Conductivity of random resistor-diode networks

S. Redner
Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215
(Received 7 December 1981)

We investigate the behavior of the conductivity in random lattice networks of resistors and *Ohmic* diodes—elements which behave like an ideal resistor in the forward direction. We study the case where the orientation of the diodes is fixed, corresponding to the geometrical model of directed percolation. New critical behavior is predicted because the structure of the underlying conducting network is characterized by two independent orthogonal correlation lengths. We use the node picture of a percolating network to derive the anisotropic scaling relation between directed percolation and conductivity exponents. In the mean-field limit, we find a directed conductivity exponent $t_+=2$, in contrast to an isotropic conductivity exponent of $t=3$. In two dimensions, we employ the renormalization group to study the critical behavior of the directed conductivity. We predict that the conductivity should approach zero with an infinite slope ($t_+<1$) as the transition is approached from above. This is consistent with the intuitive expectations developed from the node picture. We also briefly discuss conduction in networks with *superconducting* diodes (which have an infinite conductivitivity in the forward direction), and possibility of a directed conductor-superconductor transition. Additionally, we describe a more general network with Ohmic and superconducting diodes within the renormalization-group framework by introducing a larger parameter space which includes "leaky" diodes.

I. INTRODUCTION

One important application of the percolation problem[1,2] has been the description of conductivity in random conductor-insulator and conductor-superconductor mixtures.[3] In the former case, as the concentration of insulating material is increased from zero, the system undergoes a continuous transition from a conducting to an insulating state. This transition can be modeled quite simply by bond percolation in which the conductor is represented by a regular lattice whose bonds have a finite conductivity. The dilution by insulating material corresponds to randomly removing conducting bonds from the lattice. As the bond concentration p approaches the percolation threshold p_c from above, the conductivity $G(p)$ vanishes as

$$(p-p_c)^t \ . \tag{1.1}$$

Similarly, the superconducting transition can be described by a percolation process in which the lattice bonds have infinite conductivity with probability p, and finite conductivity with probability $1-p$. In this case, as p approaches p_c from below, $G(p)$ diverges as

$$(p-p_c)^{-s} \ . \tag{1.2}$$

In the mean-field limit, valid for spatial dimension $d \geq 6$, $t=3$ and $s=0$,[4-6] while for $d<6$ these exponents have been calculated by a wide variety of approaches.[3,7-14] In $d=2$ and 3, theoretical estimates are in good agreement with current experimental results.[15-17] Thus there has been considerable progress in understanding the conductivity in random, but isotropic systems.

In this article, we investigate the conductivity of networks in which diodes (one-way bonds) are introduced. The motivation for our study is twofold: (1) The geometrical properties of random networks in which diodes are added becomes considerably more interesting[18-34] than networks consisting only of resistors (two-way bonds). A study of conduction in diode networks thus complements previous investigations of the geometrical properties. (2) There may exist physical systems where directionality constraints enter in a fundamental way, and these constraints may be modeled by a random network containing resistors and diodes. One possibility that has been considered is conduction due to electron hopping in strong electric fields,[35,36] and other examples have been suggested theoretically.[20] It is hoped that our study of conductivity in diode networks will stimulate further investigations, both theoretically and experimentally.

The organization of the remainder of this article is as follows: In Sec. II, we define the random networks of resistors and diodes that are treated in later sections. In Sec. III, we present an effective-medium approach for the conductivity. It gives fairly good estimates for the locations of conductivity transitions, along with the standard result of conductivity exponents equal to unity. This effective medium approach displays some shortcomings, however, which

seem to be indicative of some rather interesting effects. In Sec. IV, a scaling theory for the conductivity based on the node picture[37,38] of the percolating network is presented. We use this to compare the critical behavior of the conductivity in a random diode and a random resistor network. In addition, we derive an anisotropic scaling relation between the conductivity and correlation length exponents of the diode network, and thereby obtain the mean-field limit for the exponents.

In Sec. V, we present a position-space renormalization-group study of the critical behavior of the conductivity in two dimensions. Our calculations indicate that the conductivity should vanish with an infinite slope near the transition, a result that is consistent with the expectations developed from the node picture. In Sec. VI, we treat networks with *superconducting* diodes, bond elements which possess an infinite conductivity in the forward direction. In this case, the network conductivity is quite interesting because several combinations of electrical responses in the forward and reverse directions are possible. To describe the system self-consistently by the renormalization group, we introduce a larger parameter space which includes "leaky" diodes. The exponents associated with a directed conductor-superconductor transition are also calculated. Finally, Sec. VII contains a brief summary and discussion.

II. MODEL SYSTEMS

A. Fully directed percolation

For concreteness, we shall consider networks on a d-dimensional hypercubic lattice. In *fully directed* percolation, each edge in the lattice may be either occupied by a diode with probability p_+, or empty with a probability $1-p_+$. The diodes are oriented so that they conduct only along the positive direction with respect to a given Cartesian axis [Fig. 1(a)]. Thus they may be thought of as being "aligned" or "polarized" along an external bias or anisotropy parallel to the diagonal $(1,1,\ldots,1)$.

The orientational order of the diodes produces a macroscopic anisotropy near the directed percolation threshold. Asymptotically, cluster shapes may be characterized by two independent diverging correlation lengths,[20,22–23,28–32] one parallel and one transverse to the anisotropy axis [Fig. 1(b)]. The divergences of these two lengths are governed by the longitudinal and transverse correlation length exponents, $\nu_\parallel$ and $\nu_\perp$, respectively. Because $\nu_\parallel > \nu_\perp$ in general, the pair connectedness becomes long-ranged only along the anisotropy axis at the percolation threshold. Above the threshold, a cone-shaped region of finite opening angle about the anisotropy axis opens up, within which it is possible to percolate.

