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Interface fluctuations and the Ising model in a random field

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With the inclusion of the effects of interface capillary waves, the lower critical dimension has recently been shown to be 3. This paper contains a detailed derivation of the interface model, with the use of the replica method. Based on this model we then present a detailed discussion of a renormalization-group calculation in $d=3+\epsilon$ dimensions.

I. INTRODUCTION

Critical behavior of systems with random impurities has been a subject of considerable interest in recent years. In certain cases the presence of impurities may result in a random field which is coupled linearly to the order parameter. Such a coupling is present in charge-density-wave systems, where impurities introduce a pinning potential. It has also been shown that by applying a magnetic field to antiferromagnetic systems with (magnetic as well as nonmagnetic) impurities, one induces a random field.[1,2] Such a field affects the critical behavior of the system rather drastically. It changes the upper critical dimension from $d_u=4$ for the pure system to $d_u=6$. Moreover, the critical behavior of the pure system in $d=4-\epsilon$ dimensions is expected to be the same as that of the random system in $d=6-\epsilon$ dimensions. This correspondence has been demonstrated to all orders in pertubation theory,[3] first by carrying out a diagramatic expansion and then by exploiting the supersymmetry of the model.[4] Using domain-wall energy arguments, it has been shown by Imry and Ma[2] that for isotropic systems with $m\geq 2$-component order parameter, the random field destroys long-range order for $d<4$, while for Ising-like systems, long-range order is destroyed for $d<2$. It is therefore clear from this argument that the lower critical dimension d_l satisfies d_l $(m\geq 2)$ ≥ 4, and d_l $(m=1)$ ≥ 2. Renormalization-group studies show that d_l $(m\geq 2)$ is, in fact, equal to 4. The critical behavior in $d=4+\epsilon$ dimensions was found to be the same as that of the pure system in $d=2+\epsilon$ dimensions. This result suggests that the dimensionality shift by 2, obtained by perturbation expansion near $d=6$, may hold down to the lower critical dimension. In this case the lower critical dimension of the Ising model should be d_l $(m=1)$ $=3$ and not 2, as suggested by the Imry-Ma argument.

Unfortunately, one cannot extend the renormalization-group (RG) studies near the lower critical dimension for $m\geq 2$ model to include the Ising case. These studies use the continuous symmetry of the $m\geq 2$-component models and the spin-wave excitations associated with it. Such excitations do not exist in the Ising model. In a recent letter Wallace and Zia[5] have argued that capillary waves which describe large-distance deviations from the planar of an essentially sharp interface can be interpreted as the Goldstone modes whose fluctuations lower the critical temperature to zero as $d\rightarrow 1+$. The capillary interface waves therefore play the same role for a system with discrete symmetry as spin waves for a system with continuous internal symmetry. A RG calculation was carried out in $d=1+\epsilon$ with $T_c\sim\epsilon$, showing that the lower critical dimension is 1 for the discrete Ising model.

Using these ideas, Pytte, Imry, and Mukamel[6] have constructed an interface model associated with the Ising model in a random field. RG studies of this model show that the lower critical dimension is indeed d_l $(m=1)=3$. It was also found that the critical exponent ν in $d=3+\epsilon$ dimensions is equal, to first order in ϵ, to that of the pure Ising model in $d=1+\epsilon$ dimensions, suggesting that the dimensionality shift by 2 holds even for the Ising model. An important assumption in this analysis is that the interface model is invariant under both translations and rotations. This study may therefore be applied for an Ising model on a discrete lattice only if the interface associated with this model is rough, at low temperatures. For the pure Ising model the roughening transition occurs[7] at $T_R=0$ for $d\leq 2$ dimensions. It has been shown[6] that a random field results in a dimensionality shift by 2 for the roughening transition, and hence one expects that $T_R=0$ for $d\leq 4$. This makes it possible to carry out the ϵ expansion calculations in $d=3+\epsilon$ dimensions.

In a recent paper,[8] Binder, Imry, and Pytte presented qualitative arguments which also support the idea that the lower critical dimension is 3. They expanded the domain argument of Imry and Ma by considering the domain-wall free energy and exploiting the fact that the domain wall is rough in three dimensions in the presence of the random field.

In this paper we present a detailed study of the interface model associated with the Ising model in a random field. We first use symmetry arguments to construct the model. The critical behavior of this model is then studied using an ϵ expansion calculation in $d=3+\epsilon$ dimensions.

II. THE INTERFACE MODEL

In this section we derive the interface Hamiltonian associated with the Ising model in a random field. For clarity and completeness we first consider the Hamiltonian corresponding to the pure Ising model.

