Flux-line instability in a pin-free superconducting cylinder with longitudinal current

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The flux-line lattice in a long cylindrical type-II superconductor carrying an electric current parallel to an applied axial magnetic field is always unstable if there is no volume pinning.

I. INTRODUCTION

The stability of the flux-line lattice (FLL) has recently been investigated for longitudinal currents flowing through the volume or near a planar surface of a type-II superconductor.[1-3] In both cases the FLL is unstable against the growth of helical perturbations when the current along the flux lines (FL's) exceeds a critical value J_c. For weak pinning of the FL's by material inhomogeneities, J_c is proportional to the fourth root of the shear modulus of the FLL, and for stronger pinning to the fourth root of the volume-pinning strength.[1,3]

These results were obtained for planar geometry, but they apply also to cylindrical specimens with radius a large compared with the pitch length of the helices and with the width of the helically distorted FL layer. These two characteristic lengths of the helical mode increase with decreasing pinning strength and even diverge in the limit of zero pinning. As a consequence, the critical current of a wire with very weak pinning requires consideration of cylindrical geometry. This is done in the present paper.

II. HELICAL INSTABILITY

We show that for long wires with weak pinning a simple elastic mode which displaces the FL's homogeneously in each cross section of the cylinder, has a critical longitudinal current which is much smaller than the value derived in Refs. 1–4.

Consider a long pin-free cylinder with radius a and length L placed in an axial magnetic field $B_{\mathrm{appl}}\hat{z}$ ($\hat{z}$ is the unit vector along the axis). When the applied field exceeds the lower penetration field of the superconductor, magnetic flux enters in the form of FL's which fill the specimen homogeneously and yield a constant induction $\vec{B}(\vec{r})=B_z\hat{z}=\text{const}$. This situation does not change when we apply an axial transport current $J\hat{z}$ since no Lorentz force is exerted by a current density parallel to the FL's. [Initially the current flows in a thin surface layer. We will see below that for the mode (1) only the total current matters, not its distribution.]

Now we allow the FL's to distort into helices with equal pitch length, phase, and amplitude. The displacement vector then depends only on the

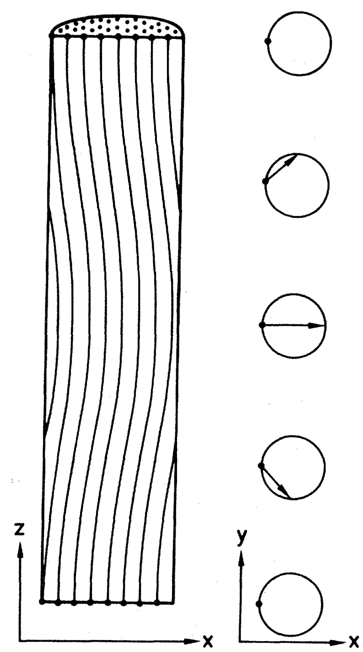

FIG. 1. Illustration of the homogeneous helical mode, Eq. (1), in a long cylindrical superconductor with the flux lines pinned to the ends. Left: helical distortion of the flux lines. The distortion is constant for any cross section $z=\text{const}$. Some flux-line sections enter and others leave on the opposite side. Right: the displacement vector $\vec{s}(z)$ for various cross sections.

coordinate z:

$$\vec{s}(z)=s_0\hat{x}\sin(kz)+s_0\hat{y}\cos(kz) \tag{1}$$

(Fig. 1). If s_0 exceeds the distance of the first FLL layer from the surface, approximately equal to the FL spacing,[5] some FL sections cut through the surface and new FL sections of the same total length enter on the opposite side. The influence of a possible surface barrier on the exit and entrance of these FL sections shall be discussed below. If we omit it, the energy of the compression-free and shear-free distortion (1) is composed of the tilt energy of the FLL, the stray-field energy (of the radial-field component outside the cylinder, induced by the distortion of FL's near the surface), and the interaction energy of the FL's with the applied current,