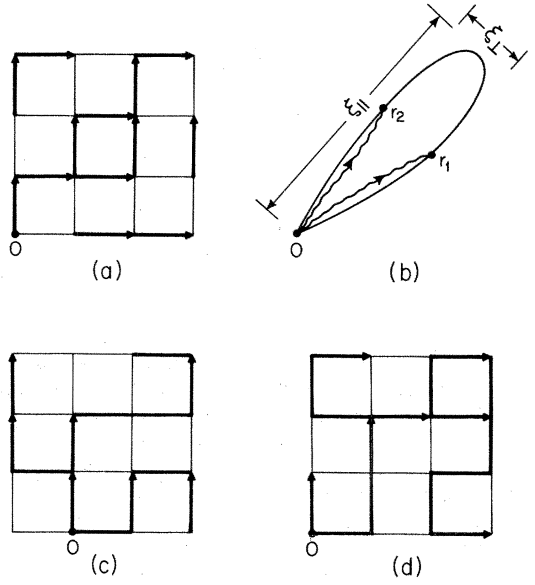

FIG. 1. (a) Fully directed percolation in two dimensions. Each pair of nearest-neighbor sites may be joined by a directed bond (diode) which conducts either upward or to the right. (b) Schematic picture of a typical cluster in directed percolation near the transition. Notice that two points, r_1 and r_2, on opposite sides of the cluster are connected only in the sense that they are joined to a common origin. (c) Partially directed percolation in two dimensions. Resistors may occupy the horizontal edges of the lattice while diodes which conduct upward may occupy the vertical edges. The occupation probabilities for the two bond elements are equal. (d) Oriented resistor-diode network. Each occupied bond may be a diode (which conducts either upward or to the right) or a resistor. There is a connected path which goes from the origin to the right edge by a path of resistors and diodes.

This represents a transition from an insulating to a diodelike state whose study is one of the aims of this paper.

B. Partially directed percolation

In this model, the diodes are aligned along an external bias that is parallel to a particular Cartesian axis [Fig. 1(c)], while perpendicular to this axis, the system is isotropic. The isotropy in the transverse directions may be accomplished either by randomly occupying the lattice edges by resistors, or by diodes with no average orientational order. For simplicity, we shall consider only the first possibility, with the further simplifying condition that the occupation probabilities for the resistors and diodes are equal. We define this as *partially directed* percolation.[39] As we shall see in the next section, the partially directed model is very convenient for applying effective-medium theory.

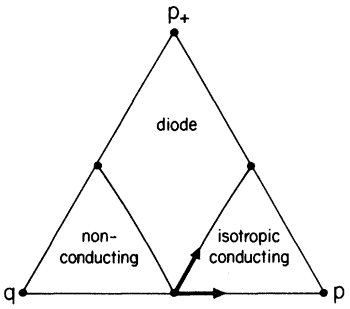

FIG. 2. Phase diagram for the oriented resistor-diode network. For any point inside the composition triangle, the perpendicular distance to an edge gives the relative concentration of the bond element labeled at the opposite corner. Fixed points of the renormalization transformation are indicated by heavy dots. The two second-order lines divide the figure into the three phases shown. The second-order lines meet to define a tricritical point at the isotropic percolation threshold. Here, there are two relevant eigenvectors (arrows) related to concentration and orientational variables.

C. Oriented resistor-diode network

This model is a simple generalization of fully directed percolation. Each bond may be occupied by a resistor with a probability p, by a positively oriented diode with a probability p_+, or the bond may be unoccupied with a probability $q = 1 - p - p_+$ [Fig. 1(d)]. The system exhibits three distinct phases depending on the relative concentrations of the bond elements (Fig. 2). There may exist an isotropic infinite cluster (conducting phase), or a unidirectional infinite cluster, in which connected paths propogate predominantly in one direction (diode phase). The network may also consist of only finite clusters (insulating phase). By varying p and p_+, one can pass between these phases, and probe several interesting conductivity transitions. First, there is the transition between the diode and nonconducting state, already discussed in Sec. II A above. In addition, there is a transition from the conducting to the diode phase, where the conductivity *opposite* to the orientation of the diodes, vanishes. Furthermore, the three phases of the system meet at a common point where they are simultaneously critical, that is, a tricritical point where new conductivity exponents may be defined.[40]

D. Electrical response of the diodes

To discuss directed conductivity transitions in more detail, we need to specify the electrical response of the diodes comprising the network. Two natural choices are: *Ohmic* diodes and *superconducting* diodes (Fig. 3). The former are defined to have an infinite resistance if the voltage is biased in the backward

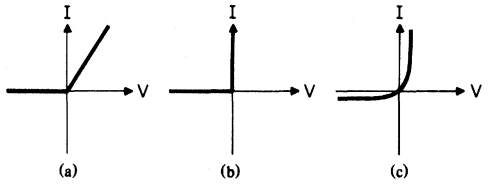

FIG. 3. The current-voltage characteristics of (a) the model "Ohmic" diode, (b) the model "superconducting" diode, and (c) a typical laboratory diode.

direction, but a finite, nonzero resistance under a forward-bias voltage. Such a response approximates that of a real diode and resistor in series. On the other hand, the superconducting diode is defined to have zero resistance in the forward direction. This represents a rough approximation to a real diode which allows a very small leakage current to flow in the backward direction, but which becomes highly (and nonlinearly) conducting in the forward direction [Fig. 3(c)].

In the following sections, we will be primarily interested in networks with Ohmic diodes. Several conductivity transitions can occur where directionality effects play an essential role. We define these transitions in the context of the oriented resistor-diode network, which encompasses both fully and partially directed percolation. In the transition from the diode to the insulating phase, the forward conductivity, $G_+(p,p_+)$, should vanish continuously as

$$(p_+ - p_{+_c})^{t_+} , \tag{2.1}$$

as $p_+ \to p_{+_c}$ from above. This defines the forward or directed conductivity exponent t_+. On the other hand, as one goes from the conducting to the diode phase, the reverse conductivity, $G_-(p,p_+)$, should vanish continuously as

$$(p - p_c)^{t_-} , \tag{2.2}$$

as $p \to p_c$ from above. Finally, at the tricritical point, both conductivity processes, (2.1) and (2.2), are simultaneously critical, although with potentially different numerical values for the exponents.

