Third sound in layered superfluids: H↓ on ^{4}He

R. A. Guyer and M. D. Miller
Laboratory for Low Temperature Physics, Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
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In order to minimize the possibility of spin flips due to interactions with and on the walls of a sample cell, spin-polarized hydrogen (H↓) is studied in a ^{4}He sample cell, i.e., a sample cell which has been coated with a film of superfluid ^{4}He. At low temperatures it is expected that the most dense H↓ sample will first form in the physisorbed state on the ^{4}He film. In this paper we examine in detail the hydrodynamic consequences of this layered film configuration. We derive the expression for third sound in the general case that the films are compressible and both films have undergone superfluid transitions. In this case two propagating modes will be seen which will provide an uneqivocal demonstration of a superfluid transition in the H↓ film. We specialize to various limiting cases: incompressible films, saturated and unsaturated films, and the normal fluid H↓ blanket. In our discussion we stress the point that even if the H↓ has not gone superfluid it will still shift the ^{4}He third-sound speed, that this shift is a direct measure of the H↓ film areal density, and that the magnitude of this shift is expected to be well within the precision of present-day third-sound experimental techniques. Thus third sound is expected to be a primary diagnostic tool in the study of H↓ films.

I. INTRODUCTION

Understanding the relationship between the λ transition in ^{4}He and Bose-Einstein condensation is certainly one of the major outstanding problems in low-temperature condensed-matter physics. Part of the excitement generated by the possibility of producing spin-polarized hydrogen (H↓) in the laboratory is due to the feeling that because H↓ does not have a gas-to-liquid transition (even at $T=0$ K) its (anticipated superfluid behavior will be relatively easily understood in terms of the weakly nonideal Bose gas model. Thus the H↓ system might prove to be a valuable stepping stone to a deeper understanding of superfluid and ^{4}He.

Spin-polarized hydrogen, H↓, is *atomic* hydrogen interacting in the electronic spin-polarized $b^3\Sigma_u^+$ molecular state.[1] The presence in a sample of an atom with the wrong electronic spin causes recombination to the H_2 ground state, the release of ~ 4.5 eV of energy, and rapid degradation of the sample. There are two basic mechanisms for producing a wrong-spin H atom: first, intrinsic modes (e.g., two-body processes on the walls or three-body processes in the bulk), and, second, external stimuli (e.g., magnetic impurities in the walls of the sample cell). A He film helps to minimize wall recombinations because of its relatively weak binding of a H atom and in addition it serves as a magnetically inert buffer.

The interaction of atomic hydrogen with liquid helium, although weak, is sufficient to produce a physisorbed bound state. The binding energy has recently been measured by Morrow *et al.*[2] who find ~ 0.1 K. A binding energy of this magnitude should produce a large surface population at temperatures which are easily accessible in the laboratory. Thus one is led to expect a Kosterlitz-Thouless-Nelson superfluid transition in the H↓ film.[3,4] Perhaps the logical first place to look for macroscopic quantum behavior in H↓ is the H↓ film above a ^{4}He surface.

In this paper we investigate the hydrodynamic modes of layered superfluid films. The surface wave in superfluid films is known as third sound. We shall derive expressions for the third-sound modes which a layered film can support. In addition we show that even before the H↓ goes superfluid its presence as a blanket on the ^{4}He will yield a measurable shift in the ^{4}He third-sound velocity that can be used to do diagnostics on the films. (Indeed, in the thick ^{4}He film limit, the shift is simply proportional to the H↓ areal density.) Our analysis includes the possibility that the H↓ third-sound mode can be driven by the lateral compression of the H↓ gas (due to the softness of first sound in the H↓ gas) as well as the usual van der Waals restoring force. A preliminary report of this work has been published.[5] In Sec. II we derive general expressions for third sound in layered films. In Sec. III we discuss the physics of various particular and limiting cases: compressible and incompressible, saturated and unsaturated films. Section IV is the conclusion.