Let $M_0(z)$ be a function which describes an interface perpendicular to the z axis. It satisfies

$$M_0(z=\pm\infty)=\pm 1 . \tag{1}$$

This function may, for example, be obtained by solving an $n=1$-component Landau-Ginzburg-Wilson (LGW) model with the appropriate boundary conditions. Consider now an interface, $M(\vec{r})$ defined by

$$M(\vec{r})=M_0(\hat{a}\cdot(\vec{r}-\vec{t})) , \tag{2}$$

where $\hat{a}$ and $\vec{t}$ are constant vectors, with $\hat{a}\cdot\hat{a}=1$. The interface $M(\vec{r})$ is obtained from $M_0(z)$ by applying a translation and a rotation, defined by the vectors $\vec{t}$ and $\hat{a}$, respectively. Since the Ising model under consideration is assumed to be invariant under both translations and rotations, the two interfaces $M(\vec{r})$ and $M_0(z)$ have the same energy per unit area. The low-lying excitations associated with $M_0(z)$ may therefore be obtained by considering interfaces $M(\vec{r})$ with $\hat{a}$ and $\vec{t}$ being slowly varying functions of $\vec{r}$.

Let $F\{M(\vec{r})\}$ be the energy density associated with the interface $M(\vec{r})$. In general, F may depend on $M(\vec{r})$ and its derivatives. Rotational and translational invariance ensure that F does not depend explicitly on $\hat{a}$ and $\vec{t}$. The interface energy is given by

$$E=\int d^d r\, F\{M(\vec{r})\} . \tag{3}$$

We now derive the interface Hamiltonian by rewriting Eq. (3) as

$$E=\int d^{d-1}x\, \mathscr{H}(\hat{a},\vec{t}) , \tag{4}$$

$$\mathscr{H}(\hat{a},\vec{t})=\int dz\, F\{M_0(\hat{a}\cdot(\vec{r}-\vec{t}))\} , \tag{5}$$

and evaluating the integral in Eq. (5). Here $\vec{r}=(x_1,\ldots,x_{d-1},z)$. In the following we assume that $\hat{a}$ and $\vec{t}$ are independent of $\vec{r}$. Only at the end of the calculation do we allow $\hat{a}$ and $\vec{t}$ to be slowly varying functions of $\vec{r}$. It is clear, since F is not an explicit function of $\vec{t}$, that the integral in Eq. (5) is independent of $\vec{t}$. We may therefore take $\vec{t}=0$ in Eq. (4), which is equivalent to shifting the origin of the coordinate system to $\vec{r}=\vec{t}$. Hence, we consider the energy

$$E=\int d^{d-1}x \int dz\, F\{M_0(\hat{a}\cdot\vec{r})\} . \tag{6}$$

Let

$$\hat{a}=\hat{z}\cos\theta-\hat{a}_1\sin\theta , \tag{7}$$

where $\hat{z}$ and $\hat{a}_1$ are two unit vectors parallel and perpendicular to the z axis, respectively. We now introduce a new coordinate system $\vec{r}_0$ (see Fig. 1) whose $\hat{z}_0$ axis is parallel to $\hat{a}$. Let R be a rotation which transforms $\vec{r}$ into $\vec{r}_0$, namely

$$\vec{r}_0=R\vec{r} . \tag{8}$$

This transformation satisfies

$$\vec{z}_0=R\vec{a} . \tag{9}$$

In this coordinate system the energy Eq. (6) takes the form

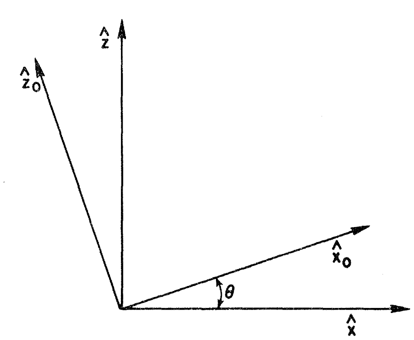

FIG. 1. Coordinate systems used to derive the interface model for the pure Ising model.

$$E=\int d^{d-1}x_0 \int dz_0 F\{M_0(z_0)\} . \quad (10)$$

The integral $\int dz_0 F\{M_0(z_0)\}$ is a constant $(=c)$, independent of $\hat{a}$ and $\vec{t}$. We therefore have

$$E=c\int d^{d-1}x_0 \quad (11)$$

or equivalently,

$$E=c\int d^{d-1}x\frac{dz}{dz_0} . \quad (12)$$

To calculate dz/dz_0 we note that

$$z_0=\hat{a}\cdot\vec{r}=z\cos\theta-(\hat{a}_1\cdot\vec{r})\sin\theta . \quad (13)$$