For networks with superconducting diodes, there exists a transition where the forward conductivity diverges continuously as the concentration of diodes approaches a threshold value from below, while the reverse conductivity remains analytic. This divergence may be described by the power law.

$$(p_+ - p_{+_c})^{-s_+} , \tag{2.3}$$

which serves to define a forward superconducting exponent, s_+.

In the following sections, we shall study the conductivity properties of the networks defined above.

III. EFFECTIVE-MEDIUM THEORY

In effective-medium theory (EMT), the macroscopic conductance of a random system is determined in terms of the conductance of an equivalent periodic system using a procedure reminiscent of the effective-field approximation in magnetism. We will apply EMT ideas to obtain results for fully and partially directed percolation in this section. To begin, we outline the basic steps of conventional EMT for a random resistor network.[3,41,42]

First, we replace the random network in which each bond has a finite conductivity g_0 with probability p and zero conductivity with probability $1-p$, by an equivalent uniform effective medium in which every bond has the same conductivity, g_m. Across each bond parallel to the field, a mean current i_m flows due to a mean voltage drop V_m. These quantities are related by

$$i_m = g_m V_m \ . \tag{3.1a}$$

Next, one bond in the effective medium is now assigned a different conductivity g, and the external voltage is adjusted to maintain a total current I passing through the lattice (Fig. 4). In this case, an additional current δi passes through the conductor g. Consequently, a current $-\delta i$ must flow from one terminal of g—defined to be a—through the remainder of the lattice, to the other terminal of g—defined to be b. Because of this current perturbation, there is an additional voltage drop, δV, across g. From these considerations we find

$$-\delta i = G_{ab}\delta V \ , \tag{3.1b}$$

$$i_m + \delta i = g(V_m + \delta V) \ , \tag{3.1c}$$

where G_{ab} is the conductivity of the rest of the lattice between the terminals of g [Fig. 4(b)]. Finally, we require that δV averaged over the conductivity probability distribution for each bond, $p\delta(g-g_0)+(1-p)\delta(g)$, is zero. This gives[41,42]

$$\langle \delta V \rangle = V_m \left[\frac{(g_m - g_0)p}{(G_{ab}+g_0)} + \frac{g_m(1-p)}{G_{ab}} \right] = 0 \ . \tag{3.2}$$

It is convenient to write $G_{ab} = \alpha g_m$, where α is a lattice-dependent constant of order unity. With this definition, we find

$$g_m = g_0[p(1+\alpha)-1]/\alpha \ . \tag{3.3}$$

Thus the conductivity vanishes at $p_c = 1/(1+\alpha)$ with a conductivity exponent $t=1$.

An important feature for obtaining a self-consistent EMT for directed percolation is that we must apply the electric field parallel to the anisotropy axis. If the field and the anisotropy axis are not parallel, EMT apparently fails, and we discuss this below in more detail. Accordingly, we first treat partially directed percolation in two dimensions. In evaluating the voltage fluctuation across the conductor g, an essential feature is that the current $-\delta i$ between a and b can flow only within a $1 \times \infty$ strip of the lattice [Fig. 5(a)]. Thus only the conductivity of this strip enters into the effective medium equation (3.1b). In this case, elementary considerations give $G_{ab} = (\sqrt{3}-1)g_m$. Inserting this value into (3.2), we find

$$g_m = g_0(\sqrt{3}p_+ - 1)/(\sqrt{3}-1) \ . \tag{3.4}$$

Thus we find that the forward conductivity, G_+, vanishes as $(p_+ - p_{+_c})^{t_+}$, with $p_{+_c} = 1/\sqrt{3} = 0.5773\ldots$, and $t_+ = 1$. This value for the critical concentration is a good approximation to the estimate $p_{+_c} = 0.555 \pm 0.002$ found by low-density series.[43]

To obtain an EMT for fully directed percolation, we apply the field along the diagonal [Fig. 5(b)]. In this case, it appears most natural to focus on two next-nearest-neighbor sites in order to evaluate the voltage fluctuation. Between these two sites, there are only two paths of nonzero conductivity. We employ a two-bond approximation in which the conductivity of both bonds on one of these paths is given by the correct probability distribution. For this path then, a conductivity $\frac{1}{2}g_0$ occurs with probability p_+^2, while a conductivity 0 occurs with probability $1-p_+^2$. Since $G_{ab} = \frac{1}{2}g_m$, the analog of Eq. (3.2) is

$$\langle \delta V \rangle = 2V_m \left[\frac{g_m - g_0}{g_m + g_0} p_+^2 + (1-p_+^2) \right] \ . \tag{3.5}$$

Setting $\langle \delta V \rangle = 0$ leads to

$$g_m = g_0(2p_+^2 - 1) \ . \tag{3.6}$$

That is, $p_{+_c} = 1/\sqrt{2} = 0.7071\ldots$, and $t_+ = 1$. The value p_{+_c} obtained by EMT should be compared with

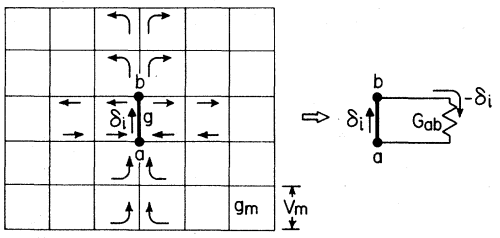

FIG. 4. In the effective-medium approximation, we replace the random network by the regular lattice shown, in which each bond has a conductivity g_m. When the conductivity of one bond (shown as a heavy line and labeled by the end points, a and b) in the effective medium is changed, a voltage fluctuation is produced. The arrows indicate schematically the additional current due to this fluctuation. The uniform current i_m is not shown. In analyzing this situation, the lattice is equivalent to the system shown to the right.

