far.

The basic idea of the dynamic Monte Carlo renormalization-group method has been adequately discussed by Tobochnik *et al.*[3] and we follow their calculational procedure. Namely, we determine $z$ by "matching" one or more time correlation functions, say

$$E(N,m,T_2;t)=E(Nb^d,m+1,T_1;b^zt) \tag{1}$$

and

$$C(N,m,T_2;t)=C(Nb^d,m+1,T_1;b^zt)\ , \tag{2}$$

where

$$E(N,m,T;t)=(N^{(m)})^{-1}\left\langle \sum_{<ij>} S_i^{(m)}(t)S_j^{(m)}(0)\right\rangle_{T,N} \tag{3}$$

and

$$C(N,m,T;t)=(N^{(m)})^{-1}\left\langle \sum_i S_i^{(m)}(t)S_i^{(m)}(0)\right\rangle_{T,N}\ , \tag{4}$$

using the notation of TSC. We generate a sequence $m=1,2\ldots$ of block spin configurations by a sequence of majority rule transformations on an initial Ising lattice with nearest-neighbor interactions, whose reduced Hamiltonian is $\mathcal{H}=T^{-1}\sum_{<ij>}S_iS_j$. Since we are concerned with a fixed point solution, we have chosen $T_1=T_2=T_c$, where $T_c$ is the critical temperature of the infinite system. (We have not considered any small shift in $T_c$ arising from finite-size effects.) We have computed the time correlation functions (3) and (4) using periodic boundary conditions and employing $1.80\times10^6$ Monte Carlo steps (MCS) and $4.0176\times10^6$ MCS for the $16\times16$ and $32\times32$ lattices, respectively. In contrast to TSC, we have performed these averages using several independent runs, with 20 runs of $9\times10^4$ MCS and 64 runs of $6.48\times10^4$ MCS, respectively, rather than using one long run. Thus we are sure we have statistically independent runs from which to compute our averages and estimate our errors. Our results for $E(t)$ and $C(t)$ for the two different lattices are shown in Tables I and II, together with our estimates for the corresponding errors as calculated by standard methods.[6] In general our errors are comparable to those reported in Ref. 3.

We have encountered some ambiguity in obtaining $z$ from the matching conditions (3) and (4), an uncertainty which is also present in the results of Tobochnik *et al.* However, since they only determined $E(t)$ and $C(t)$ for two values of the time $t$, this ambiguity is less obvious in their case. The difficulty is that if one uses the data shown in Tables I and II to estimate $z$ from (3) and (4), one obtains values of $z$ which depend on the value of $t$ at which one matches. To some extent, at least, this simply re-

TABLE I. Results for the correlation function $E(t)$ defined in Eq. (3). The quantities $N$, $m$, and $t$ are the number of spins on the original lattice, the number of renormalization transformations, and the time (in Monte Carlo steps), respectively.

| | $2^m$ | $N=32\times 32$ | | $2^m$ | $N=16\times 16$ |
|---|---|---|---|---|---|
| $E(0)$ | 1 | 1.4336 ±0.0003 | $E(0)$ | 1 | 1.4519 ±0.0009 |
| | 2 | 1.4287 ±0.0006 | | 2 | 1.4680 ±0.0018 |
| | 4 | 1.4642 ±0.0012 | | 4 | 1.5439 ±0.0032 |
| | 8 | 1.5442 ±0.0022 | | 8 | 1.7121 ±0.0044 |
| | 16 | 1.7138 ±0.0028 | | | |
| $E(60)$ | 1 | 1.9133 ±0.0046 | $E(10)$ | 1 | 2.3247 ±0.0063 |
| | 2 | 1.2465 ±0.0055 | | 2 | 2.6821 ±0.0076 |
| | 4 | 2.6308 ±0.0070 | | 4 | 3.0263 ±0.0096 |
| | 8 | 3.0091 ±0.0098 | | 8 | 3.4063 ±0.0112 |
| | 16 | 3.4059 ±0.0114 | | | |
| $E(120)$ | 1 | 1.8133 ±0.0058 | $E(20)$ | 1 | 2.1971 ±0.0069 |
| | 2 | 2.1340 ±0.0070 | | 2 | 2.5672 ±0.0083 |
| | 4 | 2.5183 ±0.0085 | | 4 | 2.9667 ±0.0105 |
| | 8 | 2.9382 ±0.0104 | | 8 | 2.3832 ±0.0132 |
| | 16 | 3.3740 ±0.0124 | | | |
| $E(180)$ | 1 | 1.7629 ±0.0070 | $E(30)$ | 1 | 2.1320 ±0.0077 |
| | 2 | 2.0766 ±0.0082 | | 2 | 2.5002 ±0.0090 |
| | 4 | 2.4563 ±0.0024 | | 4 | 2.9184 ±0.0103 |
| | 8 | 2.8885 ±0.0118 | | 8 | 3.3597 ±0.0118 |
| | 16 | 3.3384 ±0.0129 | | | |
| $E(240)$ | 1 | 1.7278 ±0.0078 | $E(40)$ | 1 | 2.0888 ±0.0083 |
| | 2 | 2.0377 ±0.0093 | | 2 | 2.4548 ±0.0099 |
| | 4 | 2.4134 ±0.0119 | | 4 | 2.8799 ±0.0120 |
| | 8 | 2.8471 ±0.0129 | | 8 | 3.3308 ±0.0129 |
| | 16 | 3.3019 ±0.0148 | | | |
| $E(300)$ | 1 | 1.7025 ±0.0086 | $E(50)$ | 1 | 2.0574 ±0.0089 |
| | 2 | 2.0060 ±0.0102 | | 2 | 2.4193 ±0.0105 |
| | 4 | 2.3765 ±0.0120 | | 4 | 2.8479 ±0.0125 |
| | 8 | 2.8072 ±0.0138 | | 8 | 3.3017 ±0.0141 |
| | 16 | 3.2613 ±0.0155 | | | |
| $E(360)$ | 1 | 1.6781 ±0.0095 | $E(70)$ | 1 | 2.0087 ±0.0101 |
| | 2 | 1.9778 ±0.0112 | | 2 | 2.3648 ±0.0118 |
| | 4 | 2.3435 ±0.0133 | | 4 | 2.7924 ±0.0142 |
| | 8 | 2.7756 ±0.0154 | | 8 | 3.2469 ±0.0154 |
| | 16 | 3.2318 ±0.0170 | | | |
| $E(420)$ | 1 | 1.6592 ±0.0100 | $E(80)$ | 1 | 1.9883 ±0.0107 |
| | 2 | 1.9556 ±0.0118 | | 2 | 2.3405 ±0.0125 |
| | 4 | 2.3185 ±0.0140 | | 4 | 2.7634 ±0.0147 |
| | 8 | 2.7440 ±0.0163 | | 8 | 3.2184 ±0.0165 |
| | 16 | 3.1876 ±0.0177 | | | |

