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Bethe-ansatz approach to the Kondo model with arbitrary impurity spin

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The Kondo model with a spin-S impurity is formulated using a generalization of the Bethe-Yang approach. The exact relation between high- and low-temperature dimensional scales is obtained for arbitrary S. The formalism is further generalized to include two species of electrons. It is found, contrary to expectations, that the screening of the impurity's magnetic moment is not dependent on the number of species: The spin of the dressed impurity is $S-\frac{1}{2}$ in both cases. To ensure that no states have been overlooked the completeness of the basis is examined in some detail.

I. INTRODUCTION

A number of exact results in the Kondo model[1-5] have been obtained recently using the Bethe-ansatz technique and quantum inverse scattering methods. As a result, we now have a greatly improved understanding of phenomena such as the screening of the impurity's magnetic moment, the scaling property of thermodynamic functions, and the crossover between weak- and strong-coupling regimes (including the exact numerical relation—Wilson's number—between high- and low-temperature scales). Although the original methods were specifically designed to treat the Kondo model with spin-$\frac{1}{2}$ impurity, Fateev and Wiegmann[4] have been able to derive the basic equations for the experimentally more relevant spin-S case, and have shown that there is only partial screening of the impurity spin (the ground state has spin $S-\frac{1}{2}$), in conformity with the predictions of Ref. 6. In their solution, imposition of periodic boundary conditions requires diagonalization of an operator which may be expressed as a trace of a product of transfer matrices, the latter having been analyzed by Baxter[7] and later by Faddeev *et al.*[8] in the context of the quantum inverse scattering method.

In the present work we wish to answer several questions which remain concerning the Kondo model with arbitrary spin. First, we want to see whether the modified Bethe-ansatz approach of Yang,[9] exploited by Andrei in the $S=\frac{1}{2}$ case, can be further generalized to handle arbitrary impurity spin. In addition to providing a conceptually somewhat simpler construction than Ref. 4 (leading, of course, to the same equations), such an approach may provide new means for attacking one-dimensional models which have not yet yielded to traditional Bethe-ansatz methods. In Sec. II we show that a generalized Bethe-Yang solution of the problem of diagonalizing the Hamiltonian is indeed possible, and in Sec. III we discuss the classification and counting of basis states.

Our second goal in this article is to generalize to arbitrary S the calculation of Ref. 2 of Wilson's number. In Sec. IV we first summarize briefly the exact treatment of the zero-temperature magnetization curve, which can be obtained using the same methods as in the $S=\frac{1}{2}$ case. In contrast to the latter case, where two distinct scales T_0 and T_H govern, respectively, the regions $H<<T_0$ and $H>>T_0$ at $T=0$, we now have a single scale for $T=0$. Hence the ratio of high- to low-temperature scales is accessible to a perturbative calculation, which we carry out. This gives us the analog of Wilson's number for arbitrary S.

As a final item of interest, we turn our attention, in Sec. IV, to the modifications (surprisingly few, it turns out) that occur when one has two electron species interacting with the spin-S impurity with the identical coupling. Of primary interest is whether the new quantum number (which we refer to as isospin) produces additional screening (beyond the half-unit of spin screened in the single-species case) of the impurity spin. Previous work on the subject has concentrated on the determination of the ground-state spin. Abstract theorems and numerical calculations[6,10] in the manner of

Wilson[11] indicate that if one has k species of electrons, $k \leq 2S$, then the ground state will have spin $S-k/2$. In our opinion, these results are an unreliable guide to the extent of screening in the model, for two reasons: (a) For finite volume L, the alleged ground state differs in energy by amounts of order L^{-1} (or less) from a number of other states of different total spin. The positions of these levels can easily depend on the cutoff prescription and boundary conditions adopted; which of these levels, if any, corresponds to a single "dressed impurity" is not always obvious, and must be determined from the structure of the spectrum (straightforward for one electron species, much less so for two species). (b) The spin of the lowest-energy state in the absence of a magnetic field is not necessarily the same as the spin of the dressed impurity as revealed by the response of the system to a weak magnetic field. The former is found by studying energy levels with excitation energies of order L^{-1}, whereas the latter is derived from a study of states with excitation energies of order μH, where $L^{-1} << \mu H << T_0$.

In Sec. V, we determine unambiguously that the spin of the dressed impurity in the two-species case is $S-\frac{1}{2}$, just as in the single-species case. This can be seen both from the structure of the low-lying spectrum and from the asymptotic weak-field behavior of the zero-temperature magnetization curve. Interestingly, the value $S-\frac{1}{2}$ differs from the ground-state spin of $S-1$ calculated in Ref. 10, and also from the value S of the spin of the lowest-energy state in our own construction.

The skeptical reader may worry that our proposed Bethe-ansatz basis may not be sufficiently rich to describe all physically relevant states in the spin-S Kondo model with two electron species. In other words: Do the Bethe-ansatz states constitute a complete set of energy eigenstates? We believe that we have satisfactorily answered this question, in the affirmative, in Sec. VI and the Appendix.

II. DIAGONALIZATION OF THE HAMILTONIAN

Our system consists of N electrons (spin-$\frac{1}{2}$ fermions) on a line segment $-L/2 \leq x \leq L/2$, with a spin-S impurity fixed at $x=0$. The Hamiltonian is

$$\mathscr{H}=-i\sum_{j=1}^{N}\frac{\partial}{\partial x_j}+2J\sum_{j=1}^{N}\delta(x_j)\vec{\sigma}_j\cdot\vec{S}\ , \tag{2.1}$$

where $\vec{\sigma}_j/2$ is the spin operator of the jth electron and $\vec{S}$ is that of the impurity, with

$$(\vec{\sigma}/2)^2=\tfrac{1}{2}(\tfrac{1}{2}+1)\ ,$$
$$\vec{S}^2=S(S+1)\ .$$

We wish to construct a complete set of mutually orthogonal, antisymmetric wave functions, which are simultaneously eigenstates of $\mathscr{H}$, $\vec{\mathscr{S}}$, and $\mathscr{S}_z$, where $\vec{\mathscr{S}}$ is the total spin,

$$\vec{\mathscr{S}}=\vec{S}+\tfrac{1}{2}\sum_{j=1}^{N}\vec{\sigma}_j\ .$$

We assume the following form for the basis wave functions of *distinguishable* electrons:

$$L^{-N/2}e^{i(k_1x_1+k_2x_2+\cdots+k_Nx_N)}\sum_{Q,\nu}\theta(x_{Q1}<\cdots<x_{Q\nu}<0<x_{Q(\nu+1)}<\cdots<x_{QN})\xi^{\nu}_{a_{Q1}\cdots a_{QN}\alpha} \tag{2.2}$$

with

$$a_i=\pm\tfrac{1}{2}\ \text{(spin index for electrons)}\ ,\quad -S\leq\alpha\leq S\,\text{(spin index for impurity)}\ .$$

The sum over Q runs through all permutations of the N electrons, whereas ν, which marks the position of the impurity within the sequence of electrons, runs through the integers from 0 to N. The coefficients $\xi^{\nu}_{a_{Q1}\cdots a_{QN}\alpha}$ and momenta are to be determined by applying the eigenvalue equations and periodic boundary conditions.

Assuming that a *complete* basis of this type can be constructed, we shall obtain immediately a complete basis for fermionic electrons by antisymmetrization. The basis wave functions will have the form

$$L^{-N/2}(N!)^{-1/2}\left[\sum_{P}(-1)^{P}e^{i(k_{P1}x_1+\cdots+k_{PN}x_N)}\right]\sum_{Q,\nu}\theta(x_{Q1}<\cdots<x_{Q\nu}<0<\cdots<x_{QN})\xi^{\nu}_{a_{Q1}\cdots a_{QN}\alpha} \tag{2.3}$$

with $k_1 < k_2 < \cdots k_N$. Note that in this basis, the plane-wave part of the wave function is antisymmetric, and this leads to the simplifying feature that all k_i must be distinct.

Postponing until later the question of completeness, let us now see what constraints the energy eigenvalue equation places on a wave function F of the form (2.2):

$$\left[\mathscr{H}-\sum_{j=1}^{N}k_j\right]F=\sum_{i=1}^{N}\delta(x_i)e^{i\sum_j k_jx_j}\sum_{\substack{Q,v\\Qv=i}}\theta(x_{Q1}<\cdots<x_i=0<x_{Q(v+1)}<\cdots<x_{QN})$$
$$\times[(\xi^{v-1}_{a_{Q1}\cdots a_{QN}\alpha}-\xi^{v}_{a_{Q1}\cdots a_{QN}\alpha})$$
$$+J(\vec{\sigma})_{a_{Qv}a'_{Qv}}\cdot(\vec{S})_{\alpha\alpha'}(\xi^{v-1}_{a_{Q1}\cdots a'_{Qv}\cdots a_{QN}\alpha'}+\xi^{v}_{a_{Q1}\cdots a'_{Qv}\cdots a_{QN}\alpha'})] \; . \tag{2.4}$$

If F is to be an eigenstate of H, the coefficient of each δ function in (2.4) must vanish, which implies

$$\xi^{v-1}_{a_{Q1}\cdots a_{QN}\alpha}-\xi^{v}_{a_{Q1}\cdots a_{QN}\alpha}+J(\vec{\sigma})_{a_{Qv}a'_{Qv}}\cdot(\vec{S})_{\alpha\alpha'}(\xi^{v-1}_{a_{Q1}\cdots a'_{Qv}\cdots a_{QN}\alpha'}+\xi^{v}_{a_{Q1}\cdots a'_{Qv}\cdots a_{QN}\alpha'})=0 \; . \tag{2.5}$$

This provides a set of linear relations which allow one to calculate all ξ^v in terms of ξ^0. A second such relation is a consequence of the perodic boundary conditions, namely, for all j,

$$F\,|_{x_j=-L}=F\,|_{x_j=+L}$$

and so we must have, from (2.2),

$$\xi^{v}_{a_j\underline{a}\alpha}=e^{ik_jL}\xi^{v-1}_{\underline{a}a_j\alpha} \; , \tag{2.6}$$

where $\underline{a}$ stands for any collection of $N-1$ indices.

