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Behavior of the Borel resummations for the critical exponents of the *n*-vector model

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We study the behavior of the Borel-resummed renormalization-group functions for $g(\vec{\phi}^2)^2$ in three dimensions under transformations between different definitions of the renormalized coupling constant. By comparing this behavior with that of certain "explicit" functions whose exact values are known, we are able to improve the accuracy of present estimates of the critical exponents for the *n*-vector model. Our results are consistent with the assumption that the renormalization-group functions are analytic in a circle in the complex g plane with a cut from the origin along the negative real axis.

I. INTRODUCTION

The critical exponents for the *n*-vector model in three dimensions are given in $g(\vec{\phi}^2)^2$ field theory [in three dimensions, and with an $O(n)$ symmetry] by the values of the renormalization-group functions evaluated at the first nontrivial zero of the β function. Le Guillou and Zinn-Justin[1,2] have applied estimates of the large-order behavior of perturbation series in $g(\vec{\phi}^2)^2$ theory to the calculation of the critical exponents using Borel-resummation methods, with encouraging results. Their method is in principle more accurate than both the Padé-Borel technique and the high-temperature series estimates and furthermore the values obtained by them are compatible with those obtained by the order methods, and also with the experimental results.

We have been motivated by the promising success of this technique to attempt to extend it in two general directions; first, to more precisely estimate the accuracy of these calculations given some finite number of terms in the perturbation expansion, and to learn how to modify the technique in order to improve this accuracy; second, to study the sensitivity of these calculations to the analytic properties (in g) of the renormalization-group functions, and in particular to examine the possibility that the Borel-resummation technique might be used to directly check the consistency of certain assumptions about these properties.

For both cases our method is to exploit the freedom, which exists in any renormalizable or superrenormalizable field theory, to adopt any of a wide range of physically equivalent definitions of the renormalized coupling constant. We have studied the behavior of the Borel-resummed renormalization-group functions at finite orders under transformations between these various definitions. First we determine whether the sum will converge more quickly and give a more accurate result in some coupling constants than in others. Second, we show in certain simple cases that the sum is sensitive at finite orders to the analytic structure of these functions. Thus we may establish which of the coupling constant transformations lead to renormalization-group functions which fail to satisfy the analyticity requirements for Borel resummation in the new (transformed) coupling constant, whereby we can test the consistency of certain assumptions about analyticity in the original (untransformed) coupling.

Our paper is organized as follows: Section II is a brief review of the method of Le Guillou and Zinn-Justin. In Sec. III we define the transformations on the coupling constant, and derive certain constraints on these transformations which will arise from the requirement that the transformed renormalization-group functions be Borel resummable in the transformed coupling constant. In Sec. IV we will apply our procedure of coupling constant transformation followed by Borel resummation to a set of explicit functions whose exact values are obtainable by other means. Two significant results will emerge here: (1) that the effect of "blindly" transforming the coupling constant, i.e., of producing transformed functions which fail to satisfy the analyticity requirements for Borel resummability, is apparent even in the first six terms of the Borel sum; (2) that certain of the proper transformations improve the apparent convergence rate and the accuracy of the Borel sum at these finite orders (and more particularly that, at a given order, the most precise value will be given by the most rapidly converging Borel sum). Finally, in Sec. V we will apply our method to the renormalization-group functions. It turns out that these functions behave under the coupling-constant transformations much like the explicit functions of Sec. IV. In particular, the behavior of the renormalization-group Borel sums with respect to the improper "blind" transformations is consistent with the assumption that these functions are analytic in a circle in the g

plane with a cut along the negative real axis (where g is the original untransformed coupling constant); and furthermore, just as for the test functions, certain of the proper transformations improve the apparent convergence rates of the Borel sums, and there is good reason[3] to believe that, as in Sec. IV, these improved convergence rates are associated with more precise values for the critical exponents. We have tabulated our best estimates for the values of the critical exponents along with those of Le Guillou and Zinn-Justin, at the end of the paper.

II. METHOD OF Le GUILLOU AND ZINN-JUSTIN

In the $g(\vec{\phi}^2)^2$ field theory large orders of the perturbative expansion of any physical quantity $A(g)$, with

$$A(g)=\sum_n a_n g^n \quad , \tag{1}$$

behave for large n as

$$a_n \sim n!(-\alpha)^n n^\epsilon \quad , \tag{2}$$

where α and ϵ have been calculated in three dimensions for the various renormalization-group functions. We now define the Borel transform $B_{b'}$ of A by

$$B_{b'}(y)\equiv\sum_k \frac{a_k y^k}{\Gamma(k+b'+1)} \quad , \tag{3}$$

where b' may be varied freely in order to control the convergence properties of the Borel sum. It follows from (2) and (3) that the first singularity of $B_{b'}(y)$ occurs at $y=-1/\alpha$, where $B_{b'}(y)$ will behave as $(1+\alpha y)^{b'-\epsilon-1}$, or as $\ln(1+\alpha y)$ for $b'=\epsilon+1$.

