Heat capacity and density of states of a normal-metal–superconductor bilayer

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The Bogoliubov equation is solved for the case of a normal metal of thickness a_n in contact with a superconductor of thickness a_s. The excitation energies and density of states are calculated for energies both less than and greater than the pair potential Δ_s in the superconductor. These results are used to calculate numerically the heat capacity of the bilayer as a function of temperature. As the thicknesses become large, the heat capacity approaches the value which would be present in the absence of a proximity effect. For layers less than a coherence length the heat capacity approaches that of a homogeneous superconductor with an effective pair potential determined by the volume average of the pair potential in the bilayer.

I. INTRODUCTION

The heat capacity of a normal metal in contact with a superconductor differs from the sum of their individual heat capacities because of the proximity effect. In the past there have been few studies of this effect because of the difficulty in preparing suitable bulk samples but in recent years fabrication techniques for preparing laminar and filamentary composite materials have improved, allowing heat-capacity measurements to be made on several bulk samples in the proximity-effect regime.[1–4] These measurements have concentrated on the discontinuity in the heat capacity at the transition temperature which, as expected, is less than that for the superconductor alone, but quantitative agreement has been elusive.

The heat-capacity measurements are usually compared to a Ginzburg-Landau theory in which the free energy is expanded as a function of an order parameter.[3,5] This type of theory should give a reasonably accurate spatial variation of the order parameter and its resulting contribution to the free energy but it completely ignores significant microscopic details such as Andreev scattering of quasiparticles near the normal-metal–superconductor interface. Andreev scattering can give rise to bound states below the energy gap[6–8] and other structure in the density of states[9–13] and should be taken into account. Such effects are important in electron tunneling measurements[10,11] and have been included in theories of electron tunneling.[9,12,13]

It is the purpose of this study to determine the effects of these microscopic details on the heat capacity. The tunneling density of states cannot conveniently be used to predict the heat capacity because it is a local density of states at the free surface and not the average or total density of states required. Instead the microscopic effects are taken into account by solving the Bogoliubov equation for a normal-metal–superconductor bilayer. In Sec. II the excitation spectrum is determined and in Sec. III the density of states is calculated. The heat capacity is calculated from the density of states in Sec. IV and the resulting discontinuity at the transition temperature is discussed in Sec. V.

II. EXCITATION SPECTRUM

The system to be investigated consists of a normal-metal film (N) of thickness a_n which is in contact with a superconductor (S) of thickness a_s at the $x=0$ plane. For simplicity it is assumed that the metals differ only in their transition temperature or electron-phonon interaction. For clean materials (where $l_e >> a_n, a_s$) the excitations of this system can be determined from the Bogoliubov equation

$$\epsilon\psi = \sigma_z\left(\frac{\hbar^2}{2_m}\nabla^2 - E_F\right)\psi + \sigma_x\Delta(x)\psi \ , \tag{1}$$

where $\psi = \begin{bmatrix} u \\ v \end{bmatrix}$ is the two-component wave function and $\Delta(x)$ is the position-dependent pair potential. To make the problem tractable it is assumed that $\Delta(x) = \Delta_s$ in the superconductor ($a_s > x > 0$) and that $\Delta(x) = 0$ in the normal metal ($0 > x > -a_n$). This should be a good approximation for thin films where a_n and a_s are less than a coherence length since the pair potential cannot vary rapidly over distances shorter than that in a given metal. On the other hand, for very large thicknesses this approximation should be reasonable since most of the variation of $\Delta(x)$ will occur within a few coherence lengths of the $x=0$ interface.

To find the excitation energies we look for a solution to Eq. (1) with the boundary condition $\psi = 0$ at

the free surfaces of the metal.[11] The solutions are of the form

$$\psi(x)=\begin{bmatrix}\alpha_+\sin k_+(x+a_n)\\ \alpha_-\sin k_-(x+a_n)\end{bmatrix}e^{i\vec{k}_\perp\cdot\vec{r}} \quad \text{for } x<0 \tag{2}$$

and

$$\psi(x)=\begin{bmatrix}\frac{\epsilon+\Omega}{\Delta_s}\gamma_+\sin q_+(x-a_s)+\frac{\epsilon-\Omega}{\Delta_s}\gamma_-\sin q_-(x-a_s)\\ \gamma_+\sin q_+(x-a_s)+\gamma_-\sin q_-(x-a_s)\end{bmatrix}e^{i\vec{k}_\perp\cdot\vec{r}} \quad \text{for } x>0 \tag{3}$$

with the following definitions:

$$k_\pm=\left[\frac{2m}{\hbar^2}\right]^{1/2}(E_F\cos^2\theta\pm\epsilon)^{1/2}\ ,$$

$$\sin\theta=k_\perp/k_F\ ,$$

$$q_\pm=\left[\frac{2m}{\hbar^2}\right]^{1/2}(E_F\cos^2\theta\pm\Omega)^{1/2}\ ,$$

$$\Omega=(\epsilon^2-\Delta_s^2)^{1/2}\ .$$

By matching ψ and $d\psi/dx$ of Eqs. (2) and (3) at $x=0$ there is a nontrivial solution for the four coefficients $\alpha_\pm$ and $\gamma_\pm$ only for values of ϵ which satisfy the equation

$$\Omega\cos\Delta k\cos\Delta q-\epsilon\sin\Delta k\sin\Delta q=\Omega\cos(\bar{k}+\bar{q})\ , \tag{4}$$

where

$$\Delta k=(k_+-k_-)a_n\simeq\frac{2a_n\epsilon}{\pi\xi\Delta_s\cos\theta}\ ,$$

$$\Delta q=(q_+-q_-)a_s\simeq\frac{2a_s\Omega}{\pi\xi\Delta_s\cos\theta}\ ,$$

$$\bar{k}=(k_++k_-)a_n\simeq 2k_Fa_n\cos\theta\ ,$$

$$\bar{q}=(q_++q_-)a_s\simeq 2k_Fa_s\cos\theta\ ,$$

$$\xi=\frac{\hbar v_F}{\pi\Delta_s}\ .$$

To arrive at the above solution it has been assumed that $\epsilon(E_F\cos\theta)^{-1}\ll 1$. We need only consider positive values of $\cos\theta$ since excitations with $\cos\theta<0$ represent the same excitations owing to the specular reflections at $x=-a_n$ and a_s.

If the energy ϵ is held fixed the right-hand side (RHS) of Eq. (4) is a rapidly oscillating function of $\cos\theta$ since the argument of the cosine is usually much greater than 1. If the RHS and left-hand side (LHS) of Eq. (4) are graphed against angle θ as in Fig. 1, then the points of intersection correspond to the values of θ for which there are solutions at this particular energy. As can be seen in Fig. 1, because the LHS is a slowly varying function there can be at most one additional solution for each increase in $\bar{k}+\bar{q}$ from $n\pi$ to $(n+1)\pi$ which corresponds to a maximum number of solutions $(\bar{k}+\bar{q})_{\max}/\pi=2k_F(a_s+a_n)/\pi$. The actual number of solutions can be even less if there are values of $\cos\theta$ for which the LHS becomes greater than Ω.

As a check on Eq. (4) we look for solutions for the case $\epsilon\gg\Delta_s$ where superconducting effects should be small. In that limit Eq. (4) becomes

$$\cos(\Delta k+\Delta q)=\cos(\bar{k}+\bar{q})$$

or

$$\Delta k+\Delta q=\pm(\bar{k}+\bar{q})+2\pi j\ ,$$

where j is an integer. Substituting the expressions for Δk, Δq, $\bar{k}$, and $\bar{q}$ leaves

$$\epsilon=\pm 2E_F\cos^2\theta+\frac{\pi\hbar^2k_F\cos\theta}{m(a_s+a_n)}j\ . \tag{5}$$

The spacing between levels is $\pi\hbar^2k_F\cos\theta\,[m(a_s+a_n)]^{-1}$ which is precisely the result for an electron confined in single dimension to a length a_s+a_n. In

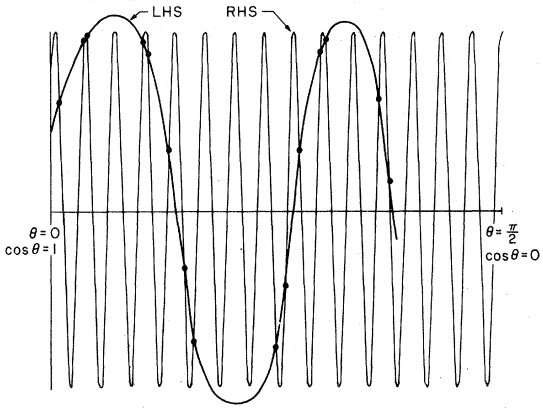

FIG. 1. Qualitative sketch of the left-hand side (LHS) and right-hand side (RHS) of Eq. (4) as function of θ or $\cos\theta$. The points of intersection correspond to different modes of excitations at a fixed energy.

addition the values of $\cos\theta$ which satisfy Eq. (5) are evenly spaced [again assuming $\epsilon(E_F\cos\theta)^{-1} << 1$] with

$$\cos\theta = \left| \frac{k_F(a_s+a_n)}{\pi} j \right| \tag{6}$$

as would be the case for the confined electron. In this case there are two solutions for each value of $\cos\theta$—one with j positive and one with j negative corresponding to the usual excitations above and below the Fermi surface.