Hence,

$$\frac{dz}{dz_0}=\frac{1}{\cos\theta} \quad (14)$$

and therefore,

$$E=c\int d^{d-1}x\frac{1}{\cos\theta} . \quad (15)$$

To complete our derivation we introduce the interface variable $f(\vec{x})$, which gives the position of the center of the interface along the z axis. Here $\vec{x}$ is a $(d-1)$-dimensional vector, perpendicular to $\hat{z}$. Using Eq. (7) we find

$$M(\vec{r})=M_0(\hat{a}\cdot\vec{r})$$
$$=M_0(\cos\theta(z-\hat{a}_1\cdot\vec{r}\tan\theta)) . \quad (16)$$

This enables us to identify the interface variable $f(\vec{x})$ as

$$f(\vec{x})=(\hat{a}_1\cdot\vec{r})\tan\theta . \quad (17)$$

We therefore have

$$\nabla f(\vec{x})=\vec{a}_1\tan\theta , \quad (18)$$

or

$$\cos\theta=\frac{1}{[1+(\nabla f)^2]^{1/2}} . \quad (19)$$

Equation (19) together with (15) yields the following effective interface Hamiltonian:

$$\frac{H}{T}=\frac{c}{T}\int d^{d-1}x[1+(\nabla f)^2]^{1/2} , \quad (20)$$

which is the model studied by Wallace and Zia. Although this result can be obtained more simply, this derivation can be readily generalized to treat random systems.

We now consider the Ising model in a random field. By replicating the partition function and averaging over the random field one obtains the following effective Hamiltonian:

$$\frac{H}{T}=-\frac{J}{T}\sum_{\alpha=1}^{n}\sum_{\langle ij\rangle}\sigma_i^\alpha\sigma_j^\alpha - \frac{\Delta}{T^2}\sum_{\alpha,\beta=1}^{n}\sum_i\sigma_i^\alpha\sigma_i^\beta , \quad (21)$$

where as usual, α and β are replica indices and $\Delta\propto\langle h_i^2\rangle$. This Hamiltonian consists of a sum of n,d-dimensional Ising models coupled via the Δ term. The coupling between different replicas results from the randomness. The interface model associated with Eq. (21) may therefore be derived by considering the energy associated with n interfaces $M_\alpha(\vec{r})$, $\alpha=1,\ldots,n$ corresponding to the n replicas. We consider first the energy associated with two interacting interfaces $M_\alpha(\vec{r})$ and $M_\beta(\vec{r})$. By analogy with the previous derivation we take

$$M_\alpha(\vec{r})=M_0(\hat{a}\cdot(\vec{r}-\vec{t}_a)) , \quad (22)$$

$$M_\beta(\vec{r})=M_0(\hat{b}\cdot(\vec{r}-\vec{t}_b)) , \quad (23)$$

where $\hat{a}$, $\vec{t}_a$, $\hat{b}$, and $\vec{t}_b$ are vectors, and where $\hat{a}$ and $\hat{b}$ satisfy $\hat{a}\cdot\hat{a}=\hat{b}\cdot\hat{b}=1$. The energy associated with the two interfaces can be written in the form

$$E=\int d^{d-1}x\int dz\, F\{M_\alpha(\vec{r});M_\beta(\vec{r})\} , \quad (24)$$

where again, due to rotational and translational invariance the energy density F does not depend explicitly on $\hat{a}$, $\hat{b}$, $\vec{t}_a$, and $\vec{t}_b$. For simplicitly, we present the derivation of the interface model for $d=2$. The model may then be generalized to an arbitrary dimension. Let

$$\hat{a}=\hat{z}\cos\theta_\alpha-\hat{x}\sin\theta_\alpha , \quad (25)$$

$$\hat{b}=\hat{z}\cos\theta_\beta-\hat{x}\sin\theta_\beta . \quad (26)$$

Assuming that $\hat{a}$ and $\hat{b}$ are not parallel to each other, the two interfaces intersect at some point in the xz plane. Taking this point at the origin of the coordinate system, the vectors $\vec{t}_a$ and $\vec{t}_b$ are eliminated from Eq. (24) and the energy is given by

$$E=\int dx\int dz\, F\{M_0(\hat{a}\cdot\vec{r});M_0(\hat{b}\cdot\vec{r})\} . \quad (27)$$