The Borel sum for $A(g)$ will be given by the integral:

$$A(g)=\int_0^\infty dt e^{-t} t^{b'} B_{b'}(gt) \quad . \tag{4}$$

The range of integration in (4) extends outside of the circle of convergence of the sum in (3); but this difficulty may easily be dealt with as follows: assuming that $B_{b'}(y)$ is anaytic in the y plane cut from $-1/\alpha$ to $-\infty$ along the negative real axis, we can map this cut plane into a circle in the x plane with

$$y=gt=x(1-\tfrac{1}{4}\alpha x)^{-2} \quad . \tag{5}$$

The range of integration in (4) will now run from $x=0$ to $x=4/\alpha$, and $B_{b'}(gt)$ will be given by a convergent series in x throughout this range.

Finally, then, we will have

$$A(g)=\sum_n B_n\left[\int_0^\infty dt e^{-t} t^{b'}[x(t)]^n\right] \quad , \tag{6}$$

where B_j will be given in terms of $\{a_i\}$ with $i\leq j$. So if we are given only the first k of the perturbation coefficients a_i, we may define an approximation $A^{(k)}(g)$ to $A(g)$ by

$$A^{(k)}(g)\equiv\sum_{n=0}^{k} B_n\left[\int_0^\infty dt e^{-t} t^{b'}[x(t)]^n\right] \quad , \tag{7}$$

where

$$A^{(k)}(g) \underset{k\to\infty}{\longrightarrow} A(g) \quad .$$

This method is first applied to the calculation of the zero g^* of the Gell-Mann–Low function, and then to the calculation of the other renormalization-group functions evaluated at g^*. The results are consistent with one another and with results obtained by other methods and thus they are consistent also with the assumption mentioned above concerning the analyticity of $B_{b'}(y)$ in y.

III. COUPLING CONSTANT TRANSFORMATIONS

Subject to certain constraints, one is free in a renormalizable field theory to adopt any of a wide range of different but physically equivalent definitions of the renormalized coupling constant. All of these, however, must preserve the first two coefficients in the perturbative expansion of the Gell-Mann–Low function, which both have direct physical significance. The higher-order terms, on the other hand, may vary freely as we redefine the coupling constant.

We consider transformations of the general form,

$$G(g)=g+O(g^2) \quad , \tag{8}$$

from one coupling constant g to a new one G. $G(g)$ is required to be nonsingular and invertable on the interval $0\leq g<g^*$, where g^* is the first nontrivial zero of the Gell-Mann–Low function $\beta(g)$. The new β function in G is given by

$$\tilde{\beta}(G)\equiv\frac{\partial G}{\partial g}\beta(g) \tag{9}$$

and we write

$$\tilde{\gamma}(G)\equiv\gamma(g(G)) \quad , \tag{10}$$

where $\gamma(g)$ represents all renormalization-group functions other than $\beta(g)$, and $\tilde{\gamma}(G)$ are corresponding functions of the new variable

t'Hooft[4] has defined a transformation of the form (8), which has been studied in detail by Khuri and McBryan,[5] for which $\tilde{\beta}_1=\beta_1$, $\tilde{\beta}_2=\beta_2$, and $\tilde{\beta}_i=0$ for $i>2$ [where $\tilde{\beta}(G)\equiv\sum\tilde{\beta}_i G^i$]. This particular transformation turns out not to be well suited to our purposes here, however, since it tends to worsen the apparent convergence rate of the $\tilde{\gamma}(G)$. We have found it useful, rather, to consider transformations of the form:

$$g=\sum_{i=1}^{m}Q_iG^i \ , \tag{11}$$

where m is some finite number. The value of Q_2 is determined by the requirement that $\tilde{\beta}_1=\beta_1$ and $\tilde{\beta}_2=\beta_2$, and is given by $Q_2=0$.

