Critical exponents of the Ashkin-Teller model

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We show that the time-continuous Hamiltonian of the two-dimensional Ashkin-Teller model is that of a massive Thirring model. This implies the existence of an infinite number of conservation laws in the critical region and enables us to determine the critical exponents of the energy, crossover, and polarization operators.

Up to now the exact solution of the two-dimensional Ashkin-Teller[1] (AT) model has not been found, although a duality transformation[2] exhibits it as a staggered version of the symmetric eight-vertex model, which has been solved by Baxter.[3] This similarity has generated a lot of results, exact and conjectured, regarding the properties of the phase diagram and critical exponents of the AT model.[4–7] In this Communication we determine the critical exponents for the crossover, energy, and polarization operators. The fundamental assumption we make is that the time-continuous Hamiltonian introduced by Fradkin and Susskind[8] lies in the same universality class, i.e., has the same long-distance physics in the critical region, as the original lattice model.

The AT model on a square lattice can be viewed as a system where two Ising spins $\sigma(\vec{r})$ and $\mu(\vec{r})$ are placed on each site interacting with the action

$$A=-\sum_{\vec{r}}\{[K_1^x\sigma(\vec{r})\sigma(\vec{r}+\hat{x})+K_2^x\mu(\vec{r})\mu(\vec{r}+\hat{x})+K_1^\tau\sigma(\vec{r})\sigma(\vec{r}+\hat{\tau})+K_2^\tau\mu(\vec{r})\mu(\vec{r}+\hat{\tau})]/2 +K_4^x\sigma(\vec{r})\sigma(\vec{r}+\hat{x})\mu(\vec{r})\mu(\vec{r}+\hat{x})+K_4^\tau\sigma(\vec{r})\sigma(\vec{r}+\hat{\tau})\mu(\vec{r})\mu(\vec{r}+\hat{\tau})\}\ , \qquad (1)$$

where $\vec{r}=(j,k)$ labels the lattice sites and $\hat{x}=(1,0)$ and $\hat{\tau}=(0,1)$ are unit vectors in the x and τ directions.

In the sequel we will only study the case $K_2=K_1=K$, since this is sufficient to obtain all the information about criticality of the isotropic model $K_a^x=K_a^\tau, a=1,2,4$.[9] Although we will be interested in the critical behavior of the isotropic model the τ-continuous limit forces us to consider different couplings in the x and τ directions. Since this limit has been discussed in Ref. 10, we only mention that in order to remain within a particular model, determined by certain values of the coupling constants, in the limit when the τ lattice spacing becomes continuous $(\tau\rightarrow 0)$ we have to take simultaneously

$$K^\tau\rightarrow\infty,\quad K^x\rightarrow 0\ , \qquad (2a)$$

with the ratios

$$\frac{K_4^\tau}{K^\tau},\quad \frac{K_4^x}{K^x} \qquad (2b)$$

fixed.[11] In this limit one loses the general self-duality of the AT model, since it mixes models with different K's. Only self-duality in the sense of Kramers-Wannier[12] (implementable by a change in temperature for the doubled Ising model $K_4=0$ and the Potts model[13] $K_1=2K_4$) survives the limit.

In this paper we only study the region of phase space where $x_1>x_2$, with $x_1=\exp(-K-2K_4)$, $x_2=\exp(-2K)$, since it contains the Baxter line of continuously varying critical exponents. The case $x_2\geqslant x_1$ will be dealt with in a forthcoming publication.[14]

In the region $x_1>x_2$ we obtain the following Hamiltonian describing the critical region of the symmetric AT model[10]

$$\begin{aligned}H=&-\frac{\lambda}{2}\sum_j[\sigma^z(j)\sigma^z(j+1)+\mu^z(j)\mu^z(j+1)]\\&-\frac{1}{2}\sum_j\sigma^x(j)-\frac{1}{2}\sum_j\mu^x(j)\\&-\lambda\Delta\sum_j\sigma^z(j)\sigma^z(j+1)\mu^z(j)\mu^z(j+1)\ ,\end{aligned} \qquad (3)$$

where λ and $\Delta=K_4^\tau/K^\tau=K_4^x/K^x$ are constants and $\sigma^x(j)$, $\sigma^z(j)$ and $\mu^x(j)$, $\mu^z(j)$ are two sets of Pauli matrices. For $K_4=0$ Eq. (3) represents two identical independent Ising systems. Notice that our Hamiltonian does not describe the Potts model[13] because of our restriction $x_1>x_2$, which excludes $K=2K_4$.