Following the derivation for the pure Ising model we introduce a rotation R along an axis perpendicular to the xz plane, to a new coordinate system $\vec{r}_0$ whose $\vec{z}_0$ axis is parallel to $\hat{a}$ [see Eqs. (8) and (9) and Fig 2]. The transformation R satisfies

$$R\hat{b}=\hat{z}_0\cos(\theta_\beta-\theta_\alpha)-\hat{x}_0 sin(\theta_\beta-\theta_\alpha)\ . \tag{28}$$

In this coordinate system, the energy takes the form

$$E=\int dx_0\int dz_0 F\{M_0(z_0);M_0(cos(\theta_\beta-\theta_\alpha)[z_0-x_0\tan(\theta_\beta-\theta_\alpha)])\}\ . \tag{29}$$

The integral $\int dz_0 F$ is a function of the two arguments which appear in M, namely $\cos(\theta_\beta-\theta_\alpha)$ and $x_0\tan(\theta_\beta-\theta_\alpha)$. The energy E may therefore be written in the form

$$E=\int dx_0\bar{g}\{\cos(\theta_\beta-\theta_\alpha);x_0\tan(\theta_\beta-\theta_\alpha)\}\ , \tag{30}$$

where $\bar{g}$ is an arbitrary function. Let $f_\alpha^0(x_0)$ and $f_\beta^0(x_0)$ be the α- and β-interface variables, respectively, in the $\vec{R}_0$ coordinate system. They give the position of the center of the interface along the z_0 axis. It is clear (see Fig. 2) that

$$f_{\alpha\beta}^0(x_0)\equiv f_\alpha^0(x_0)-f_\beta^0(x_0)=x_0\tan(\theta_\beta-\theta_\alpha)\ . \tag{31}$$

Hence,

$$E=\int dx_0\bar{g}\{\cos(\theta_\beta-\theta_\alpha);f_{\alpha\beta}^0(x_0)\}\ . \tag{32}$$

To complete the derivation we rewrite Eq. (32) in terms of x and $f_{\alpha\beta}(x)$ rather than x_0 and $f_{\alpha\beta}^0$. It is easy to verify (see Fig. 2) that $f_{\alpha\beta}^0(x)$ is related to $f_{\alpha\beta}^0(x_0)$ via

$$f_{\alpha\beta}(x)\cos\theta_\beta=f_{\alpha\beta}^0(x_0)\cos(\theta_\beta-\theta_\alpha)\ . \tag{33}$$

Equation (33), together with the fact that $dx_0=dx/\cos\theta_\alpha$, finally yields

$$E=\int dx\frac{1}{\cos\theta_\alpha}\bar{g}\left\{\cos(\theta_\beta-\theta_\alpha);f_{\alpha\beta}(x)\frac{\cos\theta_\beta}{\cos(\theta_\beta-\theta_\alpha)}\right\}, \tag{34}$$

or

$$E=\int dx\frac{1}{\cos\theta_\alpha}\tilde{g}\{\cos(\theta_\beta-\theta_\alpha);f_{\alpha\beta}(x)\cos\theta_\beta\}\ , \tag{35}$$

where $\cos\theta_{\alpha,\beta}=1/[1+(\nabla f_{\alpha,\beta})^2]^{1/2}$. The function $\tilde{g}$ is defined by Eqs. (34) and (35). It is easy to generalize this derivation for arbitrary d. The result is

$$E=\int d^{d-1}x\frac{1}{(\hat{a}\cdot\hat{z})}\tilde{g}\{(\hat{a}\cdot\hat{b});f_{\alpha\beta}(\vec{x})(\hat{b}\cdot\hat{z})\}\ . \tag{36}$$

To complete our derivation we express $\hat{a}$ and $\hat{b}$ in terms of the interface variables $f_\alpha(\vec{x})$ and $f_\beta(\vec{x})$. We have

$$\hat{a}=(a_1,\ldots,a_{d-1},a_z)=(\nabla f_\alpha,1)\frac{1}{[1+(\nabla f_\alpha)^2]^{1/2}}\ , \tag{37}$$

and

$$\hat{b}=(b_1,\ldots,b_{d-1},b_z)=(\nabla f_\beta,1)\frac{1}{[1+(\nabla_\beta)^2]^{1/2}}\ . \tag{38}$$