Equations (2.5) and (2.6) together imply that $\xi^0_{a\alpha}$ satisfies

$$Z\xi^0=\hat{\lambda}\xi^0 \; , \quad Z=Y_{0N}\tilde{P} \; , \tag{2.7}$$

where Y_{0N} and $\tilde{P}$ are matrices defined by

$$Y_{0N}=\hat{r}1+\hat{s}\sigma_{Nz}S_z$$
$$+\tfrac{1}{2}\hat{s}(\sigma_{N+}S_-+\sigma_{N-}S_+) \; ,$$
$$\hat{r}=\frac{J+i[1+S(S+1)J^2]}{J+i[1+S(S+1)J^2]} \; ,$$
$$\hat{s}=\frac{2J}{J+i[1+S(S+1)J^2]} \; ,$$
$$|\hat{r}+\hat{s}S|=1 \; ,$$
$$(\tilde{P}\xi)_{a_1\cdots a_N}\equiv\xi_{a_Na_1\cdots a_{N-1}} \; ,$$

and the eigenvalue is

$$\hat{\lambda}=e^{ik_jL} \; . \tag{2.8}$$

Equation (2.7) is quite similar, but not identical, to the discrete eigenvalue equation which arises in other models (in particular in the $S=\frac{1}{2}$ Kondo model[1]). There one could regard a spinorial tensor $\phi_{a_1\cdots a_n}$, with $a_i=\pm 1$ and $\sum a_i=N-2M$, as a wave function on a one-dimensional lattice of N sites:

$$\phi_{a_1\cdots a_N}=\varphi(y_1,\ldots,y_M) \; ,$$
$$y_1<y_2<\cdots<y_M$$

where y_j is the position of the jth downspin. The eigenvalue equation analogous to (2.7) could then be solved by the Bethe-ansatz method, as modified by Yang.[9] In order to generalize this procedure to include impurity spin $S>\frac{1}{2}$, we write

$$\phi_{a_1\cdots a_N\alpha}=\varphi_\alpha(y_1,\ldots,y_L) \tag{2.9}$$

with

$$\alpha+\tfrac{1}{2}N-L=\mathscr{S}_z=S+\tfrac{1}{2}N-M \; .$$

In terms of the wave functions (2.9), Eq. (2.7) takes the form

$$\lambda\varphi_\alpha(y_1,\ldots,y_{L-1}N)=(r-\alpha s)\varphi_\alpha(1,y_1+1,\ldots,y_{L-1}+1)$$
$$+s\sqrt{(S+\alpha)(S-\alpha+1)}\varphi_{\alpha-1}(y_1+1,\ldots,y_{L-1}+1) \; , \tag{2.10}$$

$$\lambda\varphi_\alpha(y_1,\ldots,y_L<N)=(r+\alpha s)\varphi_\alpha(y_1+1,\ldots,y_L+1)$$
$$+s\sqrt{(S-\alpha)(S+\alpha+1)}\varphi_{\alpha+1}(1,y_1+1,\ldots,y_L+1)\ , \tag{2.11}$$

where

$$\lambda=\frac{\hat\lambda}{\hat r+\hat s S}\ ,\quad r=\frac{\hat r}{\hat r+\hat s S}\ ,\quad s=\frac{\hat s}{\hat r+\hat s S}=\frac{1}{S+\frac{1}{2}+ic^{-1}}\ ,\quad c=\frac{2J}{1-S(S+1)J^2}\ .$$

In addition, the condition

$$\mathcal{S}_+\phi=0\ , \tag{2.12}$$

which we may impose without loss of generality (since basis states may be obtained from those with $\mathcal{S}_z=\mathcal{S}$ by multiple application of $\mathcal{S}_-$), takes the form

$$\sqrt{(S-\alpha)(S+\alpha+1)}\varphi_\alpha(y_1,\ldots,y_L)=-\sum_{1\le y\ne y_j}\varphi_{\alpha+1}(y_1,\ldots,y,\ldots,y_L)\ ,\quad -S-1\le\alpha\le S\ . \tag{2.13}$$

By interpreting the arguments $\alpha, y_1, \ldots, y_L$ of our wave functions (2.9) as specifying the positions of M identical down spins, $S-\alpha$ of them located at site 0, we are led to the following ansatz:

$$\varphi_\alpha(y_1,\ldots,y_L)=v_\alpha\sum_P A_P\prod_{i=1}^{S-\alpha}\Phi(\chi_{Pi})\prod_{j=1}^{L}f(\chi_{P(S-\alpha+j)},y_j)\ , \tag{2.14}$$

where the summation is over all permutations of $M=S-\alpha+L$ symbols, Φ and f are single down-spin wave functions (the former associated with site 0), and A_P and v_α are coefficients. We find that Eqs. (2.10), (2.11), and (2.13) may be satisfied by wave functions of the form (2.14), provided that we have

$$v_\alpha=\left[(2S)^{-(S-\alpha)}\begin{bmatrix}2S\\ S-\alpha\end{bmatrix}\right]^{1/2}\ , \tag{2.15a}$$

$$f(\chi_j,y)=\mu_j^{y-1}\ ,\quad \mu_j=\frac{\chi_j+i/2}{\chi_j-i/2}\ , \tag{2.15b}$$

$$\Phi(\chi)=\frac{\sqrt{2S}\left[\chi-\dfrac{i}{2}\right]}{\chi+\dfrac{1}{c}+Si}\ ,\quad c=\frac{2J}{1-S(S+1)J^2} \tag{2.15c}$$

$$\frac{A_{P'}}{A_P}=\frac{\chi_{Pj}-\chi_{P'j}-i}{\chi_{Pj}-\chi_{P'j}+i}\ ,\quad P'j=P(j+1)\ ,\quad P'(j+1)=Pj\ . \tag{2.15d}$$

The parameters χ_j, $j=1,2,\ldots,M$ must satisfy the coupled equations

$$\left[\frac{\chi_j+i/2}{\chi_j-i/2}\right]^N\left[\frac{\chi_j+1/c+Si}{\chi_j+1/c-Si}\right]=-\prod_{k=1}^{M}\left[\frac{\chi_j-\chi_k+i}{\chi_j-\chi_k-i}\right]\ . \tag{2.16}$$

Once a solution set $\{\chi_1,\ldots,\chi_M\}$ of (2.16) has been found, the corresponding eigenvalue $\hat\lambda$ of Z is

$$\hat\lambda=e^{i\eta}\prod_{j=1}^{M}\mu_j\ ,\quad e^{i\eta}=\hat r+S\hat s \tag{2.17}$$

which yields for the momenta

$$k_i=\frac{2\pi}{L}n_i+\frac{\eta}{L}-\frac{1}{L}\sum_{j=1}^{M}(2\tan^{-1}2\chi_j+\pi)\ ,\quad n_i=\text{ integer} \tag{2.18}$$

and for the energy (relative to the ground state)

$$E=\sum_{i=1}^{N}k_i=-NK+\frac{2\pi}{L}\sum_{i=1}^{N}n_i-\frac{N}{L}\sum_{j=1}^{M}(2\tan^{-1}2\chi_j+\pi)+\frac{N\eta}{L}\ , \tag{2.19}$$

where K is fixed by the condition that the ground-state energy vanishes.

The remainder of this section will be devoted to showing that Eqs. (2.10), (2.11), and (2.13) are indeed satisfied by wave functions of the form (2.14) with (2.15), (2.16), and (2.17).