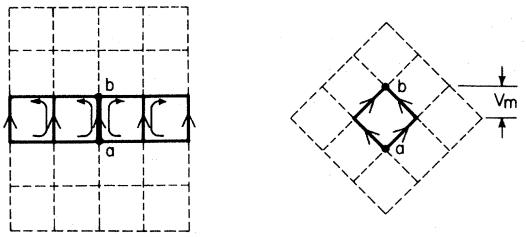

FIG. 5. (a) In partially directed percolation, the current perturbation is confined to a $1 \times \infty$ strip of the lattice. The remainder of the lattice is shown dashed. (b) For fully directed percolation, the lattice is rotated by $\frac{1}{4}\pi$. In this case, there are only two paths of nonzero conductivity joining a and b that enter into the effective-medium calculation. The irrelevant part of the lattice is shown dashed.

the estimates $p_{+_c} = 0.6445-0.6447$ obtained by more accurate numerical methods.[19,21-23] The agreement for the critical concentration here is somewhat worse than in partially directed percolation. This is not surprising given the crudeness of our approximation in which only four bonds in the entire lattice play a role in EMT.

If the lattice is oriented so that the field is parallel to a Cartesian axis, then the EMT presented here breaks down. This stems from the fact that the conductivity G_{ab} between two sites separated by one lattice spacing parallel to the field is infinite, and $\langle \delta V \rangle$ vanished identically.

A related pathology seems to occur in numerically calculating the conductivity for fully directed percolation. Consider a diode which forms a "bridge" joining two otherwise independent parallel conducting paths (Fig. 6). It is not possible to determine whether the diode in the bridge is forward biased—i.e., active—without already knowing the voltages at both ends of the diode. This requires the solution to the entire network problem. It is possible that a substantial fraction of these bridge diodes may be back biased. Thus at a given total concentration of diodes, the concentration of active diodes will be somewhat less. This may lead to interesting behavior if the ratio of active to total diodes has anomalous behavior. However when the anisotropy axis and the field are parallel, there is considerably less ambiguity because almost every diode in the conducting backbone is active.[44] Work is presently underway in simulating the conductivity of this simpler system.[45] It is interesting that the calculational problems outlined here mirror the situation encountered in EMT.

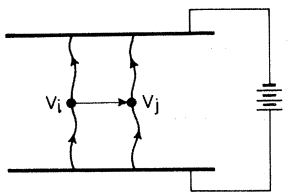

FIG. 6. A schematic picture of a conducting network in which one diode forms a bridge between two conducting paths. The difference between the voltages V_i and V_j determines whether the diode in question conducts.

IV. NODE PICTURE AND SCALING THEORY FOR DIRECTED CONDUCTIVITY

An appealing way to study the conductivity of a random network is based on a simple "node" geometrical picture.[37,38] Above the percolation threshold, it is often very convenient to schematically view the structure of the random system as a network of connected paths joining together an irregular array of nodes—points at which there are at least three independent paths leading to infinity. The average separation between neighboring nodes scales as the correlation length, ξ. In the de Gennes–Skal–Shklovskii scaling theory,[37,38] a basic hypothesis is to replace the irregular network of nodes by a *regular* lattice with a concentration-dependent lattice spacing also equal to ξ. Various transport properties of the random network above the percolation transition can then be described simply in terms of those in a spatially regular network. Based on this equivalence, scaling relations between conductivity and percolation exponents can be derived. Our goal here is to apply these ideas to directed percolation and obtain the appropriate anisotropic scaling relations between directed percolation and conductivity exponents.

Consider a system of linear dimension L which is bounded by two $(d-1)$-dimensional capacitor plates of area L^{d-1}, across which the potential is applied (Fig. 7). Near the directed percolation threshold, we

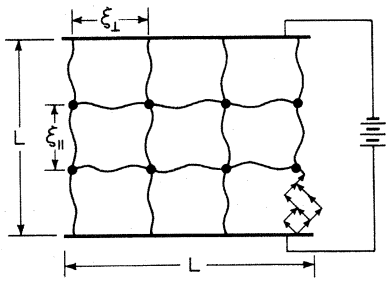

FIG. 7. Schematic picture of the node structure of a directed network above the percolation threshold. Two distinct lengths are required to characterize the structure of the infinite network. Notice the noncollinearity of the node structure with the microscopic lattice structure (lower right).

represent the conducting network as a regular array in which the lattice spacing parallel to the electric field is $\xi_{\parallel}$, while the lattice spacing perpendicular to the field is $\xi_{\perp}$.[46] [In terms of the original lattice structure, however, the field points along the diagonal $(1,1,\ldots,1)$.] Thus the effective one-dimensional conducting paths parallel to the field are separated by $\xi_{\perp}$, on average, leading to $(L/\xi_{\perp})^{d-1}$ such independent parallel paths. On the other hand, each conducting path consists of $(L/\xi_{\parallel})$ node-to-node segments. Furthermore, the conductivity of each node-to-node segment should vanish as the percolation threshold is approached because the node separation diverges. We define this segment conductivity to vanish as $(p_+-p_{+_c})^{\zeta_+}$. As a result, the conductivity of a single path joining the parallel plates varies as

$$(p_+-p_{+_c})^{\zeta_+}(\xi_{\parallel}/L)\sim(p_+-p_{+_c})^{\zeta_+-\nu_{\parallel}} .$$

Superposing the $(L/\xi_{\perp})^{d-1}$ independent paths leads to a lattice conductance that scales as $(p_+-p_{+_c})^{t_+}$ with

$$t_+=\zeta_++(d-1)\nu_{\perp}-\nu_{\parallel} . \tag{4.1}$$

For comparison, the same considerations for an isotropic network lead to $t=\zeta+(d-2)\nu$.