Going over to fermion variables we now transform the Hamiltonian to the one of a massive Thirring model[15] on a spatial lattice. To this end we first effect a Jordan-Wigner[16] transformation on the σ and μ variables independently:

$$\eta_1(j)=\prod_{k<j}\sigma^x(k)\sigma^z(j),\quad \tilde{\zeta}_1(j)=\prod_{k<j}\mu^x(k)\mu^z(j) \;,$$
$$\eta_2(j)=i\prod_{k\leq j}\sigma^x(k)\sigma^z(j),\quad \tilde{\zeta}_2(j)=i\prod_{k\leq j}\mu^x(k)\mu^z(j) \;. \tag{4}$$

These spinors satisfy the commutation relations

$$\{\eta_a(j),\eta_b(k)\}=2\delta_{ab}\delta_{jk} \;,$$
$$\{\tilde{\zeta}_a(j),\tilde{\zeta}_b(k)\}=2\delta_{ab}\delta_{jk} \;, \tag{5}$$
$$[\eta_a(j),\tilde{\zeta}_b(k)]=0 \;,$$

where $a,b=1,2$, $\{\ \}$ is the anticommutator and

$$\delta_{ij}=\begin{cases}1, & \text{if } i=j\\ 0, & \text{if } i\neq j\end{cases} \;.$$

In order to obtain a Dirac spinor we need two anticommuting Majorana spinors η_a and ζ_a. ζ_a can be obtained from $\tilde{\zeta}_a$ by a Klein transformation

$$\zeta_a(j)=\prod_{l=1}^{N}\sigma^x(l)\tilde{\zeta}_a(j) \tag{6}$$

for a lattice of N sites. The spinor

$$\psi(j)=\frac{1}{2}\begin{pmatrix}\eta_1(j+1)-i\zeta_1(j+1)\\ i\eta_2(j)+\zeta_2(j)\end{pmatrix} \tag{7}$$

satisfies now the desired commutation relations

$$\{\psi_a(j),\psi_b(k)\}=0,\quad \{\psi_a(j),\psi_b^*(k)\}=\delta_{ab}\delta_{jk} \;. \tag{8}$$

The Hamiltonian, whose expression in terms of η and ζ is

$$H=\sum_j\left(\frac{(\lambda-1)}{2}i[\eta_1(j+1)\eta_2(j)+\zeta_1(j+1)\zeta_2(j)]+\frac{i}{2}\{[\eta_1(j+1)-\eta_1(j)]\eta_2(j)+[\zeta_1(j+1)-\zeta_1(j)]\zeta_2(j)\}\right.$$
$$\left.+\lambda\Delta[\eta_1(j+1)\eta_2(j)\zeta_1(j+1)\zeta_2(j)]\right) \tag{9}$$

can now be expressed in terms of ψ as

$$H=\sum_j\{(\lambda-1)\bar{\psi}(j)\psi(j)-i\bar{\psi}(j)\gamma^1\partial_1\psi(j)$$
$$-g_0[\bar{\psi}(j)\psi(j)^2]\} \;, \tag{10}$$

where we introduced the 2×2 matrices

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix} \;,\quad \gamma^1=\begin{pmatrix}i&0\\0&-i\end{pmatrix} \tag{11}$$

satisfying

$$\{\gamma^\mu,\gamma^\nu]=2g^{\mu\nu},\quad g^{00}=-g^{11}=1,\quad g^{01}=g^{10}=0 \;, \tag{12}$$

and the coupling constants g_0 given by

$$g_0=2\lambda\Delta \;. \tag{13}$$

The spatial derivative is defined as

$$\partial_1\psi(j)=\begin{pmatrix}\psi_1(j)-\psi_1(j-1)\\ \psi_2(j+1)-\psi_2(j)\end{pmatrix} \;, \tag{14}$$

and we notice that $\partial_1\psi(j)$ cannot be written as a difference of ψ's at neighboring points.[17]

Equation (10) exhibits H as a lattice version of the massive Thirring model with mass $(\lambda-1)$ and coupling g_0. An immediate consequence of this representation is the existence of an infinite number of conserved charges of the AT model in the critical region $(x_1>x_2)$.

At this stage we may take the spatial lattice spacing to zero in order to obtain critical exponents from the continuum theory. Since these exponents are mass-independent for a suitable renormalization, we may compute them from the massless model.

Probably the fastest way to get the exponents is provided by bosonization,[18] which expresses the renormalized field $\psi(x)$ in terms of a free canonical massless fiels $\phi(x)$:

$$\psi_1(x)=\left(\frac{m}{2\pi}\right)^{1/2}:\exp\{i[\alpha\phi(u)+\delta\phi(v)]\}:$$
$$\psi_2(x)=\left(\frac{m}{2\pi}\right)^{1/2}:\exp\{-i[\delta\phi(u)+\alpha\phi(v)]\}: \;, \tag{15}$$

where m is an auxiliary mass; the double dots indicate the usual Wick ordering with respect to the field $\phi(x)$, which satisfies

$$[\phi(\tau,x),\dot{\phi}(\tau,x')]=i\delta(x-x') \;,$$
$$\partial^2\phi=0 \;, \tag{16}$$
$$\phi(\tau,x)=\phi(x+\tau)+\phi(x-\tau)=\phi(u)+\phi(v) \;,$$

and α and δ are constants depending on the renormalized coupling constant.