flects the inaccuracy in our data and possibly finite-size effects. This problem is most acute if one fits the data to a simple exponential decay, e.g., $E(t)\simeq Ae^{-t/\tau}$, which is in fact an excellent representation of the data for sufficiently large $t$, as shown in Fig. 1 for $E(t)$. The best fit then yields amplitudes which are slightly different for the two lattices ($16\times 16$ and $32\times 32$). That is, if $\ln E(t)\simeq b-mt$ for sufficiently long times, one finds $b_{16}=1.2374$ and $b_{32}=1.2398$. This means that one does not really have an exact fixed point solution. Thus the $z$ that one obtains from this fit and Eq. (3) varies with $t$,

TABLE II. Results for the correlation function $C(t)$ defined in Eq. (4), using the same notation as in Table I.

| | $2^m$ | $N=32\times32$ | | $2^m$ | $N=16\times16$ |
|---|---|---|---|---|---|
| $C(60)$ | 1 | $0.4788\pm0.0011$ | $C(10)$ | 1 | $0.5845\pm0.0016$ |
| | 2 | $0.5637\pm0.0013$ | | 2 | $0.6838\pm0.0018$ |
| | 4 | $0.6663\pm0.0016$ | | 4 | $0.7928\pm0.0020$ |
| | 8 | $0.7793\pm0.0020$ | | 8 | $0.8826\pm0.0024$ |
| | 16 | $0.8768\pm0.0023$ | | | |
| $C(120)$ | 1 | $0.4535\pm0.0015$ | $C(20)$ | 1 | $0.5505\pm0.0017$ |
| | 2 | $0.5342\pm0.0017$ | | 2 | $0.6470\pm0.0020$ |
| | 4 | $0.6327\pm0.0020$ | | 4 | $0.7586\pm0.0023$ |
| | 8 | $0.7454\pm0.0024$ | | 8 | $0.8643\pm0.0029$ |
| | 16 | $0.8564\pm0.0026$ | | | |
| $C(180)$ | 1 | $0.4408\pm0.0017$ | $C(30)$ | 1 | $0.5336\pm0.0019$ |
| | 2 | $0.5195\pm0.0020$ | | 2 | $0.6276\pm0.0022$ |
| | 4 | $0.6156\pm0.0024$ | | 4 | $0.7387\pm0.0024$ |
| | 8 | $0.7268\pm0.0028$ | | 8 | $0.8509\pm0.0028$ |
| | 16 | $0.8406\pm0.0030$ | | | |
| $C(240)$ | 1 | $0.4323\pm0.0020$ | $C(40)$ | 1 | $0.5225\pm0.0021$ |
| | 2 | $0.5095\pm0.0023$ | | 2 | $0.6151\pm0.0024$ |
| | 4 | $0.6040\pm0.0027$ | | 4 | $0.7250\pm0.0029$ |
| | 8 | $0.7142\pm0.0031$ | | 8 | $0.8389\pm0.0031$ |
| | 16 | $0.8285\pm0.0035$ | | | |
| $C(300)$ | 1 | $0.4256\pm0.0022$ | $C(50)$ | 1 | $0.5146\pm0.0022$ |
| | 2 | $0.5016\pm0.0025$ | | 2 | $0.6056\pm0.0026$ |
| | 4 | $0.5946\pm0.0030$ | | 4 | $0.7151\pm0.0030$ |
| | 8 | $0.7033\pm0.0034$ | | 8 | $0.8289\pm0.0034$ |
| | 16 | $0.8170\pm0.0037$ | | | |
| $C(360)$ | 1 | $0.4195\pm0.0024$ | $C(70)$ | 1 | $0.5023\pm0.0025$ |
| | 2 | $0.4945\pm0.0028$ | | 2 | $0.5914\pm0.0029$ |
| | 4 | $0.5860\pm0.0033$ | | 4 | $0.6991\pm0.0034$ |
| | 8 | $0.6934\pm0.0038$ | | 8 | $0.8129\pm0.0037$ |
| | 16 | $0.8087\pm0.0042$ | | | |
| $C(420)$ | 1 | $0.4148\pm0.0025$ | $C(80)$ | 1 | $0.4971\pm0.0027$ |
| | 2 | $0.4890\pm0.0029$ | | 2 | $0.5853\pm0.0031$ |
| | 4 | $0.5798\pm0.0035$ | | 4 | $0.6916\pm0.0036$ |
| | 8 | $0.6866\pm0.0041$ | | 8 | $0.8057\pm0.0041$ |
| | 16 | $0.7975\pm0.0045$ | | | |