Proof of (2.13). Suppose $\varphi_\alpha(y_1,\ldots,y_L)$ has the form (2.14) for all α and L. Since $f(\chi_j,y)$ is simply a power of μ_j, the summation over y can be performed explicitly using the standard formula for geometric series. One obtains

$$\sum_{1\le y\ne y_j}\varphi_{\alpha+1}(y_2,\ldots,y,\ldots,y_L)$$

$$=v_{\alpha+1}\sum_P A_P\prod_{l=1}^{S-\alpha-1}\Phi(\chi_{Pl})\left[\frac{1-\mu_{P1'}^{y_2-1}}{1-\mu_{P1'}}\prod_{j=2}^{L}f(\chi_{Pj'},y_j)\Bigg|+\sum_{j=2}^{L-1}\prod_{i<j}f(\chi_{Pi'},y_{j+1})\left[\frac{\mu_{Pj'}^{y_j}-\mu_{Pj'}^{y_{j+1}-1}}{1-\mu_{Pj'}}\right]\prod_{k>j}f(\chi_{Pk'},y_k)\right.$$

$$\left.+\prod_{i=1}^{L-1}f(\chi_{Pi'},y_{i+1})\left[\frac{\mu_{PM}^{y_L}-\mu_{PM}^{N}}{1-\mu_{PM}}\right]\right]\ , \tag{2.20}$$

where $i'=i+S-\alpha-1$. Now (2.15d) and (2.16) imply the following identities for the coefficients A_P:

$$A_P\left[\frac{\mu_{P(j-1)}^{y_j-1}\mu_{Pj}^{y_j}}{1-\mu_{Pj}}-\frac{\mu_{P(j-1)}^{y_j-1}\mu_{Pj}^{y_j-1}}{1-\mu_{P(j-1)}}\right]+A_{P''}\left[\frac{\mu_{Pj}^{y_j-1}\mu_{P(j-1)}^{y_j}}{1-\mu_{P(j-1)}}-\frac{\mu_{Pj}^{y_j-1}\mu_{P(j-1)}^{y_j-1}}{1-\mu_{Pj}}\right]=0\ , \tag{2.21}$$

where $Pj=P''(j-1)$ and $P(j-1)=P''j$, and

$$A_{\tilde P}\equiv A_{P2P3\cdots PMP1}=\frac{A_P}{(\rho_{P1}\mu_{P1}^{N})}\ , \tag{2.22}$$

where

$$\rho_j\equiv\frac{\chi_j+1/c+Si}{\chi_j+1/c-Si}$$

With the aid of (2.21) and (2.22) and some algebra, the right-hand side of (2.20) can be considerably simplified:

$$\sum_{1\le y\ne y_j}\varphi_{\alpha+1}(y_1,\ldots,y,\ldots,y_L)$$

$$=v_{\alpha+1}\sum_P A_P\prod_{l=1}^{S-\alpha}\Phi(\chi_{Pl})\prod_{j=2}^{L}f(\chi_{Pj'},y_j)\left[\frac{1}{(1-\mu_{P1'})\Phi(\chi_{\rho 1'})}-\frac{1}{\rho_{P1}\Phi(\chi_{P1})(1-\mu_{P1})}\right]. \tag{2.23}$$

Substituting the expressions for Φ, μ_j, and ρ_j in terms of χ_j, the quantity in large parentheses in (2.23) reduces to

$$\frac{i}{\sqrt{2S}}(\chi_{P1'}-\chi_{P1}+2Si)=\frac{i}{\sqrt{2S}}\sum_{j=1}^{S-\alpha-1}(\chi_{P(j+1)}-\chi_{Pj}+i)-\frac{1}{\sqrt{2S}}(S+\alpha+1)\ . \tag{2.24}$$

Substituting (2.24) and (2.23), one finds that only the last (χ-independent) term in (2.24) survives the sum over P. We thus end up with

$$\sum_{1\leq y\neq y_j} \varphi_{\alpha+1}(y_1,\ldots,y,\ldots,y_L) = -\frac{S+\alpha+1}{\sqrt{2S}} v_{\alpha+1}\sum_P A_P \prod_{l=1}^{S-\alpha}\Phi(\chi_{Pl})\prod_{j=2}^{L} f(\chi_{Pj'},y_j) . \tag{2.25}$$

Since by (2.15a)

$$\frac{1}{\sqrt{2S}}(S+\alpha+2)v_{\alpha+1} = \sqrt{(S+\alpha+1)(S-\alpha)}\,v_\alpha$$

we obtain, finally, Eq. (2.13). Note that in the case $\alpha=-S-1$, the left-hand side vanishes.

Proof of (2.10) and (2.11). Before proceeding to the general case, let us set $M=1$ and $\alpha=S$, so that (2.10) and (2.11) reduce to

$$\lambda f(\chi,N) = (r-\alpha s) f(\chi,1) + s\sqrt{2S}\,\Phi(\chi) , \tag{2.26}$$

$$\lambda f(\chi,y<N) = f(\chi,y+1) . \tag{2.27}$$

Equations (2.26) and (2.27) are easily verified using the definitions (2.15b) and (2.15c) and the relation (2.16). The generalization to $\alpha=S$ and arbitrary M is trivial. From here one proceeds by induction on $S-\alpha$. Suppose (2.9) is valid for all impurity spin projections $\geq\alpha$. Then for $1\leq y\notin\{y_1,y_2,\ldots,y_{L-1},N\}$ we have

$$\begin{aligned}\lambda\varphi_\alpha(y_1,\ldots,y,\ldots,y_{L-1},N) &= (r-\alpha s)\varphi_\alpha(1,y_1+1,\ldots,y+1,\ldots,y_{L-1}+1)\\ &\quad + s\sqrt{(S+\alpha)(S-\alpha+1)}\,\varphi_{\alpha-1}(y_1+1,\ldots,y+1,\ldots,y_{L-1}+1) .\end{aligned} \tag{2.28}$$

If we now sum both sides of (2.28) over y, omitting $y_1, y_2, \ldots, y_{L-1}$ and N, and substitute (2.13), we obtain (2.10) for spin projection $\alpha-1$. Equation (2.11) is proved in a similar fashion.

III. CLASSIFICATION OF STATES

In this section we discuss the physically relevant solutions of (2.16). Since much of the ground has been covered before, in the $S=\frac{1}{2}$ case[1] and in Ref. 12, we shall keep the discussion quite brief. We begin by restricting our attention to solutions of (2.16) for which all χ_j are real. It is convenient to take the logarithm of both sides of the equation to obtain

$$2N\tan^{-1}(2\chi) + 2\tan^{-1}\left[\frac{\chi+c^{-1}}{S}\right] - 2\sum_{j=1}^{M}\tan^{-1}(\chi-\chi_j) = 2\pi J(\chi) . \tag{3.1}$$

We are interested in those solutions χ of (3.1) for which $J(\chi)$ takes on one of the values

$$-\frac{N-M}{2}, -\frac{N-M}{2}+1, \ldots, \frac{N-M}{2} . \tag{3.2}$$

M of these solutions comprise the set $\{\chi_1,\ldots,\chi_M\}$, and are known as 1-strings (the terminology will become clear shortly); the remaining $N+1-2M$ solutions are called 1-string holes. As in the $S=\frac{1}{2}$ case,[1] the ground state corresponds to an absence of holes (provided, as we shall assume, N is odd). Hence we have

$$M=\tfrac{1}{2}(N+1) , \quad \mathscr{S}=S-\tfrac{1}{2} , \tag{3.3}$$

so that for $S>\frac{1}{2}$, the ground state is $2S$-fold degenerate. To determine the 1-strings in this state, we exploit the fact that the solutions of (3.1) become dense on the χ axis for N tending to infinity, so that for sufficiently large N it is an excellent approximation to replace the summation in (3.1) by an integral,

$$\sum_j \tan^{-1}(\chi-\chi_j) \to \int d\chi'\sigma(\chi')\tan^{-1}(\chi-\chi') .$$

If holes are located at $\mathcal{X}_1^h,\ldots,\mathcal{X}_{M_h}^h$, then $\sigma(\mathcal{X})$ is related to $J(\mathcal{X})$ by

$$\frac{dJ(\mathcal{X})}{d\mathcal{X}}=\sigma(\mathcal{X})+\sum_{i=1}^{M_h}\delta(\mathcal{X}-\mathcal{X}_i^h)\ . \tag{3.4}$$

Differentiating (3.1) with respect to $\mathcal{X}$ and inserting (3.4), one obtains

$$\begin{gathered}\sigma(\mathcal{X})+\int d\mathcal{X}'\sigma(\mathcal{X}')K(\mathcal{X}-\mathcal{X}')=f(\mathcal{X})-\sum_{i=1}^{M_h}\delta(\mathcal{X}-\mathcal{X}_i^h)\ ,\\ K(\mathcal{X})=\frac{1}{\pi}\frac{1}{1+\mathcal{X}^2}\ ,\ \ f(\mathcal{X})=\frac{1}{\pi}\left[\frac{N/2}{(\frac{1}{2})^2+\mathcal{X}^2}+\frac{S}{S^2+(\mathcal{X}+c^{-1})^2}\right],\end{gathered} \tag{3.5}$$

which may be solved by Fourier transformation:

$$\tilde{\sigma}(p)=\tilde{\sigma}_0(p)-\sum_{j=1}^{M_h}\frac{e^{i\mathcal{X}_j^h p}e^{|p|/2}}{2\cosh(p/2)}\ . \tag{3.6}$$

The first term in (3.6) corresponds to the ground-state distribution

$$\tilde{\sigma}_0(p)=\frac{N}{2\cosh(p/2)}+\frac{e^{ip/c}e^{-(S-1/2)|p|}}{2\cosh(p/2)}\ . \tag{3.7}$$

Formula (3.6), when substituted in (2.19), allows us to compute the energy contribution of each hole, i.e.,

$$E(\mathcal{X}_j^h)=\frac{2N}{L}\tan^{-1}(e^{\pi\mathcal{X}_j^h})\ . \tag{3.8}$$

Since this is positive, we verify that the state without holes is indeed the minimum-energy state (among those with only real $\mathcal{X}_j$). More detailed analysis shows that, as in the $S=\frac{1}{2}$ case, the energy relative to the ground state is always given by a sum of contributions (3.8); moreover, in order to have nonreal $\mathcal{X}_j$, one must have at least one 1-string hole, hence positive energy relative to the alleged ground state. Note that there is no mass gap: The energy contribution of a hole at the extreme negative end of the $\mathcal{X}$ axis is only of order $1/L$.