II. HYDRODYNAMICS

In the following we shall consider a H↓ "film" physisorbed to the surface of a ^{4}He film as shown schematically in Fig. 1. The use of the word film to

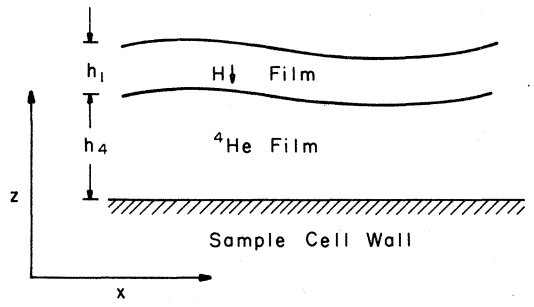

FIG. 1. Schematic drawing of layered H↓-^{4}He films: $h_4(x,t)[h_1(x,t)]$ is the helium (hydrogen) film thickness. The z axis is oriented normal to the plane of the substrate (the xy plane). Symmetry allows us to neglect the y direction with no loss of generality.

describe a one-atom-thick layer of H↓ might seem unreasonable at first glance, however its appropriateness is supported by an impressive number of *submonolayer* experiments[6] on ^{4}He whose results are interpreted within this hydrodynamic picture. The linearized hydrodynamic equations for the superfluid film are given by

$$\dot{\rho}_\alpha + \vec{\nabla}\cdot\vec{g}_\alpha = 0 \ , \tag{1}$$

$$\dot{V}_{s\alpha} + \vec{\nabla}\mu_\alpha = 0 \ , \tag{2}$$

$$\dot{s}_\alpha = 0 \ , \tag{3}$$

where we have neglected all dissipative terms, ρ is the mass density, $\vec{g} = \rho_s \vec{V}_s$ is the momentum density, μ is the chemical potential per unit mass, and the subscript $\alpha = 1, 4$ depending on whether Eqs. (1)–(3) are being applied to the H↓ or ^{4}He film. In the following we shall be considering temperatures low enough such that the entropy conservation equation, Eq. (3), and mass exchange with the vapor can be neglected. That is, we shall assume that temperatures are so low that $\rho_v/\rho_f \ll 1$ where ρ_v and ρ_f are the mean densities in the vapor and in the film (equivalently, for the ^{4}He film we consider $TS/L \ll 1$ where L is the helium latent heat). For ^{4}He films this neglect is justified for $T < 0.5$ K. We have also set the normal fluid velocity $\vec{v}_n = 0$ in Eqs. (1)–(3). This "clamping" of the normal fluid is due to the large size of the viscous penetration depth relative to the thickness of the film.[7]

The films as shown in Fig. 1 are considered to be completely phase separated. This phase separation follows from the large zero-concentration chemical potential[8] for H in ^{4}He (~ 40 K). Thus we apply Eqs. (1) and (2) to each film separately with the two films joined at the interface by an appropriate boundary condition.

We solve the set of equations, Eqs. (1) and (2), using the procedure of Bergman.[9] To this end we film average Eqs. (1) and (2), introduce appropriate boundary condition, and introduce an equation of state. We sketch the method of solution; details of the method are given in Ref. 9.

We denote a film averaged quantity by brackets which are defined by

$$\langle \cdots \rangle \equiv \frac{1}{h}\int_0^h \cdots dz \ , \tag{4}$$

where the z axis is normal to the plane of the substrate as indicated by Fig. 1.

Film averaging the continuity equations yields

$$\frac{\partial}{\partial t}(h_1\langle\rho_1\rangle) + h_1\frac{\partial}{\partial x}\langle g_{x1}\rangle = 0 \tag{5a}$$

and

$$\frac{\partial}{\partial t}(h_4\langle\rho_4\rangle) + h_4\frac{\partial}{\partial x}\langle g_{x4}\rangle = 0 \ , \tag{5b}$$

where the mass flux boundary conditions for (5a) are

$$\begin{aligned} &g_{z1}(x,h_4) = \rho_1(x,h_4)\dot{h}_4 \ , \\ &g_{z1}(x,h_1+h_4) = \rho_1(x,h_1+h_4)(\dot{h}_1+\dot{h}_4) \ , \end{aligned} \tag{6a}$$

and the boundary conditions for (5b) are

$$g_{z4}(x,h_4) = \rho_4(x,h_4)\dot{h}_4 \ , \quad g_{z4}(x,0) = 0 \ . \tag{6b}$$

In Eqs. (6) and hereafter dots stand for partial derivatives with respect to time. We shall also consistently neglect terms which are second order in small quantities. Because of azimuthal symmetry about $\hat{z}$ we can without loss of generality neglect the y components of all equations.