To compute the thermal properties of this bilayer it will be necessary to know how the excitation energies vary with Δ_s. The derivative $\partial\epsilon/\partial\Delta_s^2$ is easily calculated by differentiating Eq. (4) with respect to Δ_s^2 which gives

$$\frac{\partial\epsilon_j}{\partial\Delta_s^2} = \frac{1}{2} \frac{a_s\Omega\cos\Delta k\sin\Delta q + a_s\epsilon\sin\Delta k\cos\Delta q - \frac{1}{2}\pi(\epsilon/\Omega)\xi\Delta_s\cos\theta\sin\Delta k\sin\Delta q}{(a_n\Omega^2+a_s\epsilon^2)\sin\Delta k\cos\Delta q + \epsilon\Omega(a_s+a_n)\cos\Delta k\sin\Delta q - (\Delta_s^3/\Omega)\xi\cos\theta\sin\Delta k\sin\Delta q} . \tag{7}$$

The label j has been added to the derivative as a reminder that the quantity depends on the quantum number j (or angle $\cos\theta$) in addition to the energy. In the extreme cases a_n or $a_s = 0$, Eq. (7) reduces to $\partial\epsilon/\partial\Delta_s^2 = (2\epsilon)^{-1}$ and 0, respectively, as one would expect.

De Gennes and Saint-James[6] were the first to point out that a proximity-effect bilayer such as we are considering should give rise to bound states with energies less than Δ_s. For these states Ω is imaginary and it is more convenient to define $\rho = i\Omega = (\Delta_s^2-\epsilon^2)^{1/2}$. Then Eq. (4) becomes

$$\rho\cos\Delta k\cosh\Delta q' - \epsilon\sin\Delta k\sinh\Delta q' = \rho\cos(\bar{k}+\bar{q}) , \tag{8}$$

where $\Delta q' = i\Delta q \simeq 2a_s\rho(\pi\xi\Delta_s\cos\theta)^{-1}$. The hyperbolic functions often cause the LHS of Eq. (8) to be greater than ρ so that there are usually far fewer modes for a given $\epsilon < \Delta_s$ than for $\epsilon > \Delta_s$. Putting the hyperbolic functions in exponential form this becomes

$$e^{\Delta q'}\sin\left[\Delta k - \cos^{-1}\left(\frac{\epsilon}{\Delta_s}\right)\right] - e^{-\Delta q'}\sin\left[\Delta k + \cos^{-1}\left(\frac{\epsilon}{\Delta_s}\right)\right] = -\frac{2\rho}{\Delta_s}\cos(\bar{k}+\bar{q}) .$$

In the limit $a_s >> \xi\cos\theta$ so that $\Delta q' >> 1$ this reduces to

$$\sin\left[\Delta k - \cos^{-1}\left(\frac{\epsilon}{\Delta_s}\right)\right] \simeq 0$$

which agrees with the calculation of De Gennes and Saint-James who assumed an infinitely thick superconducting layer.

III. DENSITY OF STATES

The density of states for these excitations can be calculated from the excitation spectrum of Sec. II using[6]

$$n_j(\epsilon) = \frac{Vk_F^2}{2\pi(a_s+a_n)}\left|\cos\theta_j\sin\theta_j\frac{d\theta_j}{d\epsilon}\right| = \frac{2\pi N(0)V}{k_F(a_s+a_n)}E_F\left|\cos\theta_j\frac{d\cos\theta_j}{d\epsilon}\right| , \tag{9}$$

where V is the volume of the bilayer and the quantum number j indicates the mode. The total density of states is the sum of Eq. (9) over all quantum numbers $n(\epsilon) = \sum_j n_j(\epsilon)$.

For the case $\epsilon >> \Delta_s$ the derivative required for Eq. (9) can be calculated directly from Eq. (5). The result is that $n_j(\epsilon) = \pi N(0)V[k_F(a_s+a_n)]^{-1}$ and is independent of mode. The total density of states is then

$$n(\epsilon) = \sum_j n_j(\epsilon) = n_j(\epsilon)\frac{2k_F(a_s+a_n)}{\pi} = 2N(0)V .$$

The factor of 2 comes from the counting of excitations both above and below the Fermi surface.