It remains now to consider which of the transformations will preserve the Borel resummability of the renormalization-group functions. If a function $\tilde{\gamma}(G)$ is to be Borel resummable on an interval $0\leq G\leq G^*$ it is necessary that the large-order perturbation coefficients for $\tilde{\gamma}(G)$ in G behave as[6]

$$\tilde{\gamma}_n\sim n!(-c)^n n^d \ , \tag{12}$$

where c and d are constants, and that $\tilde{\gamma}(G)$ be analytic in the region

$$0\leq|G|\leq|G^*|;\ \left[\frac{\pi}{2}+\epsilon\right]\geq\phi\geq\left[-\frac{\pi}{2}-\epsilon\right] \ , \tag{13}$$

where $\epsilon>0$ and $G\equiv|G|e^{i\phi}$. These conditions are known to be satisfied by the renormalization-group functions[7] for $g(\vec{q}^{\,2})^2$ in the original variable g, with g^* the first nontrivial zero of $\beta(g)$.

If now we assume that all of the renormalization-group functions are analytic in the old coupling constant g throughout a circle in the g plane with a cut from 0 along the negative real axis, we may easily derive necessary conditions on the Q_i in (11) such that the new $\tilde{\gamma}(G)$ will satisfy the Borel analyticity conditions (13) in G. Consider for example the transformation

$$g=G+\delta G^3,\quad \delta>0 \ . \tag{14}$$

It turns out that (14) will map no part of the negative real g axis into the region (13) so long as

$$\delta<\frac{\sin\frac{\pi}{3}}{|g^*|^2} \tag{15}$$

and similar bounds may easily be found on the case of negative δ, and for more complicated transformations as well. In what follows we will test the consistency of the assumption above regarding the analyticity of the renormalization-group functions in g by comparing the results of transformations which respect the analyticity conditions in G [i.e., of, say, the form (14) and (15)], with the results of transformations carried out blindly; without respecting these conditions.

Furthermore, it is not difficult to show for any transformation of form (11) with $|Q_i|<1$, $\forall i>1$ that

$$\tilde{\gamma}_j=\gamma_j\left[1+O\left[\frac{1}{j}\right]\right] \quad \text{as } j\to\infty \ , \tag{16}$$

where $\tilde{\gamma}(G)$ is defined as in (10).

Finally, recall that in order to carry out the mapping (5) which yields a convergent expression for $B_{b'}(gt)$, Le Guillou and Zinn-Justin assume without proof that $B_{b'}(gt)$ is analytic in the gt plane cut from $-1/\alpha$ to $-\infty$ along the negative real axis. For our present purposes we shall require not only that the untransformed $B_{b'}(gt)$ have this property, but that the new, transformed $\tilde{B}_{b'}(GJ)$ [i.e., the inverse Laplace transforms of the $\tilde{\gamma}(G)$] have it as well. It can be shown for a wide variety of explicit functions that if $B_{b'}(t)$ has this analytic structure in gt then $\tilde{B}_{b'}(GJ)$ will have the same structure in GJ.[8] Furthermore, recent studies[9] of the effects of inserting complex singularities (which violate this assumption) into a variety of explicit functions suggest that, although such singularities will spoil the rigorous convergence of the full Borel sums, they will *not* introduce any substantial error into results obtained from low-order partial sums such as are calculated in the present paper.

IV. TRANSFORMATION PROPERTIES OF SOME EXPLICIT FUNCTIONS

We consider in this section a set of functions $P_Q(g)$ of the general form

$$P_Q(g)\equiv K\int_0^\infty e^{-t}\frac{1}{(\alpha+gt)^Q}dt \ , \tag{17}$$

where Q, α, and K are positive real numbers. We have in general adjusted α and K (for a given Q) so as to make $P_Q(g^*)\approx\gamma(g^*)$ where g^* is the zero of $\beta(g)$, and also to set the perturbation coefficients $P_{Q_n}\approx\gamma_n$. The functions $P_Q(g)$ are Borel resummable and analytic in the complex g plane with a cut along the negative real axis from 0 to $-\infty$.

We will study the behavior of these functions under transformations on the coupling constants, $G(g)$, which generate transformed functions $\tilde{P}_Q(G(g))=P_Q(g)$. We define the stability S of $\tilde{P}_Q^{(N)}$ [where $\tilde{P}_Q^{(N)}$ is defined as in (7)] with respect to a given transformation $G(g)$ by

$$S\equiv\frac{\Delta G}{\Delta\tilde{P}_Q^{(N)}(G)} \ , \tag{18}$$

where

$$\begin{aligned}\Delta G&\equiv|G(g)-g| \ ,\\ \Delta\tilde{P}_Q^{(N)}&\equiv|\tilde{P}_Q^{(N)}(G(g))-P_Q^{(N)}(g)| \ ,\end{aligned} \tag{19}$$

so that, for example, $S\to\infty$ as $N\to\infty$.