e.g., it decreases from $z\simeq2.23$ at $t\simeq67$ to $z\simeq2.17$ as $t\to\infty$. A similar effect holds if one analyzes $E(t)$ for the renormalized $4\times4$ lattice and $C(t)$ for the $2\times2$ and $4\times4$ data. In fact the effect is least severe for the results shown in Fig. 1. On the other hand, it seems clear that a sensible analysis of $E(t)$ and $C(t)$ should require a fixed point behavior if a meaningful $z$ is to be determined. Thus we have analyzed the behavior of $E(t)$ in Fig. 1 subject to the constraint that $b_{16}=b_{32}$, in which case it follows from (3) and the exponential form that $z=\ln(m_{16}/m_{32})/\ln2$. We have in fact found that our data for $E(t)$ is consistent with such a constraint, within our statistical error. We have determined that there is a range of such fits, for parameters in the domain $b_{16}=b_{32}\simeq1.2374-1.2398$, $m_{16}\simeq8.56\times10^{-4}-8.95\times10^{-4}$, $m_{32}\simeq1.82\times10^{-4}-1.89\times10^{-4}$. The value of $z$ is essentially constant within this range and is given by $z\simeq2.24$. We have also performed a similar analysis of our data for $C(t)$ renormalized down to a $2\times2$ lattice. In this case, for parameters in the range $b_{16}=b_{32}\simeq-0.1403$ to $-0.1386$ we find $m_{16}$

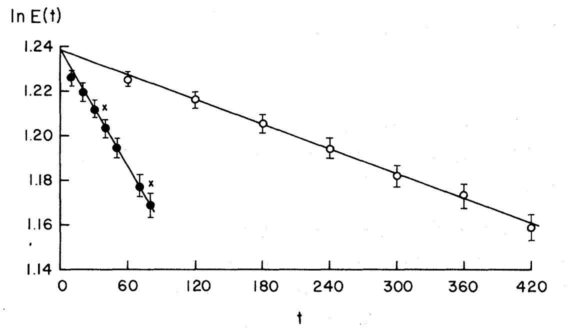


FIG. 1. Plot of $\ln E(t)$ vs $t$ for a renormalized lattice of $2\times 2$ spins. The data for initial lattices of $16\times 16$ and $32\times 32$ spins are denoted by circles and crosses, respectively. The results from Ref. 3 are indicated by squares.

$\simeq -9.50\times 10^{-4}$ to $-9.74\times 10^{-4}$, $m_{32}\simeq 2.03\times 10^{-4} - 2.08\times 10^{-4}$. This leads to a $z$ of 2.22. Together, our data for $C(t)$ and $E(t)$ would yield an average value of $z\simeq 2.23$. This estimate is in good agreement with the estimate of Tobochnik *et al.* that $z=2.22\pm 0.13$ with a most probable value of $z=2.17$. This agreement is in fact somewhat surprising, given that there exists a discrepancy between our results for $E(t)$ for the $16\times 16$ lattice and those of TSC, as shown in Fig. 1. We have at the moment no explanation for such a discrepancy. Except for the possibility of a small error in the computer program, the only possibility for such a difference is that the TSC study involved averages computed over one long run, rather than over several runs of shorter duration, as in our case.

We conclude with two observations. The first is that given the uncertainty in determining $z$, further theoretical attention should be given to improving the MCRG method. The second remark is that even with this qualification the MCRG seems to be the first real-space method proposed so far powerful enough to provide a systematic theory of critical dynamics. Our study has not indicated any major difficulty with this method, apart from suggesting the need for very accurate statistics and care in interpreting the results. Applications of this method to noncritical behavior and to other systems would seem most worthwhile.

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