For the case of arbitrary energy the derivative $d\cos\theta j/d\epsilon$ can be determined by differentiating Eq. (4) with respect to ϵ. This leads to [still assuming $\epsilon(E_F\cos\theta)^{-1} << 1$]

$$n_j(\epsilon) = \frac{\pi N(0)V}{k_F(a_s+a_n)}\left|\frac{\Omega(a_n\Omega^2+a_s\epsilon^2)\sin\Delta k\cos\Delta q + \epsilon\Omega^2(a_s+a_n)\cos\Delta k\sin\Delta q - \frac{1}{2}\pi\Delta_s^3\xi\sin\Delta k\sin\Delta q}{(a_s+a_n)\Omega^3\sin(\bar{k}+\bar{q})}\right| . \tag{10}$$

In the extreme cases a_n or $a_s=0$ this expression reduces to the appropriate value [i.e., $2N(0)V/(\epsilon^2-\Delta_s^2)^{1/2}$ for $\epsilon>\Delta_s$ and $2N(0)V$, respectively] but in general a numerical calculation is required. The calculation of Eq. (10) is a rapidly varying function of ϵ. For $\epsilon \gtrsim 2\Delta_s$ the density calculated using this equation can fluctuate 5–10% about a smooth curve and for lower values of energy the fluctuations can be much greater. To obtain a smoothed curve it is useful to either average $n(\epsilon)$ for several values of nearly the same energy or to solve Eq. (4) and calculate the derivative in Eq. (9) numerically. The results of these smoothed calculations are shown in Figs. 2–4.

Unlike the case of an isolated superconductor the density of states for the bilayer in general does not diverge as $\epsilon \to \Delta_s$ although there is a discontinuity at $\epsilon=\Delta_s$ when $a_s \gtrsim \xi$. For film thicknesses much greater than a coherence length, the peak near $\epsilon=\Delta_s$ becomes large and approaches what one would expect in the absence of a proximity effect, namely,

$$n(\epsilon)=2N(0)V\left[(1-p_s)+\frac{\epsilon}{(\epsilon^2-\Delta_s^2)^{1/2}}p_s\right] \quad \text{for } \epsilon>\Delta_s$$

$$=2N(0)V(1-p_s) \quad \text{for } \epsilon<\Delta_s \ ,$$

which is just the sum of the density of states for an isolated normal metal and superconductor. Here $p_s=a_s(a_s+a_n)^{-1}$ which is the fraction of the bilayer which is a superconductor. For films thinner than a coherence length there is no longer a discontinuity at $\epsilon=\Delta_s$.

For the bound states with $\epsilon<\Delta_s$, substitution of Eq. (8) into Eq. (9) leads to

$$n_l(\epsilon)=\frac{\pi N(0)V}{k_F(a_s+a_n)} \times \left| \frac{\rho(a_n\rho^2-a_s\epsilon^2)\sin\Delta k\cosh\Delta q'+(a_s+a_n)\epsilon\rho^2\cos\Delta k\sinh\Delta q'+\frac{1}{2}\pi\Delta_s^3\xi\cos\theta\sin\Delta k\sinh\Delta q'}{(a_s+a_n)\rho^3\sin(\bar{k}+\bar{q})+(\rho^2/2E_F\cos^2\theta)[(a_n+a_s)\epsilon\rho\sin\Delta k\cosh\Delta q'+(a_n\epsilon^2-a_s\rho^2)\cos\Delta k\sinh\Delta q']} \right| . \tag{11}$$

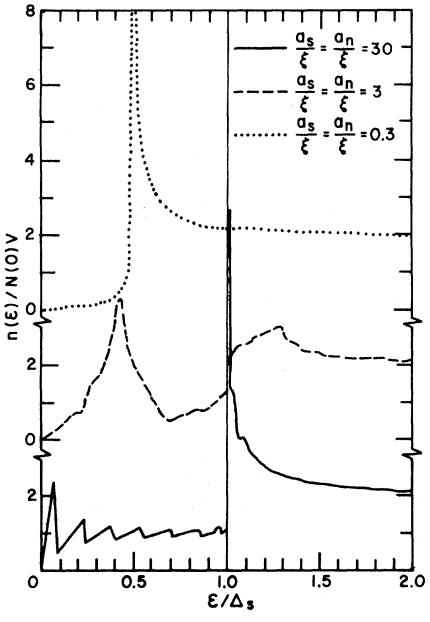

FIG. 2. Calculation of the normalized density of states vs energy for bilayers of several different thicknesses.