Film averaging the superfluid equations of motion yields

$$\frac{\partial}{\partial t}\langle V_{s1x}\rangle = -\frac{\partial}{\partial x}\langle\mu_1\rangle \ , \tag{7a}$$

$$\frac{\partial}{\partial t}\langle V_{s4x}\rangle = -\frac{\partial}{\partial x}\langle\mu_4\rangle \ . \tag{7b}$$

We can eliminate the chemical potentials from Eq. (7) using

$$d\mu = \frac{1}{\rho}dP \ , \tag{8a}$$

and the equation of state for the films, P_1 and P_4. The equilibrium constraint that there is no net force on the fluid at z yields

$$\frac{dP}{dz}(z) = \rho(z)f(z) \ , \tag{8b}$$

where P is the equilibrium pressure and

$$f(z) \equiv -\frac{dU}{dz} \tag{8c}$$

is the local force per unit mass. Here $U(z)$ is the energy in the fluid at z due to the external fields, i.e., either the van der Waals energy or a combination of gravitational energy and the van der Waals energy. Upon integrating (8b) from the gas to a point in ei-

ther the hydrogen or helium film we find

$$dP_1 = -\rho_1(h_4+h_1)f_1(h_4+h_1)(dh_4+dh_1) \ , \tag{9a}$$

$$dP_4 = [-\rho_4(h_4)f_4(h_4)+\rho_1(h_4)f_1(h_4) - \rho_1(h_4+h_1)f_1(h_4+h_1)]dh_4 - \rho_1(h_4+h_1)f_1(h_4+h_1)dh_1 \ . \tag{9b}$$

Using (9) and (8) in (7) we find

$$h_1\frac{\langle\rho_{s1}\rangle}{\langle\rho_1\rangle}\frac{\partial}{\partial t}\langle V_{s1x}\rangle = -\rho_1(h_1+h_4)\left\langle\frac{1}{\rho_1}\right\rangle C_{11}^2(h_4'+h_1') \tag{10a}$$

and

$$h_4\frac{\langle\rho_{s4}\rangle}{\langle\rho_4\rangle}\frac{\partial}{\partial t}\langle V_{s4x}\rangle = -\left\langle\frac{1}{\rho_4}\right\rangle[(\rho_4(h_4)C_{44}^2-\rho_1(h_4)C_{41}^2 + \rho_1(h_1+h_4)C_{14}^2h_4' + \rho_1(h_1+h_4)C_{14}^2h_1'] \ , \tag{10b}$$

where we have defined the following speeds

$$C_{44}^2 = -\frac{\langle\rho_{s4}\rangle}{\langle\rho_4\rangle}h_4f_4(h_4) \ , \tag{11a}$$

$$C_{11}^2 = -\frac{\langle\rho_{s1}\rangle}{\langle\rho_1\rangle}h_1f_1(h_1+h_4) \ , \tag{11b}$$

$$C_{14}^2 = -\frac{\langle\rho_{s4}\rangle}{\langle\rho_4\rangle}h_4f_1(h_1+h_4) \ , \tag{11c}$$

$$C_{41}^2 = -\frac{\langle\rho_{s4}\rangle}{\langle\rho_4\rangle}h_4f_1(h_4) \ . \tag{11d}$$

The primes in Eq. (10) henceforth signify partial derivatives with respect to x. Carrying out the indicated differentiations in Eq. (5) we find

$$h_1\frac{\langle\rho_{s1}\rangle}{\langle\rho_1\rangle}\frac{\partial}{\partial x}\langle V_{s1x}\rangle + \dot{h}_1 + \frac{\rho_1(h_1+h_4)}{\langle\rho_1\rangle}\frac{U_{11}^2}{U_{T1}^2}(\dot{h}_4+\dot{h}_1) = 0 \ , \tag{12a}$$

$$h_4\frac{\langle\rho_{s4}\rangle}{\langle\rho_4\rangle}\frac{\partial}{\partial x}\langle V_{s4x}\rangle + \dot{h}_4 + [\rho_4(h_4)U_{44}^2-\rho_1(h_4)U_{41}^2+\rho_1(h_1+h_4)U_{14}^2]\frac{\dot{h}_4}{\langle\rho_4\rangle U_{T4}^2} + \frac{\rho_1(h_1+h_4)}{\langle\rho_4\rangle U_{T4}^2}U_{14}^2\dot{h}_1 = 0 \ , \tag{12b}$$

where analagous to Eq. (11) we have introduced

$$U_{44}^2 = -h_4f_4(h_4) \ , \tag{13a}$$

$$U_{11}^2 = -h_1f_1(h_1+h_4) \ , \tag{13b}$$

$$U_{14}^2 = -h_4f_1(h_1+h_4) \ , \tag{13c}$$

$$U_{41}^2 = -h_4f_1(h_4) \ . \tag{13d}$$

We have also used

$$\frac{\partial\rho_\alpha}{\partial t} = \frac{1}{U_{T\alpha}^2}\frac{\partial P_\alpha}{\partial t} \ , \tag{14}$$

in Eq. (12). Here $\alpha = 1, 4, U_{T\alpha}^2$ is the ordinary isothermal speed of sound and Eqs. (9) are used to eliminate the dP.