Let us now identify our composite operators. The energy density is given by

$$\epsilon^{AT}=\sigma^z(j)\sigma^z(j+1)+\mu^z(j)\mu^z(j+1)$$
$$=(\psi_1^*\psi_2+\psi_2^*\psi_1)(j)=\bar{\psi}\psi(j) \quad . \tag{17}$$

In the continuum limit this operator is given by $:\cos(\alpha+\delta)\phi:$, with dimension[19]

$$d_{\bar{\psi}\psi}=2\dim\psi+\frac{\alpha\delta}{2\pi} \quad . \tag{18}$$

Since ψ's dimension is

$$d_\psi=\dim\psi=\frac{\alpha^2+\delta^2}{8\pi} \tag{19}$$

we get

$$x_\epsilon^{AT}=d_{\bar{\psi}\psi}=\frac{(\alpha+\delta)^2}{4\pi} \quad . \tag{20}$$

The crossover operator[6] is given by

$$O_{CR}^{AT}=\sigma^z(j)\sigma^z(j+1)-\mu^z(j)\mu^z(j+1)$$
$$=\psi_1^*(j)\psi_2^*(j)-\psi_1(j)\psi_2(j) \tag{21}$$

with dimension

$$x_{CR}^{AT}=\frac{(\alpha-\delta)^2}{4\pi} \quad . \tag{22}$$

We notice that only for the symmetric AT model ($K_1=K_2=K$) has our Hamiltonian (10) a U(1) symmetry

$$\psi(j)\to e^{ia}\psi(j),\quad a=\text{const.} \tag{23}$$

The crossover operator drives the system away from this point and can therefore not be U(1) invariant.

The constants α and δ are not independent for a "spin"$-\frac{1}{2}$ field, but satisfy

$$S_\psi=\frac{\alpha^2-\delta^2}{8\pi}=\frac{1}{2} \quad . \tag{24}$$

This equation immediately yields

$$x_\epsilon^{AT}=\frac{1}{x_{CR}^{AT}} \tag{25}$$

a relation which has been conjectured to be true by several authors.[4,5]

The polarization operator $P=\sigma^z(j)\mu^z(j)$ can be expressed in terms of a lattice field $\phi(\tau,j)$ satisfying

$$[\phi(\tau,j),\dot{\phi}(\tau,k)]=i\delta_{jk} \quad . \tag{26}$$

We get

$$P=\sigma^z(j)\mu^z(j)=\tfrac{1}{2}(\exp[\{i[(\alpha+\delta)/2]\phi(\tau,j)\}+\text{H.c}) \tag{27}$$

which has the following continuum limit:

$$P(x)=:\cos\frac{(\alpha+\delta)}{2}\phi(x): \quad . \tag{28}$$

Whereas Eqs. (20) and (22) could have been obtained independently of bosonization, Eq. (28) obviously depends on this procedure and its limitations.[19] The anomalous dimension of P is

$$x_P^{AT}=\frac{(\alpha+\delta)^2}{16\pi}=\frac{x_\epsilon^{AT}}{4} \quad , \tag{29}$$

which is a second conjectured relation.[20]

The relations (24) and (29) are renormalization independent. To obtain their ($x_\epsilon^{AT}, x_{CR}^{AT}, x_P^{AT}$) dependence on K_4 we go back to our isotropic model on the square lattice. There we know how x_{CR} of the AT model is related to x_ϵ of the equivalent 8V model,[2,21] namely,

$$x_{CR}^{AT}(K_4)=x_\epsilon^{8V}(\bar{K}_r) \quad , \tag{30}$$

where $\bar{K}_4$ is the marginal coupling of the 8V model given by[5]

$$\tanh(2\bar{K}_4)=\frac{\tanh(2K_4)}{\tanh(2K_4)-1} \quad . \tag{31}$$

Then, we have

$$x_\epsilon^{AT}(K_4)=\frac{1}{x_\epsilon^{8V}(\bar{K}_4)}=\left(4x_P^{AT}(K_4)=\frac{1}{x_{CR}^{AT}(K_4)}\right) \tag{32}$$

or

$$x_\epsilon^{AT}(K_4)=\left[1-\frac{2}{\pi}\sin^{-1}\left(\frac{\tanh(2K_4)}{\tanh(2K_4)-1}\right)\right]^{-1} \quad , \tag{33}$$

which completes the determination of these three exponents.

A detailed discussion of our results and their extension to $x_2\geqslant x_1$ are planned to be presented in a forthcoming publication.[14]

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