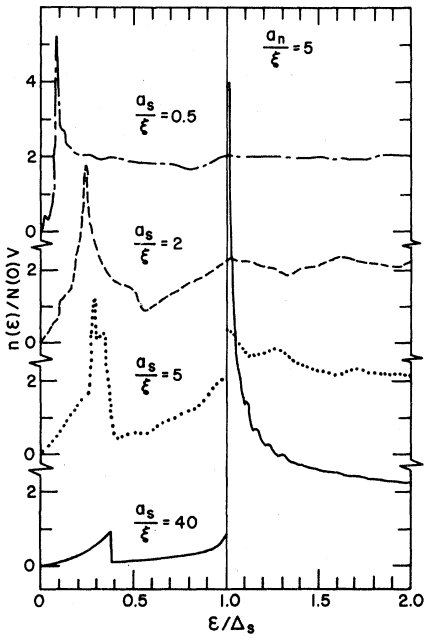

FIG. 3. Calculation of the normalized density of states vs energy for several different superconductor thicknesses. For all curves the normal-metal thickness is held constant at $a_n=5\xi$.

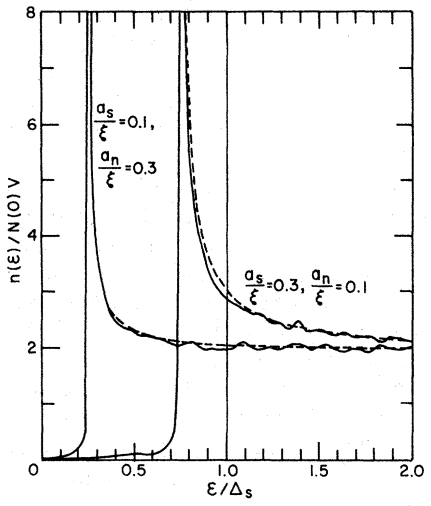

FIG. 4. Calculation of the normalized density of states vs energy for thin films which differ in the fraction of superconducting material in the bilayer. The dashed lines are the densities expected for a BCS superconductor of energy gap $\Delta_{\text{eff}}=p_s\Delta_s$.

As with the expression for $\epsilon > \Delta_s$ the density is a rapidly fluctuating function of ϵ and averaging is often useful to obtain a smooth curve. For large values of the superconductor thickness the density approaches the sawtooth curve of De Gennes and Saint-James[6] (see Fig. 2). The abrupt cutoff of the peaks is because of the absence of a given bound state mode above a certain energy. As the superconducting layer is reduced in width, the sawtooth becomes less distinct until the structure is washed out completely except for a single peak at low energy. For layers of less than a coherence length in thickness the density approaches a BCS-like density with an effective pair potential Δ_{eff} or energy gap given by the volume average of the pair potentials in the bilayer (i.e., $\Delta_{\text{eff}}=p_s\Delta_s$) as shown in Fig. 4. The reason for this can be seen from Eq. (3) which shows that the quasiparticle states of energy less than Δ_s can extend within the superconductor a distance of order $\sim\xi\cos\theta$. If this is greater than the superconductor thickness then the value of $|\psi|^2$ is nearly position independent and the probability of the excitation being localized in the superconductor is just p_s. The minimum energy of an excitation essentially confined to the superconductor would be Δ_s so one would expect the minimum energy of an excitation with a probability p_s of being in the superconductor to be $p_s\Delta_s$. For θ close enough to $\frac{1}{2}\pi$, the value of $\xi\cos\theta$ will always be smaller than a_s and this argument will break down. As a result there will always be a small number of states arbitrarily close to moving parallel to the interface in the normal metal with energies approaching zero. These states will not always show up in the calculation of Eq. (11) since that was based on an approximation which is not valid for angles too close to $\theta=\frac{1}{2}\pi$. But with the exception of these states, many of the properties (e.g., heat capacity, thermal conductivity) of these thin films should approach those of a BCS superconductor of the appropriate "effective" energy gap.

In Eqs. (9)–(11), the Fermi energy E_F comes in explicitly. As would be expected, varying E_F by as much as an order of magnitude results in only minimal changes in the density of states and heat capacity. The values of Δ_s/E_F used for these calculations were in the range 4×10^{-4} to 4×10^{-3} which are somewhat larger than would be the case for most materials but have the advantage of reducing the number of modes and hence the time required for the calculations. This trade-off seems justified considering the insensitivity of the calculation to this parameter.