Equations (10) and (12) are now easily solved by eliminating one of the variables, $\langle V_{sx}\rangle$ say, and then using Fourier transforms to solve the coupled wave equations for h_4 and h_1. When this is done, one obtains after some algebra

$$C_3^4\left[1+\left[\frac{\rho_4(h_4)U_{44}^2-\rho_1(h_4)U_{41}^2+\rho_1(h_1+h_4)U_{14}^2}{\langle\rho_4\rangle U_{T4}^2}\right]+\left[1+\frac{\rho_4(h_4)U_{44}^2-\rho_1(h_4)U_{41}^2}{\langle\rho_4\rangle U_{T4}^2}\right]\left[\frac{\rho_1(h_1+h_4)U_{11}^2}{\langle\rho_1\rangle U_{T1}^2}\right]\right]$$

$$-C_3^2\left[\left\langle\frac{1}{\rho_4}\right\rangle[\rho_4(h_4)C_{44}^2-\rho_1(h_4)C_{41}^2+\rho_1(h_1+h_4)C_{14}^2]+\left\langle\frac{1}{\rho_1}\right\rangle\rho_1(h_1+h_4)C_{11}^2 + \left[\left\langle\frac{1}{\rho_1}\right\rangle\frac{\langle\rho_{s1}\rangle}{U_{T4}^2}+\left\langle\frac{1}{\rho_4}\right\rangle\frac{\langle\rho_{s4}\rangle}{U_{T1}^2}\right]\frac{U_{11}^2}{\langle\rho_1\rangle\langle\rho_4\rangle}\rho_1(h_1+h_4)[\rho_4(h_4)U_{44}^2-\rho_1(h_4)U_{41}^2]\right]$$

$$+\left\langle\frac{1}{\rho_4}\right\rangle\rho_1(h_1+h_4)\left\langle\frac{1}{\rho_1}\right\rangle C_{11}^2[\rho_4(h_4)C_{44}^2-\rho_1(h_4)C_{41}^2] = 0 \ , \tag{15}$$

where $C_3 \equiv \omega/k$ is the third-sound (phase) velocity. Equation (15) is the general expression for coupled hydrodynamic surface waves in compressible layered films in the low-temperature, long-wavelength limit. In the

following sections this unwieldy expression will be taken apart and analyzed in various important physical cases.

III. DISCUSSION

Let us pause for a moment to be reminded of the quantities that appear in Eq. (15). There are densities, ρ_1 and ρ_4, film heights, h_1 and h_4, and velocities, C_{11}^2, C_{14}^2, C_{41}^2 C_{44}^2 [Eqs. (11)], $U_{11}^2, U_{14}^2, U_{41}^2, U_{44}^2$ [Eqs. (13)], U_{T1}^2 and U_{T4}^2 [below Eq. (14)]. The velocities U_{T1}^2 and U_{T4}^2 are the isothermal sound velocities of hydrogen and helium. They involve the forces intrinsic to the hydrogen (helium) fluid that work to prevent its compression (increase in density). If the hydrogen (helium) fluid is incompressible $U_{T1}^2 (U_{T4}^2) \to +\infty$. The velocities U_{11}^2 and C_{11}^2, U_{14}^2 and C_{14}^2, etc. are related simply to one another by density ratios that are a measure of the superfluid fraction in a film; e.g., $C_{11}^2 = \langle \rho_{s1} \rangle U_{11}^2 / \langle \rho_1 \rangle$. The velocities U_{11}^2, U_{14}^2, U_{41}^2, and U_{44}^2 are velocities associated with the external forces that drive the film, e.g., the van der Waals force on the upper edge of the ^{4}He film, on the lower edge of the H↓ film, etc. See Fig. 2.