For dimensions greater than or equal to an upper critical dimension of $d_c=5$, mean-field theory predicts[20,32] $\nu_{\parallel}=1$, $\nu_{\perp}=\frac{1}{2}$, and from considering the Cayley tree, one can obtain $\zeta_+=1$.[47] From these exponents, we find the mean-field limit, $t_+=2$. This is in contrast to $t=3$ of the isotropic random resistor network[4-6] for $d\geq d_c=6$.[48] Below five dimensions, the general inequality $t_+<t$ appears to be valid. This is based on (4.1) and the relationship between $\nu_{\parallel}$ and $\nu_{\perp}$ with the isotropic correlation length exponent ν. For all d, numerical calculations indicate that $\nu<\nu_{\parallel}$.[19,23,32] In addition, for $d=2$, the numerical estimate for $\nu_{\perp}$ is less than ν, while in $d=5$, $\nu_{\perp}=\nu$. The result $\nu_{\perp}\leq\nu<\nu_{\parallel}$, in conjunction with (4.1), suggests that $t_+<t$. Consequently, at a fixed distance from the phase transition, the directed conductivity should have a sharper concentration dependence than the isotropic conductivity.

Physically, there are two effects which contribute to this sharper conductivity variation. The first is that long tortuous paths which lower the conductivity of an isotropic system near the threshold cannot occur here. Because of the overall orientation of a conducting path of diodes, its length must scale linearly with the straight-line distance joining the end points of the path. Secondly, due to the reduced transverse lattice spacing, $\xi_{\perp}$, of the node picture, a higher density of independent conducting paths can be packed into a system of fixed size compared to the isotropic case. These two effects lend support to the inequality $t_+<t$. This feature is corroborated to some extent by our renormalization-group approach.

V. POSITION-SPACE RENORMALIZATION GROUP

As we have seen in Sec. IV, we expect different critical behavior for the conductivity in directed random networks than for isotropic networks. To obtain a rough quantitative estimate of this difference, we calculate conductivity exponents using the position-space renormalization group (PSRG). Our method is based on the cell approach first introduced by Reynolds *et al.*[49-51] for isotropic percolation, in conjunction with a conductivity rescaling, following methods developed by many authors.[8-10,12,13] For directed conductivity, there are two new features which must be addressed: The first is that there is a direction associated with the microscopic bond conductivity. A second, more interesting point is the presence of two independent diverging correlation lengths. Because of this, care is required in making the connection between length rescaling and the critical exponents of bulk thermodynamic functions, such as the conductance.

To perform the rescaling, we first break up the lattice into $b\times b$ cells, and map them to elementary 1×1 cells (Fig. 8). To renormalize the occupation probability, we consider all configurations that "percolate" across a given cell. If the configuration percolates both ways across the cell, it maps to a resistor, while if the configuration percolates in only one direction, it maps to a diode. The total probabilities for each of these two cases give the recursion relations for p' and p'_+, respectively. We write these as

$$p'=R_p(p,p_+) , \tag{5.1a}$$

$$p'_+=R_{p_+}(p,p_+) . \tag{5.1b}$$

To renormalize the conductivity, consider the binary probability distribution for the conductivities

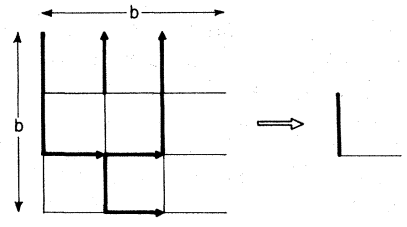

FIG. 8. Rescaling procedure in the cell PSRG approach. We show a typical configuration of diodes and resistors in the $b\times b$ cell on the left, and its rescaled counterpart in the 1×1 cell on the right.

of each bond element. For example, the forward conductivity for each diode may be written as, $P(\sigma_+) = (1-p_+)\delta(\sigma_+) + p_+\delta(\sigma_+ - \sigma_+^0)$. After rescaling, the conductivity distribution for the cell becomes more complicated, being a sum over many δ functions. To simplify the situation, we replace the sum of δ functions by a new binary distribution. For cell configurations which map to a diode, we write for the transformed binary distribution, $P(\sigma'_+) = (1-p'_+)\delta(\sigma'_+) + p'_+\delta(\sigma'_+ - \overline{\sigma}_+)$, with $\overline{\sigma}_+$ being the mean of the rescaled distribution.[8–10,12,13] As discussed by Bernasconi *et al.*,[52] the geometric mean of the conductivity is expected to give the best numerical results. Rescaling the geometric mean is equivalent to calculating the following recursion relation for the conductivity:

$$\mathcal{P}' \ln\sigma = \sum_i \mathcal{P}_i \ln\sigma_i \ , \tag{5.2}$$

where $\mathcal{P}_i$ is the probability of a configuration with conductivity σ_i, and $\mathcal{P}'$ is the rescaled probability which may be either p' or p'_+.

To relate the conductivity rescaling with the forward conductivity exponent, t_+, consider the macroscopic forward conductance in a system of linear dimensions $L_\parallel$ and $L_\perp$, parallel and perpendicular to the anisotropy axis, respectively. This quantity should vary as

$$G_+(p_+) \sim (L_\perp^{d-1}/L_\parallel)\sigma_+ \Delta p_+^{t_+} \ . \tag{5.3a}$$

Here σ_+ is the microscopic forward conductivity of each bond, and $\Delta p_+ = p_+ - p_{+c}$. Upon rescaling, the lengths in the system transform anisotropically.[22,23] For rescaling factor b, $L'_\parallel = L_\parallel/b$, but $L'_\perp = L_\perp/b^\theta$, with $\theta = \nu_\perp/\nu_\parallel < 1$ characterizing the anisotropy. From this, the conductance transforms as

$$G'_+(p'_+) \sim [(L_\perp/b^\theta)^{d-1}/(L_\parallel/b)]\sigma'_+ \Delta p_+^{\prime t_+} \ . \tag{5.3b}$$

Near the critical point, this conductance should remain invariant under rescaling. Thus equating (5.3a) and (5.3b) leads to

$$t_+ = \frac{\nu_\parallel \ln(\sigma_+/\sigma'_+)}{\ln b} + (d-1)\nu_\perp - \nu_\parallel \ . \tag{5.4}$$

This generalizes the scaling relation $t = [d-2 + \ln(\sigma/\sigma')/\ln b]\nu$ to an anisotropic system. Notice that (5.4) is essentially the same as (4.1) obtained from the node picture.