IV. HEAT CAPACITY

The electronic heat capacity of the *NS* bilayer can be calculated[14] from

$$C(T)=2\sum\epsilon\frac{\partial f}{\partial T}$$

$$=2k_B\beta^2\sum f(1-f)\epsilon\left[\epsilon+\beta\frac{\partial\epsilon}{\partial\Delta_s^2}\frac{d\Delta_s^2}{d\beta}\right],$$

where $f=(e^{\epsilon\beta}+1)^{-1}$ and $\beta=(k_BT)^{-1}$. The first term represents the ordinary part of the heat capacity due to the quasiparticle distribution changing with temperature while the second term takes account of the temperature dependence of the excitation energies. The sum over states can be put in the form of an integral over energies and a sum over the quantum number j characterizing the angle θ

$$C(T)=2k_B\beta^2\int_0^\infty d\epsilon\, f(1-f)\epsilon$$

$$\times\sum_j n_j(\epsilon)\left[\epsilon+\beta\frac{\partial\epsilon_j}{\partial\Delta_s^2}\frac{d\Delta_s^2}{d\beta}\right]. \quad (12)$$

Calculations using Eq. (12) and the density of states from Eq. (10) are shown in Fig. 5 normalized to the normal-state electronic heat capacity at T_c, $C_n(T_c)=\frac{2}{3}\pi^2N(0)Vk_B^2T_c$. Also shown are the heat capacities for a bilayer assuming no proximity effect (npe) and that of an ordinary BCS superconductor.

The calculation with thicknesses of $5\xi_0$ is representative of the thick film case. Here Δ_s should have essentially the magnitude and temperature dependence of a BCS superconductor. The calculated heat capacity approaches the npe result as would be ex-

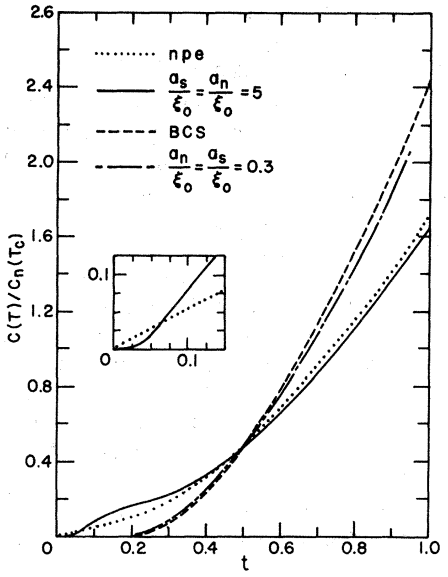

FIG. 5. Calculated normalized heat capacity vs the reduced temperature $t=T/T_c$. For the curve with $a_n/\xi_0=a_s/\xi_0=0.3$ a reduced pair potential has been assumed as discussed in the text. The inset shows greater detail for the curves near $t=0$.

pected since a similar behavior was found for the density of states. At lower reduced temperatures there is some deviation from the npe result caused by the sawtooth density of states for the bound states.[6,7] A peak in the density of states near $\epsilon\sim 0.3\Delta_s$ is responsible for the bump in heat capacity near $t\sim 0.2$. The heat capacity is roughly proportional to T^2 at still lower temperatures due to the small linear density of states near zero energy.[15] These low-temperature deviations from npe disappear as the normal-metal thickness is made greater.

For films of thickness less than a coherence length the calculation is complicated by the fact that in this Cooper limit the transition temperature and the pair potential Δ_s are expected to be reduced from the bulk values. Although the reduction in transition temperature has been calculated,[12] that for the pair potential has not. In view of the results of Sec. III in which the bilayer has a nearly BCS density of states it seems plausible to assume that Δ_{s0} (the low-temperature value of Δ_s) will be reduced in such a way as to allow $2\Delta_{\rm eff}/k_BT_c=3.53$ at low temperatures. This would make $\Delta_{s0}=3.53k_BT_c(2p_s)^{-1}$ where T_c refers to the actual transition temperature of the bilayer.

Using this value for the pair potential and assuming the same temperature dependence as in the bulk, the heat capacity (Fig. 5) is quite similar to that of a homogeneous BCS superconductor as should be expected from the BCS-like density of states. At the lowest temperatures however there are significant differences. Unlike a BCS superconductor which has a true energy gap leading to an exponential decay of the heat capacity at low temperatures the bilayer has no real gap for there are always a small number of states near $\theta=\frac{1}{2}\pi$ which can have energies arbitrarily close to zero. From Fig. 4 it can be seen that at low energy the density of states is approximately linear in ϵ leading to a roughly T^2 dependence for the low-temperature heat capacity. For thinner films the number of states below $\Delta_{\rm eff}$ would be even less, reducing this component to the heat capacity.