A. Incompressible ^{4}He

Except in the very thinnest films, the helium fluid can be taken to be incompressible.[10] (That is, for a pure ^{4}He film

$$\frac{1}{C_3^2} \approx \frac{1}{U_{44}^2} + \frac{1}{U_{T4}^2}$$

and except for the very thinnest films $U_{44}^2 \ll U_{T4}^2$.) Thus letting $U_{T4}^2 \to \infty$ in Eq. (15) we find

$$C_3^4\left[1+\frac{\rho_1(h_1+h_4)}{\langle\rho_1\rangle}\frac{U_{11}^2}{U_{T1}^2}\right]-C_3^2\left[(C_{44}^2-\frac{\rho_1(h_4)}{\langle\rho_4\rangle}C_{41}^2)\left[1+\frac{\rho_1(h_1+h_4)}{\langle\rho_1\rangle}\frac{U_{11}^2}{U_{T1}^2}\right]+\frac{\rho_1(h_1+h_4)}{\rho_4}C_{14}^2+\left\langle\frac{1}{\rho_1}\right\rangle\rho_1(h_1+h_4)C_{11}^2\right]$$

$$+\rho_1(h_1+h_4)\left\langle\frac{1}{\rho_1}\right\rangle C_{11}^2\left[C_{44}^2-\frac{\rho_1(h_4)C_{41}^2}{\langle\rho_4\rangle}\right]=0 \quad . \tag{16}$$

The compressibility corrections in Eq. (16) depend on the ratio U_{11}^2/U_{T1}^2, the ratio of the substrate force on the upper edge of the H↓ film to the compressional forces in the H↓ film. If the H↓ film is of sufficiently low density or the upper edge of this film is close to the substrate, then this ratio can be large. See Fig. 3. In the compressional dominated regime, the two modes of Eq. (16) can be written

$$C_{3+}^2 = C_{44}^2\left[1-\frac{\rho_1(h_4)}{\rho_4}\frac{C_{41}^2}{C_{44}^2}\right] ,$$

$$C_{3-}^2 = \langle\rho_{S1}\rangle\left\langle\frac{1}{\rho_1}\right\rangle U_{T1}^2 ,$$

where terms of $O[(U_{T1}^2/U_{11}^2)(C_{14}^2/C_{44}^2)]$ have been neglected. The C_{3+} mode is the shifted ^{4}He third-sound signal, the shift being due to the presence of a H↓ blanket at the interface. The C_{3-} mode is *fourth* sound in the H↓ film.

B. Both films incompressible

We now let $U_{T1}^2 \to \infty$ in Eq. (16) which yields

$$C_{3\pm}^2 = \frac{1}{2}\left[[C_{44}^2(1+\Delta)+C_{11}^2] \pm [C_{44}^2(1+\Delta)-C_{11}^2]\right.$$

$$\left.\times\left[1+\frac{4C_{11}^2\epsilon}{(C_{44}^2(1+\Delta)-C_{11}^2}\right]^{1/2}\right] , \tag{17}$$

where we have defined

$$\Delta \equiv \frac{\rho_1}{\rho_4}\frac{C_{14}^2-C_{41}^2}{C_{44}^2} , \tag{18}$$

$$\epsilon \equiv \left(\frac{\rho_1}{\rho_4}\frac{C_{14}^2}{C_{44}^2}\right)\Big/\left(1+\Delta-\frac{C_{11}^2}{C_{44}^2}\right) . \tag{19}$$

We expect that the term in the radical in Eq. (17) will be small since it roughly goes like

$$\frac{\rho_1}{\rho_4}\frac{f_1^2(h_1+h_4)}{f_4^2(h_4)} ;$$

see Fig. 2. Thus expanding the square root yields

$$\frac{C_{3+}^2}{C_{44}^2} = 1+\Delta+\epsilon\frac{C_{11}^2}{C_{44}^2} , \tag{20}$$

$$\frac{C_{3-}^2}{C_{11}^2} = 1-\epsilon . \tag{21}$$

Equations (20) and (21) show that the plus mode is the shifted ^{4}He third-sound mode and the minus mode is the shifted H↓ third-sound mode. There are two types of shift: the Δ shift is due to the hydrogen blanket and is present irrespective of whether or not the H↓ is superfluid (this will be discussed in more detail below); the ϵ shift is due to the interaction of the two superfluid modes. We note, in passing, the similarity between these results and the analogous case of coupled gravitational waves in classical liquid films.[11]