In principle, the conductivity rescaling together with the anisotropic scaling relation (5.4) forms the basis of a renormalization-group calculation. However, in implementing this procedure there is a problem in that the direction of the diodes is not collinear with the anisotropy axis. In renormalizing the bond elements, we look for connected paths which span a cell in the x or y directions. However near the percolation threshold, connected paths propogate primarily along the diagonal. Thus, the meaning of the parallel and perpendicular directions with respect to the anisotropy is somewhat ambiguous in our procedure. To understand this definitively, a method that clearly distinguishes $\nu_\parallel$ from $\nu_\perp$ is needed. Further work is required to resolve this interesting question.

Based on this renormalization approach, we now consider the oriented resistor-diode network. From a 2×2 cell renormalization, the phase diagram shown in Fig. 2 is obtained. The three phases are characterized by whether the forward or reverse conductances, G_+ and G_-, respectively, are zero or nonzero. In the diode phase, only $G_+ \neq 0$ while $G_- = 0$. In the resistor phase both $G_\pm \neq 0$, and in the vacancy phase both $G_\pm = 0$. Between the vacancy and diode phases there is a line of second-order phase transitions along which $G_+ \to 0$. On the other hand, between the diode and resistor phases, there exists a line of second-order transitions along which $G_- \to 0$. These transition lines are governed by the directed and reverse fixed points, respectively. Thus, we study conductivity rescaling near these two points in order to calculate the exponents t_+ and t_- which describe the vanishing of G_+ and G_-, respectively. Furthermore, the two critical lines in the system originate from the isotropic fixed point where both G_+ and G_- vanish simultaneously, leading to tricritical behavior.[40] If we approach the fixed point along the p axis (or from anywhere within the isotropic conductor phase), we expect that both G_+ and G_- will vanish in a way governed by the isotropic exponent t. On the other hand, there exists a second independent exponent, $\bar{t}$, which describes the vanishing of G_+ only, as the transition is approached from the positive diode phase.

To illustrate our calculation, first consider the behavior near the isotropic percolation threshold. At the fixed point, $p^* = \frac{1}{2}$, $p_+^* = 0$, the geometric mean of the isotropic conductivity rescales as $\sigma'/\sigma = 0.5425$. Employing the isotropic scaling relation for t, along with $\nu = \frac{4}{3}$, we find $t = 1.176$. If we use the value $\nu = 1.428$ from the 2×2 cell approximation instead, then we find $t = 1.260$. Our estimates for t should be compared with the value $t = 1.32 \pm 0.02$ obtained by a similar PSRG approach[12] and with values in the vicinity of 1.33 determined by numerical methods.[13,14] We see that the better estimate for t is found by using only results from the 2×2 approximation, although it is not clear that this will be optimal in general.

To obtain the exponent $\bar{t}$ describing the vanishing of G_+, we approach the fixed point along the second eigendirection, parallel to the p_+ axis (see Fig. 2). In this case, the eigenvalue is 1.5, leading to a correlation length exponent $\bar{\nu} = 1.710$. Since the system is isotropic at the fixed point, we use the isotropic scaling relation between conductivity and correlation

length exponents. These considerations lead to an exponent $\bar{t}=1.509$.

At the directed fixed point, we rescale the geometric mean of the forward conductivity. In this case, the anisotropic scaling relation evaluated at $p_{+_c}=0.6447$ in conjunction with the numerical estimates[22,23,31] $\nu_{\parallel}=1.74$ and $\nu_{\perp}=1.10$ gives $t_{+}=0.698$. This calculation indicates that $t_{+}<1$, or a forward conductance which vanishes with an infinite slope. Qualitatively, the lowering of t_{+} predicted here is in accord with the general ideas of the node picture given in Sec. IV.

At the reverse fixed point, a rescaling identical to the one given above can be formulated for the reverse conductivity exponent t_{-}. We assume that the reverse transition has the same anisotropic nature as in directed percolation, due to the self-dual nature of this model.[25,32,34] In this case, the anisotropic scaling relation (5.4) applies. Under this assumption, the rescaling of the reverse conductivity together with the numerical estimates for $\nu_{\parallel}$ and $\nu_{\perp}$ quoted previously, gives $t=0.891$. Thus the reverse conductance is also expected to vary relatively sharply near the transition.

VI. NETWORKS WITH SUPERCONDUCTING DIODES

The conductivity of a network containing superconducting diodes is considerably different than the Ohmic case. This stems from the fact that there is the possibility of several interesting combinations of conductivities in the forward and reverse directions. It thus becomes necessary to enlarge the parameter space in order to describe the system self-consistently by the renormalization group. To understand this, suppose that we initially consider a system containing only resistors, superconducting diodes, and vacancies. This is basically the oriented resistor-diode network defined in Sec. II, but with superconducting rather than Ohmic diodes. On a coarse-grained level, a finite cell may contain a percolating path of superconducting diodes and resistors in series. The electrical response of the cell is that of an Ohmic diode. Therefore under lattice rescaling, the cell should map to an Ohmic diode. A more interesting case is a cell containing two independent percolating paths, one consisting of only superconducting diodes and one consisting of only resistors. The electrical response of this cell is superconducting in one direction and Ohmic in the opposite direction—a "leaky" diode. Finally, because Ohmic diodes have been generated by rescaling, it is then possible to construct a cell configuration which is Ohmic in both directions, but with different conductivities—an "asymmetric" resistor.

Thus starting with only vacancies, resistors, and superconducting diodes, the renormalization group requires us to include leaky and Ohmic diodes, and asymmetric resistors as well. The system is described by five parameters, and in order to visualize the phase diagram, we consider only the special case of no vacancies. Furthermore, as a rough approximation, we replace an asymmetric resistor by a symmetric one by averaging over the conductivities in the two opposite directions. In this case, the phase diagram is defined by a three-dimensional composition tetrahedron within the space spanned by p, p_{+}, and s_{+}, the concentration of superconducting diodes, and 1_{+}, the concentration of leaky diodes (Fig. 9).