V. DISCONTINUITY IN THE HEAT CAPACITY

Most experimental investigations[1-3] of the heat capacity in the proximity-effect regime have focused on the discontinuity at the transition temperature so it is worthwhile to calculate this quantity explicitly. At T_c the pair potential $\Delta_s\to 0$ and the important excitations have energies much greater than Δ_s. In this limit Eqs. (7) and (10) reduce to

$$n_j(\epsilon)\to\frac{\pi N(0)V}{k_F(a_s+a_n)} \tag{13}$$

and

$$\frac{\partial\epsilon_j}{\partial\Delta_s^2}\to\frac{1}{2\epsilon}\left[\frac{a_s}{a_s+a_n}\right]-\frac{\pi\Delta_{s0}\xi_0\cos\theta\sin\Delta k\sin\Delta q}{4(a_s+a_n)\epsilon^2\sin(\Delta k+\Delta q)} . \tag{14}$$

The values of $\cos\theta$ are evenly spaced [see Eq. (6)] so the sum over quantum numbers can be replaced by an integral

$$\sum_j n_j(\epsilon)\left[\epsilon+\beta\frac{\partial\epsilon_j}{\partial\Delta_s^2}\frac{d\Delta_s^2}{d\beta}\right]\to\frac{2k_F(a_s+a_n)}{\pi}\int_0^1 d(\cos\theta)\,n_j(\epsilon)\left[\epsilon+\beta\frac{\partial\epsilon_j}{\partial\Delta_s^2}\frac{d\Delta_s^2}{d\beta}\right] .$$

Substituting the limiting behavior of Eqs. (13) and (14) into Eq. (12), the normalized heat capacity at T_c becomes

$$\frac{C}{C_n}\frac{(T_c)}{(T_c)}=\frac{6\beta_c^3}{\pi^2}\int_0^\infty d\epsilon\,f_c(1-f_c)\epsilon^2+\frac{6\beta_c^4}{\pi^2}\frac{d\Delta_s^2}{d\beta_c}\int_0^\infty d\epsilon\,f_c(1-f_c)\epsilon\int_0^1 d(\cos\theta)\left[\frac{p_s}{2\epsilon}-\frac{\pi\Delta_{s0}\xi_0\cos\theta\sin\Delta k\sin\Delta q}{4(a_s+a_n)\epsilon^2\sin(\Delta k+\Delta q)}\right] , \tag{15}$$

where the subscript c indicates that the quantity is to be evaluated at T_c. The first term is the contribution from the temperature dependence of the quasiparticle distribution and is equal to one. The second term then gives the normalized discontinuity in the heat capacity $\Delta C/C_n(T_c)$. Substituting $y=\Delta k+\Delta q$ and integrating over $\cos\theta$ leads to an expression for the discontinuity[16]

$$\frac{\Delta C}{C_n(T_c)}=\frac{3\beta_c^3}{2\pi^2}\frac{d\Delta_s^2}{d\beta_c}p_s\left[1-p_s^{-1}\int_0^\infty dy\frac{h(y)}{\sin y}\right] , \tag{16}$$

where

$$h(y)=\sin p_s y\sin(1-p_s)y\left[\tanh by-\frac{\ln\cosh by}{by}\right]y^{-2}$$

and

$$b=\frac{\Delta_{s0}\beta_c\pi\xi_0}{4(a_s+a_n)} .$$

The integral contains an infinite number of divergences but they are all integrable. The Eq. (16) can be put in a form for easier computation by noting that

$$\int_0^\infty dy\frac{h(y)}{\sin y}=\int_0^{\pi/2}dy\,\frac{h(y)}{\sin y}+\sum_{n=1}^\infty\left[\int_{(n-\frac{1}{2})\pi}^{n\pi}dy\,\frac{h(y)}{\sin y}+\int_{n\pi}^{(n+1/2)\pi}dy\,\frac{h(y)}{\sin y}\right] .$$

Using the periodicity of $\sin y$ and substituting $w=n\pi-y$ in the first integral in the parenthesis and $w=n\pi+y$ in the second, the quantity in parentheses becomes

$$(-1)^n\int_0^{\pi/2}dw\,\frac{h(n\pi+w)-h(n\pi-w)}{\sin w} .$$

Expanding $h(n\pi+w)$ and $h(n\pi-w)$ in a Taylor series about $n\pi$ we arrive at

$$\int_0^\infty dy\,\frac{h(y)}{\sin y}=\int_0^{\pi/2}dy\,\frac{h(y)}{\sin y}+2\sum_{s=0}^\infty\frac{g_{2s+1}}{(2s+1)!}\sum_{n=1}^\infty(-1)^nD^{(2s+1)}h(n\pi) , \tag{17}$$

where

$$g_{2s+1}=\int_0^{\pi/2}\frac{w^{2s+1}}{\sin w}dw .$$

Neither the first integral in Eq. (17) nor g_{2s+1} diverge. The integral of Eq. (16) is then easily evaluated and is shown in Fig. 6. Only the first few (and in many cases only the first) terms in s are important.