Now, if we carry out a set of transformations of the form (14), subject to the bound (15), for various values of δ, and if we also "blindly" carry out another set of transformations of the more general form

(11), not subject to the analyticity conditions, we find that for fixed values of g and G, $\tilde{P}_Q^6(G)$ is more stable by a factor of 10 with respect to transformations of the form (14) and (15) than with respect to those of the form (11) which do not preserve the analyticity conditions. The Borel sum is, then, already discernably sensitive at this order to the analytic structure of $\tilde{P}_Q(G)$, and this will prove a useful tool for studying the renormalization-group functions.

Furthermore, there is a connection between the apparent convergence rate C of $\tilde{P}_Q^{(6)}(G)$, defined by

$$C \equiv |\tilde{P}_Q^{(6)}(G) - \tilde{P}_Q^{(5)}(G)| + |\tilde{P}_Q^{(5)}(G) - \tilde{P}_Q^{(4)}(G)| \quad (20)$$

for the analytically proper transformations $G(g)$ of the form

$$g = G + \delta G^3, \ \delta \text{ real} \quad (21)$$

and the accuracy A of $\tilde{P}_Q^{(6)}(G)$:

$$A \equiv |\tilde{P}_Q^{(6)}(G(g)) - P_Q(g)| \ . \quad (22)$$

In particular, we find that

$$C \gtrsim A \ . \quad (23)$$

So that the best apparent convergence rate will indicate the most accurate value, and C itself may serve as an upper bound on the error associated with that value. This result is both what one would have expected and what one would have hoped for; it too will be helpful in the analysis of the renormalization-group functions.

We have typically carried out transformations $G(g)$ of magnitudes up to $\Delta G \sim O(g/2)$ [where $g \sim O(1.5)$], and the variation of the convergence rate C in this range is typically of the order of a factor of 10. Furthermore, the calculations have all been carried out for a range of values of Q, and also for various different choices of the convergence factor b' in (3), and all of these have attested to the general nature of the conclusions above.

We find, then: (1) that the analytic properties of the $\tilde{P}_Q(G)$ have discernable effects at finite orders in the Borel sum; (2) that at these same finite orders $C \gtrsim A$. These are certainly not in the nature of rigorous theorms; they are rather observations about a wide class of explicit functions which we have examined [both of the form $\tilde{P}_Q(G)$ and of other kinds], and within which we have not found notable exceptions. We shall assume that they give us a good guideline to improving the calculations of the renormalization-group functions.

V. TRANSFORMATION PROPERTIES OF THE RENORMALIZATION-GROUP FUNCTIONS

The main result of the present work is the observation that the properties of the renormalization-group functions with respect to the coupling-constant transformations are much like those of the $P_Q(g)$ of Sec. IV.

First of all, our results are consistent with the assumption that the renormalization-group functions are analytic within a cut circle around the origin in the g plane (with the cut running out from the origin along the negative real axis). In particular, we find, exactly as in the case of the $P_Q^{(6)}(G)$, that the various $\tilde{\gamma}^{(6)}(G)$ are typically more stable by a factor of 10 with respect to the analytically proper transformations than with respect to the improper ones.

Furthermore, certain of the proper transformations of the form (20) [where the original couplings g are those given by Le Guillou and Zinn-Justin as the zeros of β for the various N; $g \sim O(1.5)$] and of typical magnitudes $\Delta G \sim O(g/2)$ improve the apparent convergence rates of the $\tilde{\gamma}^{(6)}(G)$ by a factor of 10, just as for the $\tilde{P}_Q^{(6)}(G)$.

Indeed all of the essential features of the $\tilde{P}_Q^{(6)}(G)$ which are noted in Sec. IV are both qualitatively and quantitatively characteristic also of the renormalization-group functions. This persuasively suggests that (23) will be valid for the $\gamma(g)$ as well, and so provides us with a more accurate means of calculating the critical exponents and of estimating the error. We have carried out this program for the renormalization-group functions calculated by Le Guillou and Zinn-Justin.