The phases of the network can now be determined very simply. The condition $1_{+}+s_{+}=0.645$ (corresponding to the threshold for directed percolation), determines a surface of second-order transitions where the forward conductivity diverges. It divides the tetrahedron into two regions, one where the forward conductivity is infinite for $1_{+}+s_{+}>0.645$, and one where it is nonzero but finite for $1_{+}+s_{+}<0.645$. On the other hand, the condition $p+1_{+}=0.355$ (corresponding to the threshold for reverse percolation), defines another second-order surface where the reverse conductivity goes to zero continuously. Thus for $p+1_{+}>0.355$, the reverse conductivity is finite, while in the opposite case the reverse conductivity is zero. These surfaces meet obliquely, thus dividing the figure into four phases characterized by the four combinations of electrical responses possible. In the forward direction the conductivity may be either finite or infinite, while in the reverse direction the conductivity may be either zero or nonzero but finite. If vacancies are introduced, then an isotropic nonconducting phase occurs as well, and the resulting four-dimensional phase diagram is divided into five distinct regions.

There is one subspace in the phase diagram that is particularly interesting and amenable to simple calcu-

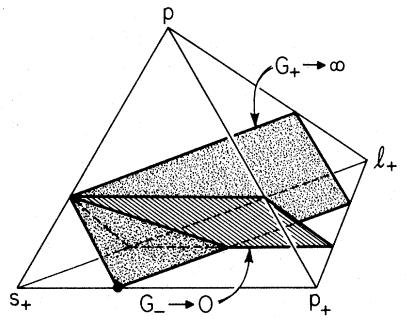

FIG. 9. Phase diagram of a network containing superconducting, Ohmic and leaky diodes of fixed orientation, and resistors. There are two second-order surfaces which divide the figure into four regimes. One surface (dotted) separates the region of infinite forward conductivity (behind the surface), from the region of finite forward conductivity. Similarly, the hatched surface separates the region of finite reverse conductivity (also behind the surface), from the region of zero reverse conductivity. On the $s_{+}-p_{+}$ axis, there is a fixed point (heavy dot) which describes the directed superconducting transition.

lation. This is a network containing only Ohmic and superconducting diodes. The system is closed in the renormalization-group sense as the reverse conductivity is always zero. Thus the network may be described by a single parameter, s_+. The divergence of the conductivity as s_+ approaches a threshold value from below represents the analog of the conductor-superconductor transition in an isotropic system. To calculate the nature of the divergence, we now rescale the bond resistivity ρ_+, using methods exactly analagous to those described in Sec. V. Furthermore, we must now write a scaling law relating the directed superconducting exponent with the correlation length exponents of directed percolation.[53] To accomplish this, we consider the macroscopic forward resistance, which near the transition should vary as

$$R_+(s_+) \sim (L_\parallel / L_\perp^{d-1}) \rho_+ \Delta s_+^{s_+} . \tag{6.1}$$

Following the same steps that led from (5.3) to (5.4) we obtain a scaling law for the superconducting exponent,

$$s_+ = \frac{\nu_\parallel \ln(\rho_+/\rho_+')}{\ln b} + \nu_\parallel - (d-1)\nu_\perp . \tag{6.2}$$

From rescaling the resistivity, and substituting $\nu_\parallel = 1.74$ and $\nu_\perp = 1.10$ into (6.2), gives $s_+ = 1.90$. This should be compared with the result $s = t \cong \frac{4}{3}$ in two dimensions for the isotropic exponents. There is a sharper divergence at the directed superconducting transition. This is consistent with the general picture already found for the directed conductor-insulator transition.

VII. DISCUSSION AND SUMMARY

We have investigated some of the simpler types of conduction processes that can occur in random networks containing resistors and diodes. Our primary interest has been to delineate a variety of interesting conductivity transitions, and study them qualitatively. The systems we have studied are constructed from diodes of two fundamentally different electrical responses. We have considered Ohmic diodes, which have a finite resistance in the forward direction and an infinite resistance in the reverse direction, and superconducting diodes, which have zero resistance in the forward direction. The former case is inherently simpler because the possible macroscopic electrical responses of the network are rather limited in number.

We have first focused on networks of Ohmic diodes, and we found that effective-medium theory yields good approximations for the locations of the conductivity transition, while the de Gennes–Skal–Shklovskii node picture gives anisotropic scaling relations and a simple way of obtaining the mean-field limit. In this limit, valid in dimensions $d \geq 5$, the directed conductivity exponent, $t_+ = 2$. This should be compared with a mean-field conductivity exponent of 3 in an isotropic random network, valid for $d \geq 6$. Thus we expect a steeper concentration dependence for the directed conductivity than for the isotropic conductivity, and this is consistent with the node picture for the structure of conducting paths.

The position space-renormalization group was applied to calculate the exponents associated with directed conductivity transitions in two dimensions. Qualitatively, our predictions are consistent with the behavior suggested by the mean-field limit. In particular, we predict a directed conductivity exponent in the vicinity of 0.7, in sharp contrast to an isotropic conductivity exponent of approximately $\frac{4}{3}$. However, the presence of two diverging length scales associated with the transition makes the interpretation of length rescaling in the renormalization group somewhat ambiguous, and our numbers may be suspect. Further work is required in order to resolve this interesting problem.

For a system of superconducting diodes, the conductivity of the network is considerably richer because there are more combinations of electrical responses possible in the forward and reverse directions. It is necessary to enlarge the parameter space and include both Ohmic and leaky diodes in order to describe the system self-consistently within the renormalization group. One special case is fairly simple, however. This is a network containing only oriented superconducting and Ohmic diodes, in which the forward conductivity diverges continuously at a critical concentration of superconductor. This represents the analog of the conductor-superconductor transition in an isotropic network. We found an exponent $s_+ \simeq 1.90$, which is considerably larger than the isotropic superconducting exponent $s \simeq \frac{4}{3}$. Thus again, there is a sharper conductivity variation of the directed network relative to the isotropic network.