$$U=U_{\text{tilt}}+U_{\text{stray}}+U_J\ . \tag{2}$$

For simplicity we assume in the following that the magnetization of the specimen is small, $B_{\text{appl}}-B_z << B_z$. This condition is satisfied in most experiments. We furthermore neglect effects of the order λ/a and λk, where $\lambda << a$ is the magnetic-penetration depth. Then $\vec{B}(\vec{r})$ is along the FL's and equals the local density of FL's times the quantum of flux ϕ_0. For FL's displaced according to (1), we find

$$\vec{B}(r\leq a)=B_{\text{appl}}\hat{z}+ks_0B_z[\hat{x}\cos(kz)-\hat{y}\sin(kz)]\ , \tag{3a}$$

or in cylindrical coordinates ($x=r\cos\varphi$, $y=r\sin\varphi$),

$$\vec{B}(r\leq a)=B_{\text{appl}}\hat{z}+ks_0B_z[\hat{r}\cos(kz+\varphi)-\hat{\varphi}\sin(kz+\varphi)]\ . \tag{3b}$$

The field outside the specimen is the stray field $B_{\text{stray}}=\nabla V$ plus the applied field. The stray field is obtained by solving $\nabla^2V=0$ in cylindrical coordinates using the continuity of the radial component B_r at $r=a$:

$$\vec{B}(r\geq a)=B_{\text{appl}}\hat{z}+s_0B_z[K_1'(ka)]^{-1}\times\nabla[K_1(kr)\cos(kr+\varphi)] \tag{4}$$

where $K_1(x)$ is a modified Bessel function and $K_1'(x)$ its derivative.

The tilt-energy density is $(c_{44}/2)(ds/dz)^2$ where

$$c_{44}=B_zB_{\text{appl}}/\mu_0\approx B_z^2/\mu_0$$

is the tilt modulus of the FLL. One easily verifies using (3) that the tilt energy U_{tilt} is just the s-dependent part of the magnetic energy inside the cylinder. The s-dependent part of the magnetic energy outside the cylinder, U_{stray}, is obtained by integration of $B_{\text{stray}}^2=(\nabla V)^2=-\nabla(V\nabla V)$ over the free space $r>a$. Adding both terms we get

$$U_{\text{tilt}}+U_{\text{stray}}=(2\mu_0)^{-1}\int_0^{2\pi}d\varphi\int_0^{\infty}dr\,r(\vec{B}-\vec{B}_{\text{appl}})^2 =(B_{\text{appl}}^2/2\mu_0)\pi a^2k^2s_0^2\{1+K_1(ka)[ka\,|K_1'(ka)|\,]^{-1}\}\ . \tag{5}$$

In the limit of long pitch length, $ka<<1$, one has $K_1(ka)\approx1/ka$ and (4) and (5) reduce to

$$\vec{B}(r\geq a)=B_{\text{appl}}\hat{z}+B_zs_0k(a/r)^2[\hat{r}\cos(kz+\varphi)+\hat{\varphi}\sin(kz+\varphi)+\hat{z}kr\sin(kz+\varphi)]\quad(kr<<1)\ , \tag{6}$$

$$U_{\text{tilt}}+U_{\text{stray}}=(B_{\text{appl}}^2/\mu_0)\pi a^2k^2s_0^2\ . \tag{7}$$

For short pitch length, $ka>>1$, one has $K_1(ka)\approx(\pi/2ka)^{1/2}\exp(-ka)$ and (4) and (5) reduce to the results of the planar problem[3]:

$$\vec{B}(r\geq a)=B_{\text{appl}}\hat{z}+B_zs_0ke^{-(r-a)k}[\hat{r}\cos(kz+\varphi)+\hat{\varphi}(kr)^{-1}\sin(kz+\varphi)+\hat{z}\sin(kz+\varphi)]\ , \tag{8}$$

$$U_{\text{tilt}}+U_{\text{stray}}=(B_{\text{appl}}^2/2\mu_0)\pi a^2k^2s_0^2[1+(ka)^{-1}]\ . \tag{9}$$