In the presence of a magnetic field H, it is no longer true that the minimum energy state corresponds to an absence of holes. Placing a hole in the 1-string distribution will, of course, cost some energy, but if H is large enough and the hole is far enough to the left on the $\mathcal{X}$ axis, the net effect is to decrease the energy. The ground state will then have holes from $-\infty$ to B, and only 1-strings to the right of B. (There are, of course, other states, with nonreal $\mathcal{X}_j$, with the same total spin $\mathcal{S}$ as this state; these states would have additional holes and higher energy.) The distribution $\sigma(\mathcal{X})$ for such a state would then be given by

$$\sigma(\mathcal{X})+\int_B^\infty d\mathcal{X}'\sigma(\mathcal{X}')K(\mathcal{X}-\mathcal{X}')=f(\mathcal{X})\ , \tag{3.9}$$

an integral equation which can be solved by the Wiener-Hopf technique.[2] The results of this calculation are summarized in Sec. IV.

The classification of general excited states of our system is completely analogous to the $S=\frac{1}{2}$ case, and will not be treated in detail here. We do wish to convince ourselves, however, that the spinorial basis which we have constructed completely spans the $(2S+1)\times 2^N$ dimensional space of tensors $\phi_{a_1\ldots a_N a}$. To this end, we follow Takahashi[13] and make the assumption (whose consistency has been thoroughly checked for states with a macroscopic number of 1-string holes, i.e., $M_h \propto L$) that apart from corrections of order $\exp(-\kappa L)$, $\kappa>0$, each $\mathcal{X}_j$ in a solution set $\{\mathcal{X}_1,\ldots,\mathcal{X}_M\}$ of (2.16) is a member of an n-string, i.e., a family of $\mathcal{X}_j$ with the same real part, of the form

$$\{\mathcal{X}_{njl}=\mathcal{X}_{nj}+i(n+1-2l)/2,\ \mathcal{X}_{nj}\text{ real},\ l=1,2,\ldots,n\}\ ,\ \ j=1,2,\ldots,M_n\ . \tag{3.10}$$

Writing Eq. (2.16) for each member of such a string and taking the product of these n equations, one obtains, as the generalization of (2.10) of Ref. 13,

$$e^{N}(2\chi_{nj}/n)\prod_{l=1}^{\min(2S,n)} e\left[\frac{2\chi_{nj}+2c^{-1}}{2S+n+1-2l}\right]=\prod_{(m,k)\neq(n,j)} E_{nm}(\chi_{nj}-\chi_{mk}),\ \ j=1,2,\ldots,M_n$$

$$E_{nm}(x)=\begin{cases} e\left[\frac{x}{|n-m|}\right]e^2\left[\frac{x}{|n-m|+2}\right]\cdots e^2\left[\frac{x}{m+n-2}\right]e\left[\frac{x}{m+n}\right], & n\neq m\\ e^2\left[\frac{x}{2}\right]e^2\left[\frac{x}{4}\right]\cdots e^2\left[\frac{x}{2n-2}\right]e\left[\frac{x}{2n}\right], & n=m\end{cases} \qquad (3.11)$$

$$e(x)=(x+i)/(x-i)\ .$$

The logarithm of (3.11) gives

$$N\Theta\left[\frac{2\chi_{nj}}{n}\right]+\sum_{l=1}^{\min(2S,n)}\Theta\left[\frac{2\chi_{nj}+2c^{-1}}{2S+n+1-2l}\right]=2\pi J_n(\chi_{nj})+\sum_{m=1}^{\infty}\sum_{k=1}^{M_n}\Theta_{nm}(\chi_{nj}-\chi_{mk})\ ,$$

where $\Theta(x)=2\tan^{-1}(x)$ and

$$\Theta_{nm}(x)=\begin{cases}\Theta\left[\frac{x}{|n-m|}\right]+2\Theta\left[\frac{x}{|n-m|+2}\right]+\cdots+2\Theta\left[\frac{x}{n+m-2}\right]+\Theta\left[\frac{x}{n+m}\right], & n\neq m\\ 2\Theta\left[\frac{x}{2}\right]+2\Theta\left[\frac{x}{4}\right]+\cdots+2\Theta\left[\frac{x}{2n-2}\right]+\Theta\left[\frac{x}{2n}\right], & n=m.\end{cases}$$

A general solution of these coupled equations has not been achieved. However, by treating the roots of the equations statistically, with the aid of densities σ_n, σ_{nh} of n-strings and n-strings holes, one can reformulate the problem of computing thermodynamic quantities at finite temperature as that of solving a set of coupled integral equations.[13,14] Our interest will be mainly in counting the number of independent solutions. According to Takahashi's argument,[13] $J_n(\chi)$ in (3.12) takes on at least

$$N-\sum_j t_{nj}M_j+\min(n,2S),$$

$$t_{nj}=\begin{cases}2\min\{n,j\}, & n\neq j\\ 2n-1, & n=j\end{cases}$$

values, M_n of which correspond to n-strings, the remainder to n-string holes. The total number of combinations is thus at least

$$\begin{bmatrix} N-\sum_j t_{nj}M_j+\min(n,2S)\\ M_n\end{bmatrix},$$

and thus the total number $n_{\mathscr{S}}$ of solutions of the full set of equations for a given value of the total spin $\mathscr{S}$ $(=\frac{1}{2}N+S-M)$ is bounded below by formula (6) of the Appendix. In the Appendix we show that the counting of roots yields precisely the right number of mutually orthogonal basis vectors for each value of $\mathscr{S}$.

IV. SCALES AND UNIVERSAL NUMBERS

In this section we follow the procedure devised by Andrei and Lowenstein[2] in order to determine the dimensional scales parametrizing the magnetization $\mathscr{M}^i$ in different asymptotic regions of the (T,H) plane, where T is the temperature and H is the magnetic field expressed as a multiple of μ, the magnetic moment of a single electron.

Let us first summarize the case $S=\frac{1}{2}$. The formulation of the finite temperature thermodynamics with the assumption $T<<D$ (scaling regime), where D is the ultraviolet cutoff, leads to a universal function for the impurity part of the free energy

$$F^i=-Tf^i(T/T_0,H/T)\ ,$$

where $T_0=D\exp[-\pi/c(J)]$ is the dynamically generated scale. By comparing the high-temperature expansion of the susceptibility in this formulation and the one calculated using usual perturbation theory it is shown that the coupling constant defined by the cutoff procedure of the present formalism and the one defined conventionally using a momentum cutoff have a nonanalytic relationship. Hence, the expression for T_0 in terms of the cutoff and coupling constant clearly depends on the particular scheme of regularization. On the other hand, there are various scales characterizing the thermodynamic functions in different asymptotic regions, such as:

$$\begin{aligned} &\text{(I)}\ T=0\ ,\ \ H<<T_0\ ,\\ &\text{(II)}\ T=0\ ,\ \ H>>T_0\ ,\\ &\text{(III)}\ T>>T_0\ ,\ \ H<<T_0\ . \end{aligned} \tag{4.1}$$

More specifically, for the magnetization $\mathcal{M}^i$ the following expansions are valid:

$$\begin{aligned} &\mathcal{M}^i(\text{I})\sim\mu\pi^{-1}\frac{H}{T_0}\ ,\\ &\mathcal{M}^i(\text{II})\sim\mu\{\,1-\tfrac{1}{2}[\ln(H/T_H)]^{-1}-\tfrac{1}{4}[\ln(H/T_H)]^{-2}\ln\ln(H/T_H)+\mathcal{O}([\ln(H/T_H)]^{-3})\,\}\ ,\\ &\mathcal{M}^i(\text{III})\sim\mu\frac{H}{T}\{\,1-[\ln(T/T_K)]^{-1}-\tfrac{1}{2}[\ln(T/T_K)]^{-2}\ln\ln(T/T_K)+\mathcal{O}([\ln(T/T_K)]^{-3})\,\}\ , \end{aligned} \tag{4.2}$$

where $T_0=T_{\text{I}}$, $T_H=T_{\text{II}}$, $T_K=$ the Kondo temperature $=T_{\text{III}}$. Notice the absence of the term of $O([\ln(\ )]^{-2})$ in the last two expressions: It has been absorbed into the definition of the scales. Now the ratios of the scales such as $T_{\text{II}}/T_{\text{I}}$, $T_{\text{III}}/T_{\text{II}}$, and $T_{\text{III}}/T_{\text{I}}$ are universal numbers, independent of the particular choice of cutoff procedure. The first ratio is calculated from the exact formula of the magnetization at $T=0$ and arbitrary H: $T_H/T_0=(\pi/e)^{1/2}$. The ratio T_K/T_H is calculated with the help of perturbation theory:

$$T_K/T_H=2\beta\gamma e^{-7/4}$$

[β and γ given in Eq. (4.17)]. Combining the two ratios above, the third one T_K/T_0 is obtained:

$$W=T_K/T_0=2\beta\gamma\pi^{1/2}e^{-9/4}\ , \tag{4.3}$$

which is the analytical expression of the number calculated numerically by Wilson.[11]

Turning now to the higher spin system, the formulation of the finite temperature thermodynamics follows analogously and the scaling property can also be shown.[5] Considering the same asymptotic regions (I), (II), and (III) above, the behavior of the magnetization function $\mathcal{M}^i$ can be studied.