Hence,

$$\hat{a}\cdot\hat{b}=\frac{1+\nabla f_\alpha\cdot\nabla f_\beta}{\{1+(\nabla f_\alpha)^2][1+(\nabla f_\beta)^2]\}^{1/2}}\ , \tag{39}$$

and

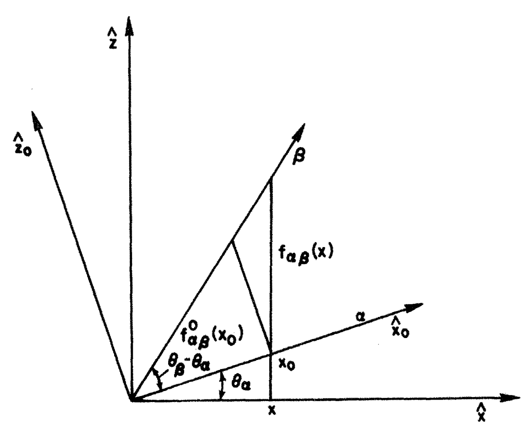

FIG. 2. Coordinate systems used to derive the interface energy for two domain walls α and β for the random-field Ising model.

$$\hat{b}\cdot\hat{z}=\frac{1}{[1+(\nabla f_\beta)^2]^{1/2}} . \tag{40}$$

The energy Eq. (36) may be simplified by noting that Eq. (33) which relates $f_{\alpha\beta}(x)$ to $f^0_{\alpha\beta}(x_0)$ is correct only for planar interfaces, namely, if $\hat{a}$ and $\hat{b}$ are $\vec{r}$ independent. In deriving this equation terms of the order of $[(d/dx)\tan\theta_\alpha]f_{\alpha\beta}$ or $[(d/dx)\tan\theta_\beta]f_{\alpha\beta}$ have been neglected. This is equivalent to neglecting all terms of the form

$$h(\nabla f_\alpha+\nabla f_\beta, f_{\alpha\beta})(\nabla f_\alpha-\nabla f_\beta) ,$$

where h is an arbitrary function of $(\nabla f_\alpha+\nabla f_\beta)$ and $f_{\alpha\beta}$. These terms (see Sec. III) turn out to be irrelevant under RG transformations.

Since [see Eqs. (39) and (40)]

$$\hat{a}\cdot\hat{b}=1+O(\nabla f_\alpha-\nabla f_\beta) \tag{41}$$

and

$$\hat{b}\cdot\hat{z}=\hat{a}\cdot\hat{z}+O(\nabla f_\alpha-\nabla f_\beta) , \tag{42}$$

we may replace $\hat{a}\cdot\hat{b}$ by 1 and $\hat{b}\cdot\hat{z}$ by $\hat{a}\cdot\hat{z}$ in the energy expression Eq. (36). Hence,

$$E=\int d^{d-1}x[1+(\nabla f_\alpha)^2]^{1/2} \times g\left\{\frac{f_{\alpha\beta}}{[1+(\nabla f_\alpha)^2]^{1/2}}\right\} , \tag{43}$$

where $g(u)$ is an arbitrary function of u. Expanding $g(u)$ in powers of u, we find that the interaction between the two interfaces α and β is governed by the Hamiltonian,

$$\frac{1}{T}H_{\text{int}}(f_\alpha,f_\beta)=\sum_{m=1}^{\infty}\int d^{d-1} \times\frac{1}{4}\frac{\Delta_m}{T^2}[1+(\nabla f_\alpha)^2]^{1/2-m} \times f^{2m}_{\alpha\beta} . \tag{44}$$

In deriving Eq. (44) we have used the fact that $g(u)$ is an even function u. Δ_m are arbitrary parameters satisfying $\Delta_m\sim\langle h_i^2\rangle$. The effective interface Hamiltonian associated with the replica model Eq. (21) is finally given by

$$\frac{1}{T}H=\frac{c}{T}\sum_{\alpha=1}^{n}\int d^{d-1}x[1+(\nabla f_\alpha)^2]^{1/2} +\sum_{\alpha,\beta=1}^{n}\frac{1}{T}H_{\text{int}}(f_\alpha,f_\beta) . \tag{45}$$

Equations (35) and (44) should, strictly speaking, have been written in a form symmetric in α and β. However, this is not necessary since in the end we sum over α and β.

III. RENORMALIZATION-GROUP CALCULATIONS

In this section we study the critical behavior of the model (45) with (44) using RG calculations in $d=3+\epsilon$ dimensions. In order to carry out this study, we expand the terms $[1+(\nabla f_\alpha)^2]^{1/2}$ and $[1+(\nabla f_\alpha)^2]^{(1/2)-m}$ in powers of $(\nabla f_\alpha)^2$. The quadratic term in Eq. (45),

$$\frac{1}{2T}\sum_\alpha(\nabla f_\alpha)^2+\frac{1}{4}\frac{\Delta_1}{T^2}\sum_{\alpha,\beta}f^2_{\alpha\beta} , \tag{46}$$

yields the following propagator[6] in the limit $n\to0$:

$$G_{\alpha\beta}(\vec{q})=\frac{T}{q^2}\delta_{\alpha\beta}+\frac{\Delta_1}{q^4} . \tag{47}$$