The interaction U_J of the FL's with the applied current may be calculated in two ways: (a) as the force $\phi_0\vec{j}\times(ds/dz)$ exerted on a given FL segment by the local current density $\vec{j}$, multiplied by the displacement $\vec{s}(z)$ of this segment; and (b) as the voltage along the specimen induced by the moving FL's, times the applied current, times the time of motion. In both pictures one has to sum over all FL segments and to integrate from the state of parallel FL's ($s\equiv0$) to the state with fully displaced FL's. The result is

$$U_J=-JB_{\text{appl}}ks_0^2/2\ . \tag{10}$$

In the derivation of (10) no use of the explicit distribution of the current density over the cross section of the cylinder has been made. $-U_J$ is the work supplied to the FLL by the source of the applied current. In this sense, $-U_J$ corresponds to the terms pV and $B_{\rm appl}\phi$ (ϕ=total magnetic flux of the sample) in the thermodynamic potentials $G=F-pV$ and $G=F-B_{\rm appl}\phi$ which are appropriate to situations with constant applied pressure or magnetic field, respectively. This interpretation shows that our energy (2) is the thermodynamic potential which has to be minimized if the applied current is fixed. Of course, this interpretation applies only up to the onset of instability. When instability sets in, the FL's move and dissipate energy. The concepts of equilibrium thermodynamics then are not applicable.

The onset of instability is determined by the condition that U can be made negative by arbitrarily small helix amplitude s_0. For long pitch lengths we get from (7) and (10)

$$U=s_0^2(B_{\rm appl}^2/\mu_0)(\pi a^2k^2-kJ\mu_0/2B_{\rm appl}) \ . \tag{11}$$

Thus, if k is sufficiently small, helical perturbations will grow spontaneously. Arbitrarily small k (or infinite pitch length k^{-1}) is possible if there is no pinning at all. The critical longitudinal current is then zero and the helical mode describes *a homogeneous shift of the entire FLL in response to an arbitrarily small tilt perturbation of the FL's.*

If the FL's are pinned to the ends of the cylinder, e.g., by increased pinning at the ends, the smallest possible k is $2\pi/L$ and helical instability sets in at a critical current

$$J_c=(B_{\rm appl}/\mu_0)2\pi a^2k=(B_{\rm appl}/\mu_0)4\pi^2a^2/L \ . \tag{12}$$

At this current the slope of the magnetic field at the surface of the cylinder, $\hat{z}B_{\rm appl}+\varphi J\mu_0/2\pi a$, is

$$t_c=B_\varphi(r=a)/B_z(r=a)=2\pi a/L \ . \tag{13}$$

[Note that this is the magnetic field *before* the FL's are allowed to distort. When the FL's distort helically, the total surface field is given by (6) to which $\varphi J\mu_0/2\pi a$ has to be added.] This means the ratio of the longitudinal current and the applied field is such that *the field lines at the cylinder surface perform just one turn along the cylinder.*

The critical current (12) is very small for long, thin cylinders. It is smaller than the corresponding value

$$J_c=(B_{\rm appl}/\mu_0)4\pi a^2[\ln(L/a)]^{-1/2}$$

of Ref. 2. Due to the contribution of the stray field outside the cylinder, the results (12) and (13) are both larger by a factor of 2 than the estimates obtained in Ref. 3 which neglected surface effects.

When the cylinder is perfectly aligned such that the FL's are exactly parallel to the surface, then the surface barrier for exit and entrance of FL's becomes important. J_c is then higher than (12) and the helical mode at the onset of instability is not the homogeneous mode (1) but a mode with helix amplitudes decreasing with increasing distance from the surface. This case is treated in Ref. 3.