For thick films, $b\ll 1$ and the $\tanh by-(\ln\cosh by)/by$ factor in $h(y)$ is large only in the region $y\gg 1$. In that range the oscillatory factors of $h(y)$ cause the integral to vanish as is seen in Fig. 6. In this limit then

$$\frac{\Delta C}{C_n(T_c)}\to\frac{3\beta_c^3}{2\pi^2}p_s\frac{d\Delta_s^2}{d\beta_c}=1.43p_s$$

which is just the npe result.

In the opposite limit of thin films $b\gg 1$ in which case the dominant contribution to the integral of Eq. (16) comes from the region $y\ll 1$. In that limit the sine functions in the integral can be approximated by

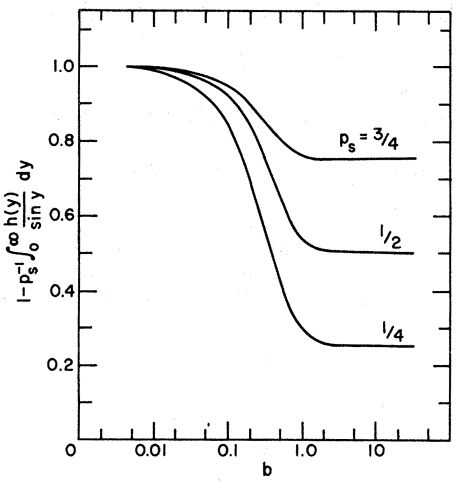

FIG. 6. Numerical calculation of the factor in Eq. (16) vs the parameter b for several different values of the fraction of superconducting material p_s.

their arguments giving

$$\int_0^\infty dy\,\frac{h(y)}{\sin y}\rightarrow\int_0^\infty dy\,p_s(1-p_s)\left[\tanh by-\frac{\ln\cosh by}{by}\right]y^{-1}$$

$$=p_s(1-p_s)\ .$$

Substituting this in Eq. (16) then gives

$$\frac{\Delta C}{C_n(T_c)}\rightarrow\frac{3\beta_c^3}{2\pi^2}p_s^2\frac{d\Delta_s^2}{d\beta_c}=\frac{3\beta_c^3}{2\pi^2}\frac{d\Delta_{\rm eff}^2}{d\beta_c}\ .$$

The last expression is just the discontinuity for a BCS superconductor with an energy gap of $\Delta_{\rm eff}$. If the magnitude of the pair potential from Sec. IV is correct (i.e., $2\Delta_{\rm eff}/k_BT_c=3.53$ at low temperatures), then the bilayer in this Cooper limit should have the same discontinuity in the heat capacity as would a BCS superconductor having the same transition temperature as the bilayer.

IV. CONCLUSIONS

The density of states and heat capacity of an *NS* bilayer have been calculated from the solutions to the Bogoliubov equation. In the limit of film thicknesses much greater than a coherence length both properties approach the npe results. For film thicknesses much less than a coherence length the density of states is BCS-like with an effective pair potential determined as a volume average over the bilayer. There is some uncertainty as to the magnitude and temperature dependence of the pair potential for the superconductor in this limit but a reasonable guess gives rise to a BCS-like heat capacity. In the intermediate thickness range numerical results are given with both the density of states and the heat capacity containing structure due to bound states.

A more realistic calculation would determine the pair potential self-consistently and take into account the finite mean free path of the excitations. Nevertheless the purpose of this calculation was to include the details of Andreev scattering and bound states in the heat capacity—details which could not be included in a Ginzburg-Landau calculation. It is believed that a more realistic calculation would arrive at qualitatively similar results.

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[15]For a fairly thick superconducting film it is not hard to show that the limiting behavior is

$$\lim_{\epsilon\rightarrow 0}\frac{n(\epsilon)}{N(0)}\rightarrow\frac{2a_n}{\xi}\frac{\epsilon}{\Delta_s}\ ,$$

$$\lim_{T\rightarrow 0}C(T)\rightarrow\frac{27\zeta(3)}{\pi^2}\frac{k_BT_C}{\Delta_{s0}}\frac{a_n}{\xi_0}t^2=1.86\frac{a_n}{\xi_0}t^2\ .$$

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