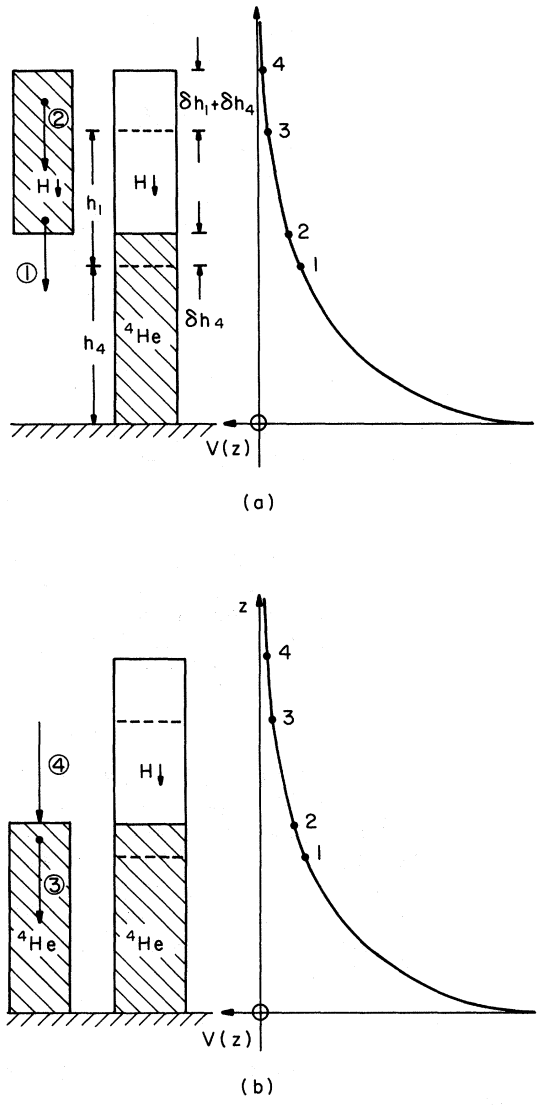

FIG. 2. Forces on columns of ^{4}He and H↓. In (a) and (b) the potential energy seen by columns of ^{4}He and H↓ in the presence of a substrate are shown as $V(z)$ vs z. In (a) a particular layered-film profile is shown. This profile departs from the static-film profile indicated by h_1 and h_4. The fluctuation δh_4 pushes the ^{4}He column height from 1 to 2 (it also pushes the lower edge of the H↓ film from 1 to 2; the fluctuation δh_1 raises the upper edge of the H↓ film from $h_1+h_4+\delta h_4$ to $h_1+h_4+\delta h_4+\delta h_1$. Due to δh_1 and δh_4 the upper edge of the H↓ film is pushed from 3 to 4. On the left of (a) the forces ① and ②, see Eq. (9a), that are exerted on the piece of H↓ film are shown. These forces arise because the top and bottom edge of the H↓ are pulled back toward equilibrium by the substrate. On the left of (b) the forces ③ and ④, see Eq. 9(b), that are exerted on a piece of ^{4}He film are shown. The force ③ arises from raising the upper edge of the ^{4}He film in the substrate field. The force ④ is the total force of the H↓ column pushing on the ^{4}He column ④ = ① + ②.

