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Critical behavior of random systems

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The critical behavior of magnetic systems with nonmagnetic impurities is analyzed. It is argued that most magnetic systems should be described by the random-uniaxial-anisotropy model (RAM), rather than the random-exchange model. The crossover exponent ψ_m associated with random uniaxial anisotropy in m-component systems satisfies $\psi_m = 2\phi_m - 2 + \alpha_m \sim 0.3 > 0$ for $m \gtrsim 2$ where ϕ_m is the anisotropy crossover exponent and α_m is the specific-heat exponent of the pure system. The critical behavior of these systems is therefore expected to be *different* from that of the pure ones. The critical behavior of the RAM with cubic- and higher-order nonrandom anisotropy terms (which are always present in models appropriate for nonamorphous compounds) is studied using renormalization-group calculations in $d = 4 - \epsilon$ dimensions. It is also argued that the multicritical behavior of randomly mixed magnets with competing order parameters is not determined by the decoupled fixed point, as suggested in previous studies.

I. INTRODUCTION

Phase diagrams and critical behavior of systems with quenched random impurities have been a subject of extensive theoretical[1,2] and experimental[3] study in recent years. A model which has been used to analyze the nature of the phase transitions in disordered systems is the random-exchange model. The Hamiltonian takes the form[2]

$$\mathcal{H} = -\sum_{\langle ij\rangle}(J_0+\Delta J_{ij})\vec{S}_i\cdot\vec{S}_j - \sum_i\sum_l O_l(\vec{S}_i) \quad , \qquad (1.1)$$

where $\vec{S}_i = (S_{i1}, \ldots, S_{im})$ in an m-component vector spin, ΔJ_{ij} is a random variable whose configurational average satisfies $\overline{\Delta J_{ij}} = 0$, and the sum $\sum_{\langle ij\rangle}$ is over nearest-neighbor sites $\langle ij\rangle$. $O_l(\vec{S}_i)$ are single-ion anisotropy terms which are allowed by the symmetry of the crystal. It is assumed that the concentration of impurities is small so that the global crystalline symmetry of the system is not destroyed by the impurities. It has been shown first by heuristic arguments due to Harris[1] and then by renormalization-group calculations[2] that the critical behavior of this model depends on the specific-heat critical exponent α_m of the pure system (for which $\Delta J_{ij} \equiv 0$). For $\alpha_m < 0$, the random exchange is an *irrelevant*[4] operator and the critical behavior of the model is expected to be the same as that of the corresponding nonrandom system. For $\alpha_m > 0$, however, the random exchange is a relevant operator, and the critical behavior is determined by a new "random" fixed point. This dependence on α_m is known as the Harris criterion. For $m \geq 2$ one has[5] $\alpha_m < 0$ in $d = 3$ dimensions, and therefore it has been argued[1,2] that random impurities will not affect the critical behavior of $m \geq 2$-component physical systems.

In this paper we point out that the random-exchange model (1.1) does not properly describe most *magnetic systems* with random impurities (except systems with Ising-like, $m = 1$, order parameter). In particular, one may not use this model to describe magnetic systems for which dipole-dipole interactions or spin-lattice coupling are non-negligible. The Harris criterion is therefore *not applicable to most magnetic systems.* The point is that random impurities locally break the symmetry of the lattice. Due to dipole-dipole interactions[2(e)] or spin-lattice coupling, this introduces off-diagonal exchange terms into the Hamiltonian. A model appropriate for magnetic systems, therefore, takes the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \quad , \qquad (1.2a)$$

with

$$\mathcal{H}_0 = -\sum_{\langle ij\rangle}(J_0+\Delta J_{ij})\vec{S}_i\cdot\vec{S}_j - \sum_i\sum_{\mu,\nu=1}^{m} D_i^{\mu\nu}S_{i\mu}S_{i\nu} \qquad (1.2b)$$