A small misalignment or deviation from cylindrical geometry will, however, decrease J_c to a value of order given by (12). This is because then some FL's will end at the cylinder surface and, by slipping their ends along the surface, can exit or enter without having to surmount the surface barrier. Furthermore, "end effects" (sharp edges, the contacts, finite length) will probably suffice to trigger the instability when J exceeds the value (12) in a pin-free sample. It thus appears to us that in existing experiments[6–9] it was always volume pinning or surface roughness, but not the surface barrier, which caused critical currents larger than the value (12).

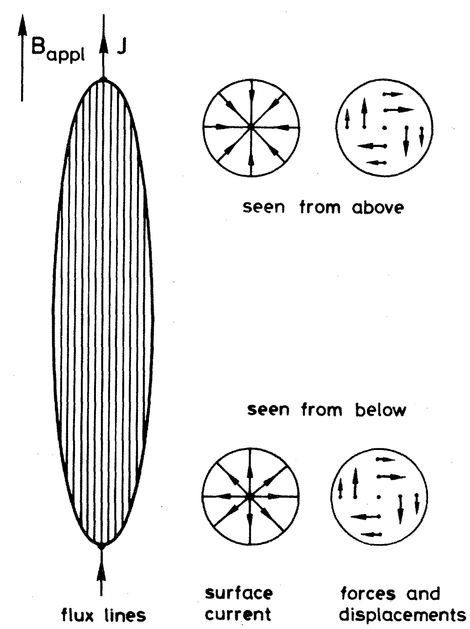

FIG. 2. Flux lines (left), surface current (middle), and forces on or displacements of (right) the flux-line ends for a long superconducting ellipsoid placed in a homogeneous axial field. The longitudinal current is applied through point contacts at the vertices of the ellipsoid.

III. EXAMPLE FOR END EFFECTS

We illustrate this statement by a further argument. Consider a long rotational ellipsoid with no pinning and with a smooth surface in an axial magnetic field. This is an ideal wire with well-defined ends (Fig. 2). The FL's are then precisely parallel since the demagnetizing field inside an ellipsoid is homogeneous. Now apply a weak longitudinal current through point contacts placed on the axis. Initially this current flows along the surface. This state is, however, not in equilibrium. The surface current j_s exerts an azimuthal force $\phi_0 j_s$ on the FL ends. As a consequence, the FL's are tilted. The tilt t, and the induction component $B\varphi = tB_z$ caused by it, is proportional to $1/r$ since FL's with distance r from the axis feel a tilting force $\phi_0 J/2\pi r$ and the tilt modulus of the FLL, $c_{44} = B_{\text{appl}} B_z/\mu_0$, is spatially constant. (We neglect here the small shear modulus of the FLL.) The total field is now

$$\begin{aligned} \vec{B}[r < R(z)] &= \hat{z}B_z + \varphi J\mu_0/2\pi r \;, \\ \vec{B}[r > R(z)] &= \hat{z}B_{\text{appl}} + \varphi J\mu_0/2\pi r \;, \end{aligned} \tag{14}$$

where $R(z) = a(L^2/4 - z^2)^{1/2}$ is the local radius.

The twist of the FLL generates an additional current density

$$j_{\text{torsion}} = J\delta(x)\delta(y) - j_s\delta(r - R(z)) \;, \tag{15}$$

where δ is the one-dimensional delta function.

The first term in (15) describes a filament along the axis carrying the total current, and the second term precisely compensates the surface current $j_s = J/2\pi R(z)$. The resulting current density is such that *the entire transport current is shifted from the surface layer to a filament on the axis,* as is easily verified by taking the curl of (14).