We proceed with the computation of the ratio $T_{\text{II}}/T_{\text{I}}$. The system at $T=0$ and arbitrary H is characterized by a density $\sigma(X)$ of real X's (Ref. 15) satisfying the generalized equation derived from Eq. (3.12) in the preceding section:

$$\sigma(X)+\int_{B(H)}^{\infty}K(X-X')\sigma(X')dX'=f(X)\ , \tag{4.4}$$

where the S dependence enters only through the driving term

$$f(X)=\frac{2}{\pi}\left[\frac{N}{1+4X^2}+\frac{2S}{(2S)^2+4(X+c^{-1})^2}\right] \tag{4.5}$$

and $K(X)$ is as before[1]

$$K(X)=\frac{1}{\pi}\frac{1}{1+X^2}\ .$$

Equation (4) can be solved using the same method as in $S=\frac{1}{2}$ case, giving for the magnetization the following:

$$\begin{aligned}\mathcal{M}&=\mu\left[N+2S-2\int_{B(H)}^{\infty}dX\,\sigma(X)\right]\\ &=\mathcal{M}^e+\mathcal{M}^i\ ,\end{aligned} \tag{4.6}$$

where

$$\mathcal{M}^e=\mu\left[\frac{2}{\pi e}\right]^{1/2}LT_0e^{\pi(B+c^{-1})}\ ,\ \ B<<0 \tag{4.7}$$

$$\mathcal{M}^i=\begin{cases}\mu\left[2S-\pi^{-3/2}\displaystyle\int_0^{\infty}\frac{dt}{t}\sin(2\pi St)e^{-2\pi(B+c^{-1})t}e^{t-t\ln t}\Gamma(\tfrac{1}{2}+t)\right]\ ,\ \ 0\le B+c^{-1}<<c^{-1}\\ \mu\left[2S-1+2^{3/2}\displaystyle\sum_{j=0}^{\infty}(-1)^j\int_0^{\infty}\frac{dt}{t}\sin[(j+S)2\pi t]\frac{e^{t(\ln t-1)}}{\Gamma(\frac{1}{2}+t)}e^{2\pi(B+c^{-1})t}\right]\ ,\ \ B+c^{-1}\le 0\ ,\ \ S>\tfrac{1}{2}\ .\end{cases} \tag{4.8}$$

The electron's part $\mathcal{M}^e$ can similarly be identified with the free-electron system, namely,

$$\mathcal{M}^e = \frac{\mu L H}{\pi} = \mu\left[\frac{2}{\pi e}\right]^{1/2} L T_0 e^{\pi(B+c^{-1})} , \tag{4.9}$$

thus relating B with H. Hence, the B dependence in the integrand of (4.8) can be written as follows:

$$e^{\mp 2\pi(B+c^{-1})t} = \begin{cases} \exp\left\{-t\ln\left[H\Big/\left[\frac{2\pi}{e}\right]^{1/2} T_0\right]\right\}, & H > T_0 \\ \exp\left\{-t\ln\left[\left[\frac{2\pi}{e}\right]^{1/2} T_0/H\right]\right\}, & H < T_0 . \end{cases} \tag{4.10}$$

Expanding the rest of the integrand around $t=0$, one can obtain asymptotic expansions for $H << T_0$, and $H >> T_0$ (still $H << D$). Absorbing the $[\ln(\ )]^{-2}$ term into the definition of scale, we get

$$\mathcal{M}^i(\mathrm{I}) = \mu(2S-1)\left\{1+\frac{1}{\ln(T_H/H)} - \frac{1}{4}\frac{\ln\ln(T_H/H)}{[\ln(T_H/H)]^2} + O\left[\left[\ln\left[\frac{T_H}{H}\right]\right]^{-3}\right]\right\}, \quad S > \tfrac{1}{2} \tag{4.11}$$

with

$$T_H = T_0\left[\frac{\pi}{e}\right]^{1/2} , \tag{4.12}$$

and

$$\mathcal{M}^i(\mathrm{II}) = \mu 2S\left\{1 - \frac{1}{\ln(H/T_H)} + \frac{\ln\ln(H/T_H)}{4[\ln(H/T_H)]^2} + O\left[\left[\ln\left[\frac{T_H}{H}\right]\right]^{-3}\right]\right\}, \quad S \geq \tfrac{1}{2} \tag{4.13}$$

with the *same* scale T_H. Therefore, for $S > \frac{1}{2}$ there is only one scale T_H characterizing the behavior at $T=0$ for both $H << T_0$ and $H >> T_0$, in contrast to the case $S=\frac{1}{2}$ where a new scale $T_{\mathrm{I}} = T_0$ is present for low field region.

The above observation implies that $T_{\mathrm{III}}/T_{\mathrm{I}} = T_{\mathrm{III}}/T_{\mathrm{II}} = T_H/T_K$ for $S > \frac{1}{2}$, and consequently it is sufficient to calculate the ratio T_H/T_K which is accessible to ordinary perturbation theory. Therefore, we have calculated the impurity part of the free energy up to second order in perturbation theory. The asymptotic magnetization formulas in regions II and III can be derived from the free energy formula, expressed in terms of the momentum cutoff $\mathcal{D}$ and coupling constant g:

$$\mathcal{M}^i(\mathrm{II}) \underset{\substack{T/H\to 0 \\ \mathcal{D} >> H,T}}{\sim} \mu 2S\left[1 - \frac{1}{\pi}g - 2\left[\frac{g}{\pi}\right]^2 \ln\left|\frac{\mathcal{D}e}{2H}\right| + O(g^3)\right] \tag{4.14a}$$

or, equivalently,

$$\mathcal{M}^i(\mathrm{II}) \sim \mu 2S\left[1 - \frac{1}{2\ln(H/T_H)} + \cdots\right], \tag{4.14b}$$

where

$$T_H = \tfrac{1}{2}\mathcal{D}e^{-\pi/2g} , \tag{4.15}$$

and

$$\mathcal{M}^i(\mathrm{III}) \underset{\substack{H/T\to 0 \\ \mathcal{D} >> H,T}}{\sim} \mu\frac{H}{T}\frac{2S(S+1)}{3}\left[1 - \frac{2}{\pi}g + 2\left[\frac{g}{\pi}\right]^2 \ln\left|\frac{T}{\mathcal{D}\beta\gamma e^{-7/4}\alpha^{[1-4S(S+1)/3]/10}}\right| + O(g^3)\right] \tag{4.16a}$$

or

$$\mathscr{M}^i(\mathrm{III}) \sim \mu \frac{H}{T} \frac{2S(S+1)}{3} \left[1 - \frac{1}{\ln(T/T_K)} + \cdots \right], \tag{4.16b}$$

where

$$\begin{aligned} T_K &= \beta\gamma e^{-7/4} \alpha^{[1-4S(S+1)/3]/10} \mathscr{D} e^{-\pi/2g}, \\ \ln\beta &= \int_0^1 dx\,(1-x^2)x[\pi^2 \csc^2(\pi x) - x^{-2}] = 0.662122\ldots, \\ \ln\alpha &= \int_0^1 dx\, x[\pi^2 \csc^2(\pi x) - x^{-2} - (1-x)^{-2}] = 0.841166\ldots, \\ \ln\gamma &= 0.577216\ldots \text{ (Euler's constant)}. \end{aligned} \tag{4.17}$$

Thus, combining (4.15) and (4.17),

$$\frac{T_K}{T_H} = 2\beta\gamma e^{-7/4} \alpha^{[1-4S(S+1)/3]/10}. \tag{4.18}$$

Details of the perturbative calculation are planned to be presented in a separate publication.

V. ADDITIONAL SPECIES OF ELECTRONS

How are the results of Sec. II modified if a second species of electron is added, without modification of the Hamiltonian (2.1)? Does this lead, as certain theoretical work suggests,[6] to the screening of an additional one-half unit of the impurity's spin? The investigation of these questions, which fortunately requires only minor modification of the formalism, will occupy our attention in this section.

We now assume that each of our N electrons has a new quantum number, which we call "isospin," and that the Hamiltonian retains its isospin-independent form (2.1). Our ansatz for energy eigenstates, analogous to (2.2), is now

$$L^{-N/2} e^{i(k_1x_1 + \cdots + k_Nx_N)} \sum_{Q,\nu} \theta(x_{Q1} < \cdots < x_{Q\nu} < 0 < x_{Q(\nu+1)} < \cdots < x_{QN}) \xi^{\nu}_{a_{Q1} \cdots a_{QN} a} \zeta_{b_{Q1} \cdots b_{QN}}, \tag{5.1}$$

where the indices of the Nth rank isospin tensor take on values $\pm\frac{1}{2}$. The remainder of the notation coincides with that of (2.2).