and

$$\mathcal{H}_1 = -\sum_i\sum_l O_l(\vec{S}_i) \quad , \qquad (1.2c)$$

where $\sum_{\mu=1}^m D_i^{\mu\mu} = 0$ and $\overline{D_i^{\mu\nu}} = 0$. This is a generalization of the random-uniaxial-anisotropy model[6] (RAM) which has been studied quite extensively in recent years.[7] We note that the off-diagonal exchange $D_i^{\mu\nu}$ may well be comparable in magnitude to ΔJ_{ij} and therefore may not be neglected.[3(c),3(e),8] (In superconductors the order parameter ψ is not coupled to the lattice degrees of freedom, and therefore $D_i^{\mu\nu}\equiv 0$. The Harris criterion is therefore applicable to these systems.) The crossover exponent ψ_m associated with the random anisotropy term $D_i^{\mu\nu}$ in magnetic systems satisfies[7(a)] $\psi_m = 2\phi_m - 2 + \alpha_m$, where ϕ_m is the anisotropic spin crossover exponent. Since[9] $\psi_m \cong 0.3$ for $m \gtrsim 2$, the uniaxial anisotropy term is *relevant*[4] and the critical behavior is expected to be different from that of the pure system. Note that (unlike the crossover exponent for the random-exchange Ising model) ψ_m is rather large. Changes in critical behavior induced by disorder should therefore be observable. The critical behavior may in general depend on the higher-order single-ion anisotropic terms $O_l(\vec{S}_i)$.

The isotropic case $[O_l(\vec{S}_i)=0]$ has been analyzed by several authors.[7] Arguments similar to those applied by Imry and Ma[10] for the random-field case show[7(b),11] that the model does not exhibit long-range order in $d<4$ dimensions. It has also been found that the model does not exhibit an accessible fixed point[7(a)] in $d=4-\epsilon$ dimensions. Recent calculations by Pytte and Aharony[7(c)] suggest that for $2<d<4$ and for sufficiently low temperature the RAM undergoes a transition into a phase characterized by zero magnetization and an infinite susceptibility. (Presumably this phase, if it exists, does not persist down to the lowest temperatures. Very plausible domain arguments[10,11] show that the correlation length of the RAM ought to be finite at zero temperature. It seems most reasonable, therefore, that a phase of the RAM with infinite susceptibility ought to give way at still lower temperature to one with a finite susceptibility. This lowest-temperature phase might well be[7(a),12] a spin-glass phase of sorts.) Clearly these conclusions do not hold for the anisotropic case; the anisotropy will certainly stabilize long-range magnetic order at sufficiently low temperatures. Since (nonrandom) single-ion anisotropic terms do exist in all nonamorphous magnets, it would be of interest to study the model (1.2) with various anisotropic terms $O_l(\vec{S}_i)$. In the present paper we study the model (1.2) using renormalization-group calculations in $d=4-\epsilon$ dimensions. We first consider the case of cubic anisotropy. We then analyze the case where the single-ion anisotropic terms are of sixth (or higher) order in $\vec{S}_i$. In both cases we find no accessible stable fixed point, from which we conclude that the paramagnetic-ferromagnetic transition is either first order or proceeds in two stages, an infinite susceptibility phase without long-range order[7(c)] intervening between the paramagnet and ferromagnet.

It has been argued that the RAM (with no higher-order anisotropic terms) may describe certain systems exhibiting helical or spin-density-wave structure.[13] These systems are more likely to exhibit the infinite susceptibility phase predicted by Pytte and Aharony.[7(c)] We consider the Landau-Ginzburg-Wilson (LGW) Hamiltonian associated with randomly mixed helimagnets[14] such as Ho, Dy, and Tb. The LGW Hamiltonian is found to exhibit anisotropic fourth-order terms. However, these terms are such that the model is *isotropic* within certain subspaces of the m-dimensional order-parameter space $(S_1,\ldots,S_m)$. The possible phase diagrams are discussed.

Multicritical behavior in random systems with competing order parameters has been analyzed in terms of random-exchange models.[15] It has recently been pointed out[3(c),16] that this analysis is not applicable to real physical systems for two reasons: (a) no account is taken of off-diagonal terms $(D_i^{\mu\nu}S_{i\mu}S_{i\nu})$ which are always present in magnetic systems with $m>1$ and (b) in the random-exchange models the two competing order parameters $\vec{S}_1$ and $\vec{S}_2$ are assumed to be coupled via an energy-energylike term $|\vec{S}_1|^2|\vec{S}_2|^2$. It turns out that in certain cases the coupling is more complicated. We argue that the model (1.2) may in fact be used to analyze the multicritical behavior of certain disordered magnets.