We may then use a diagramatic expansion in T and Δ_m to renormalize the Hamiltonian in $d=3+\epsilon$ dimensions. Since the model described has certain symmetry properties (namely, it is invariant under translations and rotations) one expects the renormalized Hamiltonian to be of the same form as the original one, but with renormalized T and Δ_m. We may thus study the recursion relations of T, Δ_1, Δ_2, etc., by renormalizing the terms $(\nabla f_\alpha)^2$, $f^2_{\alpha\beta}$, and $f^4_{\alpha\beta}$, respectively. In constructing the recursion relations we notice that the following two rules are satisfied: (a) Δ_m does not contribute to the recursion relations of $\Delta_{m'}$ with $m'>m$ to leading order in ϵ, and (b) Δ_m contributes to the recursion relation of Δ_{m-1} via the product $(\Delta_m T)$. The parameters which enter into the recursion relations are therefore $(\Delta_m T^{m-1})$ rather than Δ_m. The recursion relations take the form

$$\frac{dT}{dl}=-(d-1)T+\tfrac{1}{2}K_2\Delta_1 T , \tag{48}$$

$$\frac{d\Delta}{dl}=-\epsilon\Delta_1+\tfrac{1}{2}K_2\Delta_1^2+12K_2(\Delta_2 T) , \tag{49}$$

$$\frac{d(\Delta_m T^{m-1})}{dl}=-(\tfrac{3}{2}m-1)\epsilon(\Delta_m T^{m-1}) +O((\Delta_{m+1}T^m);(\Delta_m T^{m-1})^2) , \tag{50}$$

for $m\geq2$ where $K_2=1/2\pi$ is a phase-space integration constant. In order to verify that the renormalized Hamiltonian is of the same form as the original model we derived the recursion relations for T and Δ_1 also by renormalizing the terms $(\nabla f_\alpha)^4$ and $(\nabla f_\alpha)^2 f_{\alpha\beta}^2$, respectively. We find the same recursion relations (48) and (49) as obtained by renormalizing $(\nabla f_\alpha)^2$ and $f_{\alpha\beta}^2$. These recursion relations have two fixed points, (a),

$$T^*=0,\quad (\Delta_m T^{m-1})^*=0\ , \tag{51}$$

and (b),

$$T^*=0,\quad \Delta_1^*=\frac{2\epsilon}{K_2}\ ,$$

and

$$(\Delta_m T^{m-1})^*=0,\quad m\geq 2\ . \tag{52}$$

The fixed point (a) is a fully stable one, while the fixed point (b) has one relevant operator, Δ_1, whose critical exponent satisfies $\lambda=\epsilon$. Identifying $1/\lambda$ with the critical exponent ν for the correlation length we finally obtain

$$\nu=\frac{1}{\epsilon}+O(1)\ . \tag{53}$$

For $d<3$ the fixed point (a) has the relevant operator with a critical exponent $\nu=1/(3-d)$. Thus, the correlation length of the interface model behaves as $\xi=(1/\Delta)^{1/2}$, $\xi=1/\Delta$, and $\xi=e^{1/\Delta}$ in one, two, and three dimensions, respectively.

This calculation on the random-field interface model demonstrates that if it is assumed that the Ising model disorders by breaking up into domains, then the lower critical dimension for the Ising model is 3. In any case, it is a lower bound on the lower critical dimension, $d_l\geq 3$.

By renormalizing the Hamiltonian one generates terms which have been neglected in the derivation of the model Eq. (45). These terms have the form,

$$\frac{1}{T}\overline{H}=h(\nabla f_\alpha+\nabla f_\beta;f_{\alpha\beta})(\nabla f_\alpha-\nabla f_\beta)^l\ , \tag{54}$$

where h is a function of its two arguments and $l\geq 1$. In the following we show that these terms are strongly irrelevant and that they do not contribute to the recursion relations of T and Δ_m. Expanding h in powers of $(\nabla f_\alpha+\nabla f_\beta)$ and $f_{\alpha\beta}$, one has