The analysis leading to (2.7),

$$Z\phi = \hat{\lambda}\phi$$

requires no modification, but is now supplemented, to ensure periodic boundary conditions, by the relation

$$(\tilde{P}\zeta)_{b_1 \cdots b_N} \equiv \zeta_{b_N b_1 b_2 \cdots b_{N-1}} = \lambda' \zeta_{b_1 \cdots b_N}, \tag{5.2}$$

with (2.8) replaced by

$$\hat{\lambda}\lambda' = \exp(ik_jL). \tag{5.3}$$

Moreover, since both $\mathscr{H}$ and the total spin $\mathscr{S}$ commute with the total isospin operator $\mathscr{I}$, it is convenient to classify our states according to eigenvalues of $\vec{\mathscr{I}}^2$ and $\mathscr{I}_z$. Without loss of generality, we may limit our attention to those states with $\mathscr{I} = \mathscr{I}_z$, imposing

$$\mathscr{I}_+ \zeta = 0. \tag{5.4}$$

To solve simultaneously (5.2) and (5.4), we may proceed as before, writing

$$\zeta_{b_1 \cdots b_N} = \zeta(z_1, \ldots, z_{M'}), \tag{5.5}$$

where z_j is the position of the jth negative isospin, and

$$\mathscr{I}_z = \tfrac{1}{2}N - M'. \tag{5.6}$$

In terms of the discrete wave function

$\zeta(z_1,\ldots,z_{M'})$, Eqs. (5.2) and (5.4) become

$$\lambda'\zeta(z_1,\ldots,z_{M'-1},N)=\zeta(1,z_1+1,..,z_{M'-1}+1)\ , \tag{5.7}$$

$$\lambda'\zeta(z_1,\ldots,z_{M'-1}<N)=\zeta(z_1+1,\ldots,z_{M'}+1)\ , \tag{5.8}$$

$$\sum_{1\le z\ne z_j}\zeta(z_1,\ldots,z,\ldots,z_{M'})=0\ , \tag{5.9}$$

which allow Bethe-Yang solutions

$$\zeta(z_1,\ldots,z_{M'})=\sum_P B_P\prod_{j=1}^{M'}f(\omega_{Pj},z_j) \tag{5.10}$$

with

$$f(\omega,z)=\left[\frac{\omega+i/2}{\omega-i/2}\right]^{z-1}\ ,$$

$$\frac{B_{P'}}{B_P}=\frac{\omega_{Pj}-\omega_{P'j}-i}{\omega_{Pj}-\omega_{P'j}+i}\ ,$$

$$P'j=P(j+1)\ ,\quad P'(j+1)=Pj$$

$$\left[\frac{\omega_i+i/2}{\omega_i-i/2}\right]^N=-\prod_{j=1}^{M'}\left[\frac{\omega_i-\omega_j+i}{\omega_i-\omega_j-i}\right]\ ,\quad i=1,\ldots,M' \tag{5.11}$$

$$\lambda'=\prod_{j=1}^{M'}\left[\frac{\omega_j+i/2}{\omega_j-i/2}\right]\ . \tag{5.12}$$

The proof that (5.10)–(5.12) yield solutions of (5.7)–(5.9) parallels completely the spinorial case treated in Sec. II. We note that Eq. (5.11) is precisely the same as the corresponding equation in the Heisenberg model.[13]

Owing to the decoupling of the discrete eigenvalue problems, the classification of states is not difficult. With an appropriate choice of chemical potential K, the eigenvalues of $\mathscr{H}-KN$ may be written

$$E=\sum_{j=1}^{N}\frac{2\pi}{L}(n_j-n_j^0)+\sum_{j=1}^{M_{1h}}E(\chi_j^h)+\sum_{l=1}^{M'_{1h}}E(\omega_e^h)\ , \tag{5.13}$$

where $E(\chi)$ is given in (3.8), n_j^0, $j=1,\ldots,$ N are successive integers, and n_j are distinct integers satisfying $n_{j+1}>n_j\ge n_j^0$. Each basis state, labeled by integers n_j, $j=1,\ldots,N$, and 1-string holes χ_j^h, $j=1,\ldots,M_{1h}$, and ω_l^h, $l=1,\ldots,M_h$ may be interpreted as a scattering state containing the following.

(i) A fixed dressed impurity of spin $S-\frac{1}{2}$.

(ii) M_{1h} spin-$\frac{1}{2}$ neutral, isoscalar particles interacting with each other and with the dressed impurity, with momenta $=$energies$=E(\chi_j^h)$ when far separated from one another and from the dressed impurity.

(iii) M'_{1h} isospin-$\frac{1}{2}$, neutral spinless particles interacting only with each other, with momenta $=$energies$=E(\omega_l^h)$ when far separated from one another.

(iv) Noninteracting charged particles and antiparticles, bearing no spin or isopin, corresponding to the quantum numbers n_j.

The picture described here differs from that of the single-species case only in the presence of the particles associated with isospin, and these decouple from the dynamics of the spin-bearing particles (i) and (ii). There is, however, an important selection rule: The sum $M_{1h}+M'_{1h}$ must be an odd integer. *Thus there is no state containing only the dressed impurity.* In the single-species case, there is such a state and it is the ground state of the system. With two species, the low-lying spectrum is not so tidy. There are three candidates for "ground state," none of which is presumably a normalized state in the continuum limit. They are (omitting details) as follows:

(I) $N=$ even integer , $M_{1h}=1$, $M'_{1h}=0$, $M=M'=M'_1=\frac{1}{2}N$, $M_1=M$ (no n-strings , $n>1$) ,

total spin $\mathscr{S}=S$, isospin $\mathscr{I}=0$,

$$\text{Energy } E=\begin{cases}\dfrac{\pi}{2L}+O(L^{-2})\ , & S=\frac{1}{2}\\[2ex] \dfrac{\pi}{2L}\left[1-\dfrac{2S-1}{\ln(LT_0)}+O(\ln^{-2}LT_0)\right]\ , & S>\frac{1}{2}\ .\end{cases}$$

(II) $N=$ even integer , $M_{1h}=1$, $M'_{1h}=0$, $M_1=\frac{1}{2}N-1$, $M'_1=M'=\frac{1}{2}N$,

$$M=\tfrac{1}{2}N+1 \quad \left[\text{one 2-string at } \chi^s=\left[1-\frac{1}{2S}\right]\chi^h+\frac{1}{2S}(-c^{-1})\right] , \qquad \mathscr{S}=S-1 , \quad \mathscr{I}=0 ,$$

$$E=\frac{\pi}{2L}\left[1+\frac{2S+1}{\ln LT_0}+O(\ln^{-2}(LT_0))\right] ,$$

Note: case (II) applies only to $S>\frac{1}{2}$.

(III) $N=$ odd integer , $M_{1h}=0$, $M'_{1h}=1$, $M=M_1=\frac{1}{2}(N+1)$, $M'=M'_1=\frac{1}{2}(N-1)$,

$$\mathscr{S}=S-\tfrac{1}{2} , \quad \mathscr{I}=\tfrac{1}{2} , \quad E=\frac{\pi}{2L} .$$

Clearly (I) and (II) are the lowest-lying states containing the dressed impurity and one spin-bearing particle; in (I), the spins are aligned, whereas in (II) they are opposed to one another. On the other hand, (III) is the lowest-lying state containing the dressed impurity and one isospin-bearing particle. The three states are essentially degenerate, differing in energy by an amount only of order $(L\ln L)^{-1}$, with state (I) (spin S) slightly lower than the others for $S>\frac{1}{2}$. As discussed in the Introduction, there is no compelling reason why the spin of lowest-energy state should be equal to that of the dressed impurity alone, and here we see that the two are indeed different.

From the structure of the spectrum, we have concluded that the spin of the dressed impurity is $S-\frac{1}{2}$. This can also be seen from the first term of formula (4.11), which remains the same for the two-species case.

VI. COMPLETENESS OF THE BETHE-ANSATZ BASIS

To establish the completeness of our basis (5.1) of energy eigenfunctions, we must show that it spans the entire Hilbert space

$$\mathscr{H}=\mathscr{H}_x\otimes\mathscr{H}_{\text{spin}}\otimes\mathscr{H}_{\text{isospin}} ,$$

where $\mathscr{H}_x$ is the space of square-integrable functions on the line segment $[-L/2, L/2]$, $\mathscr{H}_{\text{spin}}$ is the $(2S+1)\times 2^N$-dimensional space of tensors $\phi_{a_1\cdots a_N\alpha}$ with the inner product

$$(\phi,\psi)=\sum_{a_1,\ldots,a_N,\alpha}\phi^*_{a_1\cdots a_N\alpha}\psi_{a_1\cdots a_N\alpha}$$

and $\mathscr{H}_{\text{isospin}}$ is the analogous 2^N-dimensional vector space of tensors $\zeta_{b_1\cdots b_N}$. Suppose we have found complete bases $\{f_l\}$, $\{\phi_m\}$, $\{\zeta_n\}$ of each of the three spaces. Then an arbitrary element of $\mathscr{H}$ can be expanded in terms of the product states

$$f_l\otimes\phi_m\otimes\xi_n . \tag{6.1}$$

It will be sufficient to show that every wave function of the form (6.1) may be written as a linear superposition of the basis functions (5.1). With the aid of (2.6) and (5.3), the latter may be written in the form

$$\psi^{kl}_{n\chi\omega}(x_1,\ldots,x_n;a_1,\ldots,a_N,\alpha;b_1,\ldots,b_N)$$
$$=L^{-N/2}e^{2\pi i\sum_{j=1}^{N}n_jx_j(1/L)}\sum_{Q,\nu}[\lambda(\chi)\lambda'(\omega)]^{(\sum_j x_j/L)+\nu}$$
$$\times\theta(x_{Q1}\cdots<x_{Q\nu}<0<x_{Q(\nu+1)}<\cdots<x_{QN})(\mathscr{S}^k_-\phi\{\chi\})_{a_{Q\tilde{P}^\nu}\alpha}(\mathscr{I}^l_-\zeta\{\omega\})_{b_{Q\tilde{P}^\nu}} , \tag{6.2}$$

where for any permutation P

$$a_P\equiv a_{P1}a_{P2}\cdots a_{PN} , \quad b_P\equiv b_{P1}\cdots b_{PN} .$$

The spin tensors $\phi\{\chi\}$ are the simultaneous eigenstates of Z, $\vec{\mathscr{S}}^2$ and $\mathscr{S}_z$ constructed in Sec. II, and the isospin tensors $\zeta\{\omega\}$ are the simultaneous eigenstates of $\tilde{P}$, $\vec{\mathscr{I}}^2$, and $\mathscr{I}_z$ constructed in Sec. V. Here the symbols χ and ω represent solution sets $\chi_1, \ldots, \chi_M, \omega_1, \ldots, \omega_{M'}$ of Eqs. (2.16) and (5.11), respectively.