The paper is organized as follows. In Sec. II we consider the model (1.2) with (A) cubic anisotropy and (B) sixth- and higher-order anisotropy. The LGW model associated with rare-earth helimagnets is discussed in Sec. II C. In Sec. III we consider the multicritical behavior of random magnetic systems with competing order parameters.

II. RANDOM MAGNETS WITH CUBIC- AND HIGHER-ORDER ANISOTROPY

A. Cubic anisotropy

A description of random magnets with cubic anisotropy requires that[17] $\mathcal{H}_1 = y\sum_{i=1}^m S_i^4$ in the Hamiltonian (1.2). Use of the replica method[2,18] in integrating out the random variables then gives rise to the following LGW Hamiltonian:

$$H=\tfrac{1}{2}(\nabla\phi_i^\alpha)^2+\tfrac{1}{2}r(\phi_i^\alpha)^2+v\phi_i^{\alpha^2}\phi_j^{\alpha^2}+y\phi_i^{\alpha^4}+u\phi_i^{\alpha^2}\phi_j^{\beta^2}+z\phi_i^\alpha\phi_i^\beta\phi_j^\alpha\phi_j^\beta+w\phi_i^{\alpha^2}\phi_i^{\beta^2}\ , \qquad (2.1)$$

where indices i and j are to be summed from 1 to m and superscripts α and β from 1 to n. Note that the w term is not generated directly by the random variables, but rather by the first RG iteration of the Hamiltonian [see Eq. (2.2)].

The recursion relations necessary to RG analysis of

TABLE I. Fixed points in $4-\epsilon$ dimensions for random magnetic systems with $m=2$ and 3 and cubic anisotropy, to $O(\epsilon)$. Only fixed points with $z^* \neq 0$ are listed. Fixed points for $m=2$ and 3 were, respectively, obtained by analytic and numerical calculation. Here $a \equiv 1/[11 \mp 3(19)^{1/2}]$, $b \equiv 1/[-115 \pm 3(2545)^{1/2}]$, and $c \equiv (-47 \pm 2545^{1/2})$. For $m=2$ (I,II,III) and (IV,V,VI) are two distinct triply-degenerate fixed points.

Fixed Point	$m=2$ $4K_4v^*/\epsilon$	$4K_4y^*/\epsilon$	$4K_4u^*/\epsilon$	$4K_4z^*/\epsilon$	$4K_4w^*/\epsilon$	$m=3$ $4K_4v^*/\epsilon$	$4K_4y^*/\epsilon$	$4K_4u^*/\epsilon$	$4K_4z^*/\epsilon$	$4K_4w^*/\epsilon$
I	0	0	$[4(3+3^{1/2})]^{-1}$	$[6(1+3^{1/2})]^{-1}$	0	0	0	$\frac{1}{19}$	$\frac{1}{19}$	0
II, III	0	0	$[4(3+3^{1/2})]^{-1}$	$[6(1+3^{1/2})]^{-1}$	0	0.026 $\mp 0.024i$	−0.039 $\pm 0.027i$	0.039 $\pm 0.005i$	0.022 $\pm 0.025i$	0.065 $\pm 0.037i$
IV	0	0	$[4(3-3^{1/2})]^{-1}$	$[6(1-3^{1/2})]^{-1}$	0	0	0	$\frac{3}{8}$	$-\frac{1}{2}$	0
V, VI	0	0	$[4(3-3^{1/2})]^{-1}$	$[6(1-3^{1/2})]^{-1}$	0	−0.245 $\pm 0.212i$	0.360 $\pm 0.239i$	0.480 $\mp 0.041i$	−0.198 $\mp 0.237i$	−0.599 $\pm 0.328i$
VII, VIII	$2a$	0	$(1 \mp 19^{1/2}/4)a$	$-\frac{3}{2}a$	0	$-18b$	0	$\frac{1}{4}cb$	$14b$	0
IX	0	0	$\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{8}$
X	0	0	$\frac{1}{16}$	$\frac{1}{8}$	$-\frac{1}{8}$	0	0	$\frac{3}{52}$	$\frac{1}{13}$	$-\frac{3}{52}$
XI	−1.651	1.356	0.537	0.575	−0.575	−559.	475.	138.	272.	−332.
XII	0.151	−0.245	0.087	−0.325	0.325	0.200	−0.445	0.027	−0.293	0.480
XIII	−0.040	0.066	0.082	0.088	−0.088	−0.040	0.090	0.094	0.059	−0.097
XIV	0.226	−0.185	0.021	−0.078	0.078	0.139	−0.118	0.048	−0.067	0.082
XV	$\frac{1}{6}$	$-\frac{1}{9}$	$\frac{1}{24}$	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{15}$	$-\frac{1}{9}$	$\frac{1}{20}$	$-\frac{1}{15}$	$\frac{1}{12}$