Finally, there is a wealth of interesting phenomena that has not yet been addressed. For example, when the orientation of the diodes is allowed to be arbitrary, a transition from a diodelike state of a given orientation to one of the opposite orientation can occur. This leads to new types of threshold behavior. We hope to treat some of these problems, where orientational degrees of freedom play a central role in the conductivity transitions, in a future publication.

ACKNOWLEDGEMENTS

I am grateful to J. S. Brooks, A. Coniglio, D. J. Frank, W. Kinzel, C. J. Lobb, M. J. Stephen, J. P. Straley, and F. Y. Wu for very helpful discussions and correspondence during the course of this work. I also thank A. Coniglio, E. T. Gawlinski, and H. E. Stanley for a careful reading of the manuscript and constructive suggestions. This work was supported in part by grants from the ARO, ONR, and NSF.

[1]D. Stauffer, Phys. Rep. 54, 1 (1979).
[2]J. W. Essam, Rep. Prog. Phys. 43, 833, (1980).
[3]S. Kirkpatrick, in *Proceedings of the Les Houches Summer School on Ill-Condensed Matter* (North-Holland, Amsterdam, 1979), and extensive references contained therein.
[4]P. G. de Gennes, J. Phys. Lett. (Paris) 37, L1 (1976).
[5]J. P. Straley, J. Phys. C 10, 3009 (1977).
[6]M. J. Stephen, Phys. Rev. B 11, 4444 (1978).
[7]M. E. Levinstein, M. S. Shur, and A. L. Efros, Zh. Eksp. Teor. Fiz. 69, 2203 (1975) [Sov. Phys. JETP 42, 1120 (1975)].
[8]R. B. Stinchcombe and B. P. Watson, J. Phys. C 9, 3221 (1976).
[9]S. Kirkpatrick, Phys. Rev. B 15, 1533 (1977).
[10]R. Rosman and B. Shapiro, Phys. Rev. B 16, 5117 (1977).
[11]R. Fisch and A. B. Harris, Phys. Rev. B 18, 416 (1978).
[12]J. Bernasconi, Phys. Rev. B 18, 2185 (1978).
[13]C. J. Lobb and D. J. Frank, J. Phys. C 12, L827 (1979).
[14]R. Fogelholm, J. Phys. C 13, L571 (1980).
[15]D. Adler, L. P. Flora, and S. D. Senturia, Solid State Commun. 12, 9 (1973).
[16]B. P. Watson and P. Leath, Phys. Rev. B 9, 4893 (1974).
[17]L. N. Smith and C. J. Lobb, Phys. Rev. B 20, 3643 (1979).
[18]S. R. Broadbent and J. M. Hammersley, Proc. Cambridge Philos. Soc. 53, 629 (1957).
[19]J. Blease, J. Phys. C 10, 917, 925, 3461 (1977).
[20]S. P. Obukhov, Physica (Utrecht) 101A, 145 (1980).
[21]J. Kertesz and T. Vicsek, J. Phys. C 13, L343 (1980).
[22]D. Dhar and M. Barma, J. Phys. C 14, L1 (1981).
[23]W. Kinzel and J. M. Yeomans, J. Phys. A 14, L163 (1981).
[24]P. J. Reynolds (unpublished).
[25]S. Redner, J. Phys. A 14, L349 (1981).
[26]S. Redner and A. C. Brown, J. Phys. A 14, L285 (1981).
[27]J. Adler, M. Moshe, and V. Privman J. Phys. A 14, L363 (1981).
[28]E. Domany and W. Kinzel, Phys. Rev. Lett. 47, 5 (1981).
[29]W. Klein and W. Kinzel, J. Phys. A 14, L405 (1981).
[30]W. Klein, J. Phys. A (in press).
[31]J. W. Essam and K. De'Bell, J. Phys. A 14, L459 (1981).
[32]S. Redner, Phys. Rev. B 25, 3242 (1982). References contained therein discuss the relationship of directed percolation with a wide range of problems.
[33]B. C. Harms and J. P. Straley, J. Phys. C (in press).
[34]D. Dhar, M. Barma, and M. K. Phani, Phys. Rev. Lett. 47, 1238 (1981).
[35]H. Böttger and V. V. Bryksin, Philos. Mag. B 42, 297 (1980).
[36]N. Van Lien and B. I. Shklovskii, Solid State Commun. 38, 99 (1981).
[37]P. G. de Gennes, Recherches 7, 919 (1976).
[38]A. S. Skal and B. I. Shklovskii, Fiz Tekh. Poluprovdn. 8, 1586 (1974) [Sov. Phys. Semicond. 8, 1029 (1975)].
[39]This model was first considered in Ref. 6.
[40]P. M. Kogut and J. P. Straley, J. Phys. C 12, 2151 (1979).
[41]S. Kirkpatrick, Rev. Mod. Phys. 45, 574 (1973).
[42]L. Turban, J. Phys. C 11, 449 (1978).
[43]S. Redner (unpublished).
[44]Naively, one may expect that every diode in the network will be active. However, it is possible to construct configurations where diodes are back biased.
[45]P. R. Mueller and S. Redner (unpublished).
[46]Some of the following considerations were also found independently in Ref. 33.
[47]A. Coniglio (private communication).
[48]G. Toulouse, Nuovo Cimento 23, 234 (1974).
[49]P. J. Reynolds, W. Klein, and H. E. Stanley, J. Phys. C 10, L167 (1977).
[50]P. J. Reynolds, H. E. Stanley, and W. Klein, J. Phys. A 11, L199 (1978).
[51]P. J. Reynolds, H. E. Stanley, and W. Klein, Phys. Rev. B 1223 (1980).
[52]J. Bernasconi, W. R. Schneider, and H. J. Wiseman, Phys. Rev. B 16, 5250 (1977).
[53]I am grateful to M. J. Stephen for some clarifying discussions on this point.