Without loss of generality, we may choose the product basis functions (6.1) to have the form

$$\tau^{kl}_{(Qvm)\chi\omega}(x_1, \ldots, x_N; a_1, \ldots, a_N, \alpha; b_1, \ldots, b_N) = f_{Qvm}(x_1, \ldots, x_N)(\mathscr{S}^k_-\phi\{\chi\})_{a_{Q\tilde{P}^v}\alpha}(\mathscr{I}^l_-\xi\{\omega\})_{b_{Q\tilde{P}^v}}, \tag{6.3}$$

where the functions $f_{Qvm}(x_1, \ldots, x_N)$, $m = 1,2, \ldots$ span the space of square-integrable functions over the sector

$$-\frac{L}{2} < x_{Q1} < \cdots < x_{Qv} < 0 < \cdots < x_{QN} < \frac{L}{2} .$$

The completeness of the spinorial basis $\mathscr{S}^k_-\phi\{\chi\}$ is shown in the Appendix; permuting the indices with $Q\tilde{P}^v$ does not affect the orthogonality and completeness relations. The isospin basis is identical to that of the one-dimensional Heisenberg model, and its completeness was treated by Takahashi.[13]

Suppose we are given an arbitrary member $\tau^{kl}_{(Qvm\chi\omega)}$ of the basis (6.3). We may apply Fourier analysis to write

$$f_{Qvm}(x)(\lambda\{\chi\}\lambda'\{\omega\})^{-v-\sum_i x_i/L} = L^{-N/2} \sum_{n_1, \ldots, n_N} C^{Qvm}_{n_1 \ldots n_N \chi\omega} \exp\left[2\pi i \sum_j n_j \frac{x_j}{L}\right] .$$

Then we obtain the desired expansion,

$$\tau^{kl}_{(Qvm)\chi\omega} = \sum_{n_1, \ldots, n_N} c^{Qvm}_{n\chi\omega} \psi^{kl}_{n\chi\omega} .$$

This establishes the completeness of our basis in the case of two species of electrons. Deleting the isospin tensors, one obtains at the same time the completeness of the basis in the case of a single species.

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APPENDIX: COMPLETENESS OF THE SPINORIAL BASIS

We wish to show that there exist $(2S+1)\times 2^N$ mutually orthogonal tensors ϕ (with components $\phi_{a_1 \ldots a_N \alpha}$, $a_i = \pm\frac{1}{2}$, $\alpha = -S, \ldots, S$) in the basis constructed in Sec. II. The argument proceeds in two steps: First we show that different solution sets $\{\chi_1, \ldots, \chi_M\}$ of (2.16) yield tensors which are orthogonal to one another; then we are left with the problem of counting the solutions of (2.16) which will be solved by generalizing the combinatorial proof used by Takahashi in the Heisenberg model.[13]

Orthogonality proof. Let $\underline{\chi} = \{\chi_1, \ldots, \chi_M\}$ and $\underline{\chi}' = \{\chi'_1, \ldots, \chi'_{M'}\}$ be two distinct solution sets of (2.16). The case $M' \neq M$ is trivial; we therefore assume $M = M'$. Orthogonality of the corresponding ϕ and ϕ', satisfying

$$Z\phi = \lambda(\underline{\chi})\phi , \qquad Z\phi' = \lambda(\underline{\chi}')\phi' , \tag{A1}$$

will be a consequence of the unitarity of Z, provided that $\lambda(\underline{\chi}) \neq \lambda(\underline{\chi}')$. By continuity, it will be sufficient to show that in every open interval of the real half-line of coupling parameters, $c > 0$, there is at least one value of c for which $\lambda(\underline{\chi}) \neq \lambda(\underline{\chi}')$.

Suppose the contrary, i.e., that for all c in some open interval, $\lambda(\underline{\chi}) = \lambda(\underline{\chi}')$, and hence, from (2.16),

$$\prod_{i=1}^{M}\left[\frac{\chi_i + c^{-1} + Si}{\chi_i + c^{-1} - Si}\right] = \prod_{i=1}^{M'}\left[\frac{\chi'_i + c^{-1} + Si}{\chi'_i + c^{-1} - Si}\right] . \tag{A2}$$

This equation holds as a relation between analytic functions in the complex c^{-1} plane. Now let $c^{-1} = -\chi'_j - Si$ for some j. The right-hand side of (A2) vanishes, which implies $\chi_i = \chi'_j$ for some i. Repetition of this procedure shows that $\underline{\chi} = \underline{\chi}'$, a contradiction. This establishes our result.

Counting argument. We shall show that for fixed N and M (hence fixed total spin $\mathscr{S}=\frac{1}{2}N-M+S$) the number $n_{\mathscr{S}}$ of distinct solution sets $\{X_1,\ldots,X_M\}$ of (2.16) is

$$n_{\mathscr{S}}=\begin{bmatrix}N\\M\end{bmatrix}-\begin{bmatrix}N\\M-2S-1\end{bmatrix}. \tag{A3}$$

That this is the desired result may be seen as follows: For given $\mathscr{S}$ and S, the total *electron* spin $\mathscr{S}_e$ can take on the values $\mathscr{S}-S, \mathscr{S}-S+1, \ldots, \mathscr{S}+S$. Since there are

$$\begin{bmatrix}N\\M_e\end{bmatrix}-\begin{bmatrix}N\\M_e-1\end{bmatrix}, \quad M_e=\tfrac{1}{2}N-\mathscr{S}_e$$

ways in which N electron spins can be added to give $\mathscr{S}_e$, we have

$$n_{\mathscr{S}}=\sum_{M_e=M-2S}^{M}\left[\begin{bmatrix}N\\M_e\end{bmatrix}-\begin{bmatrix}N\\M_e-1\end{bmatrix}\right] \tag{A4}$$

which is nothing but (A3). The total number of mutually orthogonal basis vectors is then obtained by summing over all $\mathscr{S}$, with weight factor $2\mathscr{S}+1$:

$$n=\sum_{\mathscr{S}=0}^{N/2+S}(2\mathscr{S}+1)n_{\mathscr{S}}=\sum_{M=0}^{N/2+S}(N-2M+2S+1)\left[\begin{bmatrix}N\\M\end{bmatrix}-\begin{bmatrix}N\\M-2S-1\end{bmatrix}\right]=(2S+1)\times 2^N. \tag{A5}$$

The first steps in deriving (A3) are identical to those followed by Takahashi in Ref. 13. We omit the details and merely state the important formulas using the notation of that reference. The reader who has taken the trouble to make his way through the Appendix of Ref. 13 will have no problem supplying the missing steps. The starting point is the analog of Eq. (A1) of Ref. 13, namely

$$n_{\mathscr{S}}=\sum_{\alpha_1+2\alpha_2+\cdots+M\alpha_M=M}\prod_{i=1}^{M}\begin{bmatrix}N-\sum_j t_{ij}\alpha_j+\min(i,2S)\\ \alpha_i\end{bmatrix}, \tag{A6}$$

where

$$t_{ij}=\begin{cases}2\min\{i,j\}, & i\neq j\\ 2i-1, & i=j.\end{cases}$$

Following Ref. 13, we reexpress $n_{\mathscr{S}}$ as the coefficient of x^M in

$$(1+x)^{N-M+1}\prod_{j=2}^{\infty}(1-u_j^{-1})^{-N+2M-(2S+1)}\prod_{j=2}^{2S-1}(1-u_j^{-1})^{2S-j}, \tag{A7}$$

where

$$u_2=x^{-2},\quad u_3=x^{-3}(1+x)(1-x)^2,\quad u_j=f_j^2,\quad 1-u_j^{-1}=f_{j-1}f_{j+1}f_j^{-2}, \tag{A8}$$

$$f_j=(a^{j+1}-a^{-j-1})/(a-a^{-1}),\quad a=\left[\frac{1+x}{4x}\right]^{1/2}\left[1+\left[1-\frac{4x}{1+x}\right]^{1/2}\right],$$
$$f_j=\left[\frac{1+x}{4x}\right]^{j/2}\sum_{t\,\mathrm{odd}}\begin{bmatrix}j+1\\t\end{bmatrix}\left[1-\frac{4x}{1+x}\right]^{(t-1)/2}. \tag{A9}$$

Inserting (A8) and (A9), one obtains for the products over $1-u_j^{-1}$ in (A7) the expressions

$$\prod_{j=2}^{\infty}(1-u_j^{-1})=2x\left[1-\left[1-\frac{4x}{1+x}\right]^{1/2}\right]^{-1},$$
$$\prod_{j=2}^{2S-1}(1-u_j^{-1})^{2S-j}=(1-x^2)^{2S-2}f_2^{2S-3}f_3^{-2S+2}f_{2S}=2^{-2S}(1+x)^{2S-1}\sum_{t\,\mathrm{odd}}\begin{bmatrix}2S+1\\t\end{bmatrix}\left[1-\frac{4x}{1+x}\right]^{(t-1)/2}. \tag{A10}$$