(2.1) in $4-\epsilon$ dimensions are, in the $n=0$ limit,

$$\frac{\partial v}{\partial l}=\epsilon v-4K_4[(m+8)v^2+12uv+4wv+6yv+6yz+(2m+10)vz] ,$$

$$\frac{\partial y}{\partial l}=\epsilon y-4K_4(9y^2+12yu+12yw+12yv+6yz+8vw) ,$$

$$\frac{\partial u}{\partial l}=\epsilon u-4K_4[8u^2+4uw+2vw+6uy+(2m+4)uv+(2m+2)uz+2vz+2wz+3z^2] , \tag{2.2}$$

$$\frac{\partial z}{\partial l}=\epsilon z-4K_4[(m+4)z^2+12zu+4zw+4zv] ,$$

$$\frac{\partial w}{\partial l}=\epsilon w-4K_4(8w^2+12wu+4wv+6wy+10wz+6yz) .$$

We study these recursion relations for the two physically relevant cases, $m=2$ and 3, for which in $O(\epsilon)$ there are, respectively, 25 and 29 fixed points.

For each value of m, 14 of these fixed points have $z^*=0$; these have been analyzed by Aharony[7(a)] who showed that none of them is both accessible to the physical system and stable to $O(\epsilon)$. The remaining fixed points have $z^*\neq 0$ and are listed in Table I. Though we have not listed their eigenvalues in the table, it is straightforward to show that they are all unstable. Note that the fixed points with $y^*=w^*=0$ correspond to the isotropic RAM, and have, therefore, been studied in Ref. 7(a).

The interpretation of the absence of a stable fixed point is not entirely clear. It is safe to assume that (at least for weak random uniaxial anisotropy: $z<<y$) at sufficiently low temperatures the ferromagnetic state will be stabilized by the presence of the cubic anisotropy, despite the random uniaxial term. The simplest interpretation of the lack of a stable fixed point is, therefore, the occurrence of a (fluctuation-induced) first-order transition[19] from the paramagentic to the ferromagnetic state.[20] We cannot however discount the possibility that a state without long-range order but with infinite susceptibility (as proposed[7(c)] for the isotropic RAM) separates the paramagnetic and ferromagnetic phases; in this case the system would undergo two successive phase transitions as the temperature is lowered. It is possible that a separatrix divides the space of coupling constants into two regions, one corresponding to each of the two scenarios. Starting from one of the regions one would flow under the RG across a stability boundary into a region where the Hamiltonian is unstable; a single first-order transition would result.[19] Starting from the second one would flow to the fixed point [which, if it exists, must be of $O(1)$ even when $d=4-\epsilon$] controlling the transition into the infinite-susceptibility phase; at lower temperature a second transition to the ferromagnetic state would occur. Unfortunately, the recursion relations are sufficiently tangled that we have been unable to determine if this hypothetical separatrix exist. All we can say with certainty is that there is some range of values of the initial parameters which gives rise to a single first-order transition. This is evident since a pure system [$u=w=z=0$ in (2.1)] with sufficiently strong cubic anisotropy ($v<<y$) is known[19] to undergo a fluctuation-induced first-order transition. Addition of weak randomness (u,w,z all $<<v$, for example) to such a system cannot alter this result.[21]