$$\frac{1}{T}\overline{H}=\frac{1}{T^2}\Delta_{0,1}(\nabla f_\alpha-\nabla f_\beta)^2+\frac{1}{T^2}\Delta_{0,2}(\nabla f_\alpha-\nabla f_\beta)^4+\frac{1}{T^2}\Delta_{0,3}[(\nabla f_\alpha)^2-(\nabla f_\beta)^2]^2+\cdots$$
$$+\frac{1}{T^2}\Delta_{1,1}(\nabla f_\alpha-\nabla f_\beta)^2 f_{\alpha\beta}^2+\frac{1}{T^2}\Delta_{1,2}(\nabla f_\alpha-\nabla f_\beta)^4 f_{\alpha\beta}^2+\cdots\ , \tag{55}$$

where the parameter $\Delta_{k,p}$ is associated with a term which contains $f_{\alpha\beta}^{2k}$. The various $\Delta_{k,p}$ terms are generated by renormalizing the Hamiltonian (45). Consider now the recursion relations for $\Delta_{0,p}$. It is easy to verify that to leading order in ϵ,

$$\frac{d\Delta_{0,p}}{dl}=-2\Delta_{0,p}\ , \tag{56}$$

hence, these terms are strongly irrelevant. It is also easy to verify that the recursion relations for $\Delta_{k,p}$ satisfy similar rules to those obeyed by Δ_m, namely, (a) $\Delta_{k,p}$ does not contribute to $\Delta_{k',p'}$ with $k'>k$, and (b) $\Delta_{k,p}$ contributes to $\Delta_{k-1,p'}$ only via the product $(\Delta_{k,p}T)$. The parameters which appear in the recursion relations are therefore $(\Delta_{k,p}T^k)$ (rather than $\Delta_{k,p}$). We find that to leading order in ϵ,

$$\frac{d}{dl}(\Delta_{k,p}T^k)=-2(\Delta_{k,p}T^k),\quad k\geq 1 \tag{57}$$

demonstrating that all the parameters $\Delta_{k,p}$ are strongly irrelevant. We have also checked that the parameters $\Delta_{k,p}$ do not contribute to the recursion relations of Δ_m. These terms may therefore be neglected.

The results presented in Ref. 6 were based on a Hamiltonian of the form given by Eq. (45). However, for $m=1$ the higher-order terms in $(\nabla f_\alpha)^2$, obtained by expanding the square root $[1+(\nabla f_\alpha)^2]^{1/2}$ in Eq. (44) were, incorrectly, assumed to be irrelevant as noted in Ref. 9. As a consequence some of the coefficients of the recursion relations in Eqs. (48) and (49) for T and Δ_1 differed by a numerical factor from the recursion relations given above. The critical exponent ν obtained remains unchanged as does, of course, the conclusion that 3 is the lower critical dimension. The suggestions[9] that the replica method may not be reliable, and that our Hamiltonian is not consistently renormalizable are not correct.

Recently an effective interface energy for the Ising model in a random field has been obtained by Kogon and Wallace by exploiting the supersymmetry of the model to obtain a form of the interface energy.[9] This form leads immediately to the dimensional shift by 2, as compared to the pure Ising model, with Δ playing the role of T.

IV. CONCLUDING REMARKS

In Sec. III we calculated the critical exponent ν of the correlation length ξ of the interface model. To understand the meaning of this correlation length consider the susceptibility of the interface model defined by

$$k_B T\chi_f(q)=\mathscr{T}(\langle\langle f_i f_j\rangle-\langle f_i\rangle\langle f_j\rangle\rangle_h)\;, \tag{58}$$

or in terms of replicas

$$k_B T\chi_f(q)=\mathscr{T}(\langle f_i^\alpha f_j^\alpha\rangle-\langle f_i^\alpha f_j^\beta\rangle_{\alpha\neq\beta})\;. \tag{59}$$

In Eq. (58) the inner bracket denotes the thermal average and the outer heavy bracket $\langle\cdots\rangle_h$ denotes the average with respect to the random field and $\mathscr{T}$ represents the Fourier transform. Note that the second term in the propagator Eq. (47) does not contribute to $\chi_f(q)$. Thus, in the harmonic approximation,

$$\chi_f^{-1}=q^2\;. \tag{60}$$

Including fluctuations $\chi_f(q)$ can be written in the form

$$\chi_f^{-1}(q)=q^2\mathscr{F}(q\xi_f)\;. \tag{61}$$

That is, the interface-model correlation length determines the crossover from the hydrodynamic regime $q\xi_f<<1$ to the critical regime $q\xi_f>>1$. $\mathscr{F}(q\xi_f)$ is expected to approach a (different) constant both for $q\xi_f<<1$ and $q\xi_f>>1$ as $\eta\equiv 0$ for this model.