Substituting (A10) and (A7) and expanding in powers of x and $1+x$, we obtain that $n_{\mathscr{S}}$ that is the coefficient of $x^{N-M+2S+1}$ in

$$\sum_{t\text{ odd}}\sum_{q,r,s}\begin{bmatrix}2S+1\\t\end{bmatrix}\begin{bmatrix}t-1/2\\q\end{bmatrix}\begin{bmatrix}(t-1)/2\\q\end{bmatrix}(-1)^{q+r+s}\begin{bmatrix}N-2M+2S+1\\s\end{bmatrix}\begin{bmatrix}s/2\\r\end{bmatrix}. \tag{A11}$$

We now observe that the only term in the summation over r which has a nonzero coefficient of $x^{N-M+2S+1}$ is that with $r=N-M+2S+1-q$. In addition, we may simplify (A11) using the following lemma (essentially the second half of the Appendix of Ref. 13):

Lemma A1. If $p\geq m+a$ and $b\geq 0$ then the coefficient of x^p in

$$2^{-n+2m}\sum_{s}(-1)^{m+s}\begin{bmatrix}n\\s\end{bmatrix}\begin{bmatrix}s/2\\m\end{bmatrix}x^{m+a}(1+x)^{-b-1}$$

is equal to

$$(-1)^{p-a-m}\begin{bmatrix}p-m-a+b\\b\end{bmatrix}\left[\begin{bmatrix}2m-n-1\\m-1\end{bmatrix}-\begin{bmatrix}2m-n-1\\m\end{bmatrix}\right].$$

With the relevant substitutions, (A11) becomes

$$n_{\mathscr{S}}=\sum_{t\text{ odd}}\sum_{q=0}\begin{bmatrix}2S+1\\t\end{bmatrix}\begin{bmatrix}\frac{t-1}{2}\\q\end{bmatrix}(-4)^{q-S}\left[\begin{bmatrix}N+2S-2q\\M-q\end{bmatrix}-\begin{bmatrix}N+2S-2q\\M-q-1\end{bmatrix}\right]. \tag{A12}$$

To proceed to the conclusion of our argument, we need two combinatorial lemmas which are not found in Ref. 13:

Lemma A2 (proof given below). If $2S$ and q are integers with $q\leq S$, then

$$\sum_{t\text{ odd}}\begin{bmatrix}2S+1\\t\end{bmatrix}\begin{bmatrix}(t-1)/2\\q\end{bmatrix}=4^{S-q}\begin{bmatrix}2S-q\\q\end{bmatrix}.$$

Lemma A3 (proof by mathematical induction on $2S$ and l). If $2S$ and l are integers, with $0\leq l\leq 2S+1$, then

$$\sum_{0\leq q\leq S}(-1)^q\left[\begin{bmatrix}2S-q\\l\end{bmatrix}\begin{bmatrix}l\\q\end{bmatrix}-\begin{bmatrix}2S-q\\l-1\end{bmatrix}\begin{bmatrix}l-1\\q\end{bmatrix}\right]=\begin{cases}1\,, & l=0\\0\,, & l=1,\ldots,2S\\-1\,, & l=2S+1\,.\end{cases}$$

With the aid of these lemmas, we obtain the final result:

$$\begin{aligned}n_{\mathscr{S}}&=\sum_{0\leq q\leq S}\begin{bmatrix}2q-2S-1\\q\end{bmatrix}\left[\begin{bmatrix}N+2S-2q\\M-q\end{bmatrix}-\begin{bmatrix}N+2S-2q\\M-q-1\end{bmatrix}\right]\\&=\sum_{0\leq q\leq S}\sum_{l=0}^{2S+1}\begin{bmatrix}2q-2S-1\\q\end{bmatrix}\left[\begin{bmatrix}2S-2q\\l-q\end{bmatrix}-\begin{bmatrix}2S-2q\\l-q-1\end{bmatrix}\right]\begin{bmatrix}N\\M-l\end{bmatrix}\\&=\sum_{l=0}^{2S+1}\sum_{0\leq q\leq S}(-1)^q\left[\begin{bmatrix}2S-q\\l\end{bmatrix}\begin{bmatrix}l\\q\end{bmatrix}-\begin{bmatrix}2S-q\\l-1\end{bmatrix}\begin{bmatrix}l-1\\q\end{bmatrix}\right]\begin{bmatrix}N\\M-l\end{bmatrix}=\begin{bmatrix}N\\M\end{bmatrix}-\begin{bmatrix}N\\M-(2S+1)\end{bmatrix}.\end{aligned} \tag{A13}$$

Proof of Lemma A2. Define

$$C(p,q)=\sum_{k=0}\begin{bmatrix}p+1\\2k+1\end{bmatrix}\begin{bmatrix}k\\q\end{bmatrix},\quad D(p,q)=\sum_{k=0}\begin{bmatrix}p+1\\2k\end{bmatrix}\begin{bmatrix}k\\q\end{bmatrix}.$$

We wish to prove, for all $q \le p/2$,

$$C(p,q)=2^{p-2q}\begin{bmatrix} p-q \\ q \end{bmatrix} .$$

Our proof is by induction on p. Suppose the lemma has been established for $C(p',q)$, $p'<p$. Using the identity

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} ,$$

we have the following recursion relations:

$$\begin{aligned} &C(p,q)=C(p-1,q)+D(p-1,q) , \\ &D(p-1,q)=D(p-2,q)+C(p-2,q)+C(p-2,q-1) . \end{aligned} \tag{A14}$$

Iterating the second of these formulas yields

$$D(p-1,q)=D(2q,q)+\sum_{r=2q}^{p-2}[C(r,q)+C(r,q-1)]=\sum_{r=2q}^{p-2}C(r,q)+\sum_{r=2(q-1)}^{p-2}C(r,q-1) , \tag{A15}$$

where we have used

$$D(2q,q)=2q+1=C(2q-1,q-1)+C(2q-2,q-1) . \tag{A16}$$

Thus, by (A14) and the induction hypothesis

$$C(p,q)=C(p-1,q)+\sum_{r=2q}^{p-2}C(r,q)+\sum_{r=2(q-1)}^{p-2}C(r,q-1) , \tag{A17}$$

where, using $(\ )_q$ to represent the coefficient of x^q in the following expressions in parentheses,

$$\begin{aligned} \sum_{r=2q}^{p-2}C(r,q)&=\sum_{r=2q}^{p-2}2^{r-2q}\begin{bmatrix} r-q \\ q \end{bmatrix}=\sum_{m=0}^{p-2q-2}2^m\begin{bmatrix} m+q \\ q \end{bmatrix} \\ &=\left[\sum_{m=0}^{p-2q-2}2^m(1-x)^{-m-1}\right]_q=\left[-\frac{1}{1+x}\left[1-\left[\frac{2}{1-x}\right]^{p-2q-1}\right]\right]_q . \end{aligned} \tag{A18}$$

Similarly,

$$\sum_{r=2(q-1)}^{p-2}C(r,q-1)=\left[\frac{-x}{1+x}\left[1-\left[\frac{2}{1-x}\right]^{p-2q+1}\right]\right]_q \tag{A19}$$

and

$$C(p-1,q)=\left[\frac{1}{1-x}\left[\frac{2}{1-x}\right]^{p-1-2q}\right]_q . \tag{A20}$$

Substituting (A18), (A19), and (A20) into (A17), we obtain

$$C(p,q)=\left[\frac{1}{1-x}\left[\frac{2}{1-x}\right]^{p-2q}\right]_q=2^p4^{-q}\begin{bmatrix} p-q \\ q \end{bmatrix} ,$$

which is what we wanted to establish. To complete the induction argument, we observe

$$C(1,0)=\sum_{k=0}\begin{bmatrix} 2 \\ 2k+1 \end{bmatrix}\begin{bmatrix} k \\ 0 \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \end{bmatrix}=2^14^0\begin{bmatrix} 2 \\ 0 \end{bmatrix} .$$

[1]N. Andrei, Phys. Rev. Lett. 45, 379 (1980).
[2]N. Andrei and J. H. Lowenstein, Phys. Rev. Lett. 46, 356 (1981).
[3]P. B. Wiegmann, Zh. Eksp. Teor. Fiz. Pis'ma Red. 31, 392 (1980); J. Phys. C 14, 1463 (1981); V. M. Filyov, A. M. Tzvelik, and P. B. Wiegmann, Phys. Lett. 81A, 175 (1981).
[4]V. A. Fateev and P. B. Wiegmann, Phys. Lett. 81A, 179 (1981).
[5]V. A. Fateev and P. B. Wiegmann, Phys. Rev. Lett. 46, 1595 (1981).
[6]D. C. Mattis, Phys. Rev. Lett. 19, 1478 (1967); P. Nozières and A. Blandin, J. Phys. (Paris) 41, 193 (1980).
[7]R. J. Baxter, Ann. Phys. 70, 193 (1972).
[8]L. D. Faddeev, Lomi Report No. P-2-79 (unpublished).
[9]C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).
[10]D. M. Cragg, P. Lloyd, and P. Nozières, J. Phys. C 13, 803 (1980).
[11]K. G. Wilson, in *Nobel Symposia,* edited by B. Lundquist and S. Lundquist (Academic, New York, 1974), Vol. 24, p. 68; Rev. Mod. Phys. 47, 773 (1975).
[12]J. H. Lowenstein, Surv. High Energy Phys. 2, 207 (1981).
[13]M. Takahashi, Prog. Theor. Phys. 46, 401 (1971).
[14]M. Gaudin, Phys. Rev. Lett. 26, 1301 (1971).
[15]The variable χ is related to the Λ in Ref. 2 as follows: $\chi=(\Lambda-1)c^{-1}$.