We adopt our definitions directly from their paper, namely,

$$\beta(g) \equiv (d-4)\left[\frac{\partial \ln g_0}{\partial g}\right]^{-1} , \quad (24)$$

where

$$g_0 = g\frac{Z_{(4)}}{Z^2} , \quad (25)$$

$Z(g)$ being the renormalization constant for the field and $Z_{(4)}(g)$ that for the vertex, and where the renormalized coupling g is normalized such that

$$\beta(g) = -g + g^2 + O(g^3) \ . \quad (26)$$

Then we have

$$\zeta(g) \equiv \beta(g)\left[\frac{d \ln Z(g)}{dg}\right] ,$$

$$\nu(g) \equiv \left[2 + \beta(g)\frac{d \ln Z(2)}{dg} - \zeta(g)\right] ,$$

$$\gamma(g) \equiv \nu(g)[2-\zeta(g)] , \quad (27)$$

$$\delta(g) \equiv \tfrac{1}{2}(3\nu - \gamma) ,$$

$$\omega(g) = \beta'(g) ,$$

where $Z_{(2)}$ is the renormalization constant for the $\vec{\phi}_2$ insertion. The critical exponents and the leading or-

TABLE I. Comparison of our results for the critical exponents (left column) with those of Le Guillou and Zinn-Justin (right column; Ref. 2).

	$N=1$	
γ	1.2412 ±0.0011	1.241 ±0.002
η	0.040 ±0.001	0.031 ±0.004
ν	0.6302 ±0.0003	0.630 ±0.0015
δ	0.3265 ±0.0011	0.325 ±0.0015
ω	0.81 ±0.05	0.80 ±0.04
	$N=2$	
γ	1.3172 ±0.0007	1.316 ±0.0025
η	0.041 ±0.0025	0.033 ±0.004
ν	0.672 ±0.002	0.669 ±0.002
δ	0.3479 ±0.0013	0.3455 ±0.002
ω	0.80 ±0.020	0.78 ±0.025
	$N=3$	
γ	1.390 ±0.005	1.386 ±0.0040
η	0.042 ±0.0020	0.033 ±0.004
ν	0.708 ±0.003	0.705 ±0.0030
δ	0.369 ±0.002	0.3645 ±0.0025
ω	0.782 ±0.004	0.78 ±0.02
	$N=0$	
γ	1.1617 ±0.0012	1.1615 ±0.002
η	0.034 ±0.001	0.027 ±0.004
ν	0.5884 ±0.0004	0.588 ±0.0015
δ	0.3092 ±0.0008	0.302 ±0.0015
ω	0.78 ±0.055	0.80 ±0.04

der corrections to the scaling laws are given by

$$\zeta\equiv\zeta(g^*),\quad \nu\equiv\nu(g^*),\quad \gamma\equiv\gamma(g^*),$$
$$\delta\equiv\delta(g^*),\quad \omega\equiv\omega(g^*)\ , \tag{28}$$

where g^* is the first nontrivial zero of $\beta(g)$.

Finally we remark that our technique, and the Borel-resummation method in general, are less accurate in the case of η than in that of the other, larger, exponents. The apparent convergence rates for $\eta^{(6)}$ are rather slow and are difficult to improve upon even by the coupling-constant transformations, and $\eta^{(6)}$ varies under these transformations by as much as a factor of 2, in contrast to $\nu^{(6)}$, $\sigma^{(6)}$, $\delta^{(6)}$, and $\omega^{(6)}$ which vary at most by factors of 1 ± 0.05.

For the other critical exponents our results are always slightly larger than, but in general consistent with, those of Ref. 2. In this sense our results strengthen the reliability of the Borel techniques for the present case.

Our best estimates for the critical exponents, along with those of Le Guillou and Zinn-Justin (Ref. 2) are presented in Table I. The error is the sum of two components: (1) the error in $\tilde{\gamma}(G(g^*))$ for a given value of g^*, which we determine via (23) with the minimum value for C; (2) the variation of $\tilde{\gamma}(G(g^*))$ over the range $\Delta(g^*)$ of the error in g^* itself, where we have taken $\Delta(g^*)$ directly from Ref. 2.

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[1]J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. 39, 95 (1977).
[2]J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B 21, 3976 (1980).
[3]As shall become apparent in Sec. V.
[4]G. t'Hooft, Erice Lectures, 1977 (unpublished).
[5]N. N. Khuri and Oliver A. McBryan, Phys. Rev. D 20, 881 (1979).
[6]See G. H. Hardy, *Divergent Series* (Oxford, London, 1956), p. 192.
[7]In the vicinity of the origin.
[8]More recently, we have been able to show the same for a much wider class of functions with cuts along the negative real axis (see Ref. 9).
[9]David Z. Albert, Ph.D. thesis (The Rockefeller University, New York, 1981) (unpublished).