The spin susceptibility is similarly defined by

$$\begin{aligned}k_B T\chi_s(q)&=\mathscr{T}(\langle\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle\rangle_h)\\&=\mathscr{T}(\langle S_i^\alpha S_j^\alpha\rangle-\langle S_i^\alpha S_j^\beta\rangle_{\alpha\neq\beta})\;.\end{aligned} \tag{62}$$

For the spin model Eq. (21) the propagator by analogy with Eq. (47) takes the form

$$\langle S_\alpha S_\beta\rangle=\frac{1}{\kappa^2+q^2}\delta_{\alpha\beta}+\frac{\Delta}{(\kappa^2+q^2)^2}\;, \tag{63}$$

where $\kappa^2\propto(T-T_c)$. Again the Δ term in the propagator does not contribute to the susceptibility, and we obtain the usual result in the harmonic approximation,

$$k_B T\chi_s(q)=\frac{1}{\kappa^2+q^2}\;. \tag{64}$$

With fluctuations included, again as usual,

$$k_B T\chi(q)=\frac{1}{q^{2-\eta}}\mathscr{F}(q\xi_s)\;, \tag{65}$$

where for $d>3$ when a magnetic transition takes place,

$$\mathscr{F}(q\xi_s)=\begin{cases}1, & q\xi_s>>1\\(q\xi_s)^{2-\eta}, & q\xi_s<<1\end{cases} \tag{66}$$

where ξ_s is the spin-correlation length. At the lower critical dimension $d=3$, one expects by analogy to the pure Ising model,

$$\mathscr{F}(q\xi_s)=\frac{q\xi_s}{1+q^2\xi_s^2}\quad\text{and }\eta=1\;. \tag{67}$$

We note that the functional dependence of $\chi_f(q)$ on ξ_f is very different from that of $\chi_s(q)$ on ξ_s. The correlation length ξ_f can be identified with the surface-tension length or phase-coherence length discussed by Fisher, Barber, and Jasnow.[10] If this length is assumed to be proportional to ξ_s this leads immediately[10] to the Widom scaling relation (Ref. 11), $\mu=(d-1)\nu$, where μ is the exponent for the surface tension and ν is the critical exponent of ξ_s. This relation is known to be exact for the two-dimensional Ising model[12] and appears to be well satisfied also for three-dimensional systems.[11] However, for the pure Ising model in one dimension[5] $\xi_f=T^{1/2}e^{1/T}[1+O(T)]$, while exactly $\xi_s=e^{1/T}$. Thus, the interface-model correlation length has logarithmic corrections at the lower critical dimension which are not present in the Ising model.

Finally it should be noted that in random systems the structure factor is generally very different from the susceptibility even in the disordered phase.[13] The structure factor is defined by

$$S(q)=\mathscr{T}(\langle\langle S_iS_j\rangle\rangle_h)=\mathscr{T}(\langle S_i^\alpha S_j^\alpha\rangle)\;. \tag{68}$$

Thus, in the harmonic approximation the structure factor (for $T\geq T_c$) is just given by the diagonal (in replica space) part of the propogator Eq. (63),

$$S(q)=\frac{1}{\kappa^2+q^2}+\frac{\Delta}{(\kappa^2+q^2)^2}\;. \tag{69}$$

Including fluctuations the structure factor is expected to take the form

$$\begin{aligned}S(q)=&\frac{1}{q^{2-\eta}}\mathscr{F}(q\xi_s)\\&+\Delta\frac{1}{q^{4-\eta_\Delta}}\mathscr{F}_\Delta(q\xi_s)\;,\end{aligned} \tag{70}$$

where the exponent η_Δ has been introduced.[14] For $d>3$ the limiting values of $\mathscr{F}(q\xi_s)$ are given by Eq. (66), while for $\mathscr{F}_\Delta(q\xi_s)$, we expect

$$\mathscr{F}_\Delta(q\xi_s)=\begin{cases}1, & q\xi_s>>1\\ (q\xi_s)^{4-\eta_\Delta}, & q\xi_s<<1\ .\end{cases} \tag{71}$$

In an $\epsilon=6-d$ expansion, $\eta=n_\Lambda$ to all order in ϵ.[15] For $d=3$ one might expect

$$S(q)=\frac{\xi_s}{1+q^2\xi_s^2}+\frac{\Delta\xi_s^3}{(1+q^2\xi_s^2)^2}\ , \tag{72}$$

which corresponds in Eq. (70) to $\eta=\eta_\Delta=1$, and

$$\mathscr{F}_\Delta(q\xi_s)=\frac{(q\xi_s)^3}{(1+q^2\xi_s^2)^2}\ , \tag{73}$$

with $\mathscr{F}(q\xi_s)$ given by Eq. (67).

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