If the current is fed by *ring* contacts concentric to the axis rather than by *point* contacts, a similar calculation shows that the torsion of the FLL is such that the total current flows in a thin cylindrical layer connecting the two contacts (Fig. 3). Outside the current carrying filament or cylinder the FL's are in equilibrium: they are parallel to the local field (14) and the local current density vanishes. In the region where the current density is high, the FL's in general are at an angle with the current and thus start to move. The entire specimen then switches to a dissipative flux-flow state. Thus, if the FL's can tilt freely, an arbitrarily small applied current leads to a dissipative state triggered by the spatial concentration of the current. This conclusion was arrived at also in Ref. 10 using an energy argument.

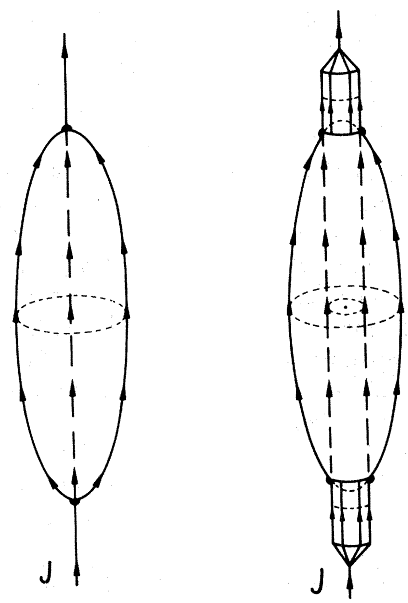

FIG. 3. Distribution of the applied current in a long ellipsoid in the superconducting state. Left: point contacts, right: ring contacts. Initially the current flows along the surface (solid lines). This surface current "pulls" at the flux-line ends and causes the flux lines to twist. Local equilibrium is reached when the current caused by the twisted flux lines just compensates the surface current. The total current flows then along the axis or a cylinder connecting the contacts (dashed lines).

IV. CONCLUSION

The mechanism of Sec. III (instability due to end effects in a long ellipsoid) and the one proposed by Campbell[2] (buckling of current filaments, or pinch effect in deformable conductors) are both based on a self-concentration of the longitudinal current. In contrast to this, the spontaneous helical deformation of the FLL of Sec. II does *not* change the longitudinal current density: to lowest order in k it generates a current density [the curl of (3)]

$$\vec{j} = (B_{\text{appl}}/\mu_0)k^2[\hat{x}\cos(kz) - \hat{y}\sin(kz)] + O(k^2) \;,$$

which is perpendicular to the longitudinal current. This shows that the tendency for current concentration is not essential for the *onset* of instability in longitudinal geometry, though it might be important during the *evolution* of the flux-flow state.

In spite of considerable effort in theory and experiment,[11] little is known as yet about the resistive flux-flow state in longitudinal geometry. The problem is similar in some respect to turbulence in hydrodynamics. Although the "simple" (but nonlinear) Navier-Stokes equation is believed to completely describe the situation in many cases, theorists have not been able to solve it even for quite general features of turbulent flow.[12] Unfortunately, the equations describing the flow of FL's in a type-II superconductor are more complex than the Navier-Stokes equation.

Our result that the longitudinal critical current is zero for an infinitely long superconducting wire with no pinning has its analog in hydrodynamics, where the critical velocity for the onset of turbulence vanishes between walls of infinite distance (the Reynolds number is then infinite). A further analogy is that the flux flow (or the turbulent flow) is very sensitive to the geometry of the superconductor and of the contacts (or of the vessel and its supply pipes).

In order to put forward the investigation of flux flow in longitudinal geometry, further experiments should be performed with well-defined shape and pinning properties of the specimen. In addition to spatially averaged properties such as voltage and magnetic moment, the electric and magnetic fields should be measured in as many points as possible on the surface. This may be performed by fixed[7] or sliding[6] contacts or a combination of these. Precise alignment of the specimen in the applied field is required. Time resolution of the signals, which turn out to be rapidly oscillating,[13] provides further valuable information. Finally, small-angle neutron scattering would offer a means to look into the specimen and obtain the temporally and spatially averaged distance and orientation of the FL's.[14]

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