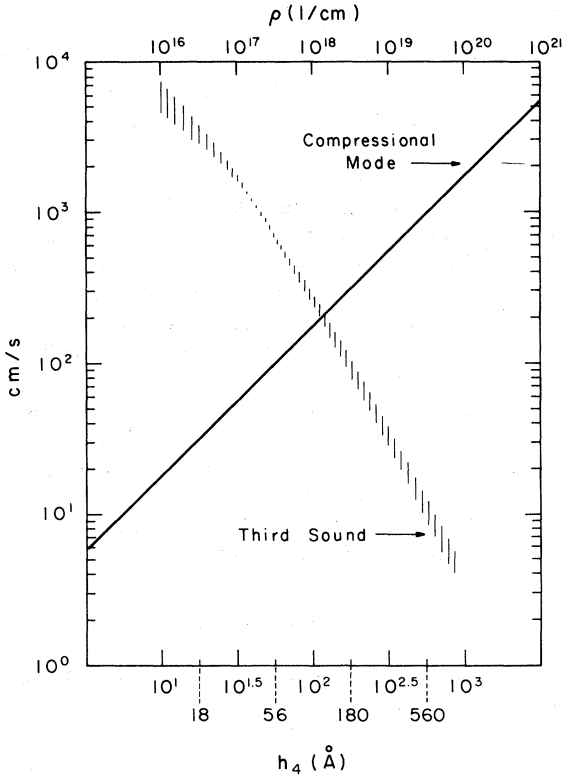

FIG. 3. Velocity in H↓ film as a function of bulk H↓ density and film thickness. For a hydrogen film of nominal thickness of 10 Å, the third-sound velocity is shown as a function of ^{4}He film thickness, h_4 (bottom scale). The velocity of the compressional mode in the film as a function of bulk density, ρ, is shown (top scale). Note crossover at $\rho \approx 10^{18}$ part/cm^3. It is straightforward to modify the discussion in the text, applicable to $\rho \gtrsim 10^{18}$ part/cm^3, to deal with $\rho < 10^{18}$ part/cm^3. For example, the simplification below Eq. (16) must be dropped.

We can now write Δ and ϵ in terms of the potential energy parameters. There are two points to be considered. First, we must identify those forces which drive the third sound and, second, we have to distinguish between experiments performed on saturated films from those performed on unsaturated films.

We shall assume that the only force which drives the third sound is the external van der Waals potential from the walls of the sample cell. In the case of layered films there is also the possibility that interfilm interactions can serve as a driving force. In order for this to be of importance one film has to be rigid with respect to motions of the other film. In recent work[12] we have examined third sound in ^{3}He-^{4}He mixture films and we conclude that even in films with a thick ^{3}He blanket riding on the ^{4}He there was no indication of film-film interactions playing an important role in the third sound. Thus we use the

general van der Waals form

$$f(h) = -\frac{3\alpha}{mh^4} \ , \tag{22}$$

where α is the substrate-film van der Waals parameter.

One may also make the important distinction between saturated and unsaturated films. A saturated film is produced in a sample cell which contains a "puddle" of bulk helium which is in equilibrium with its vapor at saturated vapor pressure. At a given temperature the thickness of a saturated film is determined by the height of the film above the bulk liquid level. An unsaturated film is formed in a container which has no bulk puddle and the thickness is determined by the vapor pressure. We shall find below that results in the saturated-film limit are somewhat simpler to analyze than for unsaturated films because we can eliminate the van der Waals interaction in favor of the gravitational potential.

C. Unsaturated films

Substituting (22) and (11) into (18) and (19) we find

$$\Delta = -\frac{\alpha_1}{\alpha_4}\frac{n_1}{n_4}\left[1 - \frac{1}{(1+h_1/h_4)^4}\right] \ , \tag{23}$$

where n_1 and n_4 are the *number* densities of the H↓ and ^{4}He film, respectively. In the thick ^{4}He film limit we may expand Eq. (23) in powers of h_1/h_4;

$$\Delta = -\left[4\frac{\alpha_1}{\alpha_4}\frac{1}{n_4 h_4}\right]\sigma_1 + O\left[\frac{h_1^2}{h_4}\right] \ , \tag{24}$$

where we have used

$$n_1 = \sigma_1/h_1 \ , \tag{25}$$

and σ_1 is the areal density of the hydrogen film. Similarly, we find

$$\epsilon = \frac{1}{1 - \frac{\langle\rho_{s1}\rangle}{\rho_1}\frac{\rho_4}{\langle\rho_{s4}\rangle}\frac{\rho_4}{\rho_1}\frac{h_1}{h_4} - \left[1 - \frac{n_4\alpha_4}{n_1\alpha_1}\right]\left[1 + \frac{h_1}{h_4}\right]^4} \ . \tag{26}$$

Expanding Eq. (26) in powers of h_1/h_4 yields

$$\epsilon = \frac{n_1\alpha_1}{n_4\alpha_4} + O\left[\frac{h_1}{h_4}\right] \ , \tag{27}$$

to lowest order.

D. Saturated films

In Fig. 4 we show schematically a sample cell whose lower region is filled with liquid ^{4}He and whose walls are coated with a thick film of helium upon which a hydrogen film is adsorbed. We now argue (following Atkins[7]) that since the film and bulk liquid are both in equilibrium at the saturated vapor pressure, chemical potential equality for an atom in the surface of the bulk relative to one in the surface of the film (a height H above the bulk liquid surface) requires that

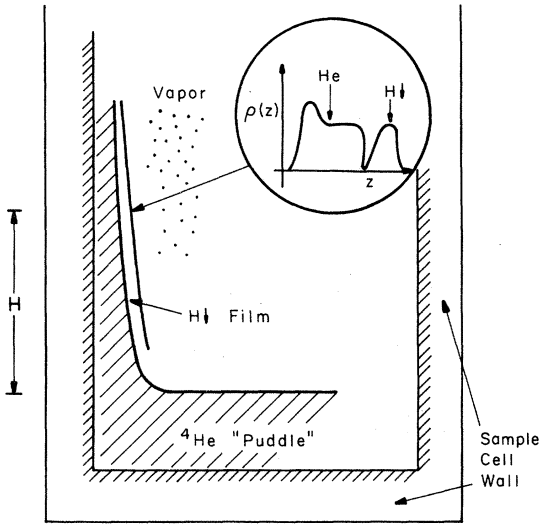

FIG. 4. Schematic drawing of the setup for a saturated-film experiment. We show a sample cell with an *in situ* ^{4}He puddle. The thickness of the He film depends on the height H above the surface of the bulk. The inset shows the density profiles of the physisorbed He and H↓ films.

$$\alpha/h^3 = mgH \ . \tag{28}$$

Using (28) we may eliminate the van der Waals coefficients (especially the unknown hydrogen coefficient) in terms of the film height H. We find

$$\Delta = -\frac{\rho_1}{\rho_4}\left[1 - \frac{1}{(1+h_1/h_4)}\right] \ , \tag{29}$$

which in the thick-film limit becomes

$$\Delta = -\left[\frac{m_1}{\rho_4 h_4}\right]\sigma_1 + O\left[\frac{h_1}{h_4}\right]^2 \ . \tag{30}$$

Similarly for ϵ

$$\epsilon = \frac{(\rho_1/\rho_4)}{1 + \left[1 - \frac{\rho_1}{\rho_4} - \frac{\langle\rho_{s1}\rangle}{\rho_1}\frac{\rho_4}{\langle\rho_{s4}\rangle}\right]\frac{h_1}{h_4}} \ , \tag{31}$$

and thus

$$\epsilon = \frac{\rho_1}{\rho_4} + O\left[\frac{h_1}{h_4}\right] \ , \tag{32}$$

where in Eqs. (29) and (31) we have neglected terms of $O(h/H)$.

E. Normal fluid blanket

The purpose of this subsection is to stress the following point. Even before H↓ goes superfluid the fact that it is riding on the He surface will shift the ^{4}He third-sound velocities by the factor Δ calculated in Eqs. (23) and (29). This shift can be used to deduce the areal density of the hydrogen film. Thus from Eq. (30) $\Delta \sim 10^{-3}$ (assuming $h_4 = 100$ Å, $\sigma_1 = 10^{-2}$ Å^{-2}) and from Eq. (24) $\Delta \sim 10^{-1}$ (assuming $h_4 = 36$ Å, $\sigma_1 = 10^{-2}$ Å^{-2} and estimating $\alpha_1/\alpha_4 \approx 2.4$ from the H-H and He-He van der Waals tails).

IV. CONCLUSION

Attaining a superfluid transition in a H↓ system would clearly be of major scientific importance. The recombination complications which render this system at best metastable make this goal a difficult technical problem. Indeed, nonstatic probes of the H↓ system must be performed with care, lest the perturbation stimulate excessive recombination. Third sound is a noninvasive tool which can be used to investigate the nature of H↓ films both above and below their superfluid transition. Third-sound experiments can be performed in either a pulsed or resonance mode (in the latter, third-sound velocities can be measured to parts in 10^5). Third sound can be observed by either capacitive means (i.e., mass motion) or resistive-edge superconducting strips (i.e., temperature pulses). In summary, then, the present-day state of the art in third-sound technology[13] is capable of yielding important quantitative information concerning the structure of H↓ films in a manner which will minimize the disturbance to the fragile H↓ equilibrium.

In the foregoing we have developed the hydrodynamic theory of layered superfluid films in the low-temperature, nondispersive limit utilizing two simplifying assumptions. First, we have assumed that the H↓ normal fluid is clamped (i.e., the propagating mode is third or fourth sound). We feel secure in this assumption by appealing to the work of Osborne,[14] showing that He vapor is carried along by the normal component of the liquid. Second, we have assumed that (at least in the $k \to 0$ hydrodynamic regime) the H↓ film can be described by a film height $h_1(\vec{\rho},t)$. This assumption is not absolutely necessary and is difficult to justify in the compression dominated regime. In work to be published elsewhere, we shall go beyond this restriction.

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