We conclude this subsection with a comment on another random model with cubic anisotropy: Aharony's RAM wherein the easy axis at each site points along one of the m-axis directions of the m-component spins.[7(a)] Aharony[7(a)] has shown that this model gives rise to a replica Hamiltonian identical to (2.1) with $z=0$. There is, therefore, no stable fixed point $O(\epsilon)$. To interpret this phenomenon consider the situation where the strength of the random cubic anisotropy is infinite, so the spins are constrained to lie along the easy axis at each site. Since the m easy axes are mutually orthogonal a spin oriented along a given axis feels only the presence of near-neighbor spins constrained to lie upon the same axis. The system therefore decomposes, in this limit, into m independent dilute Ising models. One suspects, therefore, that the $O(\epsilon^{1/2})$ fixed-point characteristic of the random Ising model[2(b)] might describe the critical behavior of this model even with finite anisotropy. It is easy to check that this is indeed the case: the fixed point[2(b)] $4K_4w^*=-(\frac{54}{53}\epsilon)^{1/2}+O(\epsilon)$, $4K_4y^*=4(\frac{6}{53}\epsilon)^{1/2}+O(\epsilon)$, $u^*=v^*=0$, is stable in the invariant subspace of (2.1) with $z=0$; Anharony's model ought to undergo a continuous transition with critical exponents characteristic of the random Ising model. The $O(\epsilon^{1/2})$ fixed point is, as is simply verifiable, *unstable* for finite z. It does not describe the critical behavior of the full Hamiltonian (2.1).

B. Higher-order anisotropy

Description of random magnets with hexagonal or higher-order anisotropy requires the addition to (1.2) to a term of order ϕ^6 or higher. Integration of the random variables then produces a Hamiltonian whose quartic terms are those of the isotropic RAM,

$$H=\tfrac{1}{2}(\nabla\phi_i^\alpha)^2+\tfrac{1}{2}r(\phi_i^\alpha)^2+v\phi_i^{\alpha^2}\phi_j^{\alpha^2}+u\phi_i^{\alpha^2}\phi_i^{\beta^2}+z\phi_i^\alpha\phi_i^\beta\phi_j^\alpha\phi_j^\beta ; \tag{2.3}$$

again greek and roman indices are to be summed from 1 to n and 1 to m, respectively. The anisotropy enters in $O(\phi^6)$ or higher and so is *irrelevant*[4] to the RG analysis of critical behavior in $4-\epsilon$ dimensions. Such analysis of (2.3) has been carried out by Aharony,[7(a)] who found no stable, accessible fixed point. Just as in the cubic case this result implies either a single first-order transition into a ferromagnetic state or two successive transitions, one into a state without long-range order but with an infinite susceptibility, and the second into the ferromagnetic phase which at low temperatures must be stabilized (for sufficiently small z) by the hexagonal or higher-order anisotropy.

The determination of the range of values of u, v, and z for which Hamiltonian (2.3) is stable is complicated by the $n \to 0$ limit. For example, putting $\phi_i^\alpha = M\delta_{\alpha 1}\delta_{i1}$ one can write the quartic terms as $(v+u+z)M^4$; this suggests that $v+u+z>0$ is a condition for stability. Different choices for ϕ_i^α give different conditions. If $\phi_i^\alpha = M$ for all i and α, for example, then the quartic terms become $nm^2(v+u+nz)$; this gives $v>0$ as a stability condition in the $n=0$ limit. It is not clear that all possible conditions obtained in this way are legitimate stability requirements of the system. To see this, note that Aharony's discussion[7(a)] of the recursion relations for (2.3) shows that there are physically allowed starting points (u, v, w) which flow under RG iteration across the prospective stability boundary $v+u+z=0$. If this is indeed a legitimate stability boundary, RG flow across it results in a fluctuation-induced first-order transition.[19] One finds, in the standard way,[19] that the order parameter $\langle \phi_i^\alpha \rangle = M\delta_{i1}\delta_{\alpha 1}$, for some M, just below the transition. But this form for $\langle \phi \rangle$ breaks the symmetry between the n replicas $\alpha = 1, \ldots, n$; it is not clear how to interpret the replica-symmetry breaking[22] in terms of the original random model without replicas. One should, therefore, treat with suspicion apparent stability conditions connected with replica-symmetry breaking. We now try to argue that such conditions for Hamiltonian (2.3) are all spurious: only the replica-symmetric condition $v>0$ is a legitimate stability requirement.

The argument can be given most cleanly for the simple random-T_c LGW model,[2]

$$H(\{r(\vec{x})\}) = \frac{1}{2}\sum_{i=1}^{m}(\nabla\phi_i)^2 + \frac{1}{2}r(\vec{x})\sum_{i=1}^{m}\phi_i^2 + v\left(\sum_{i=1}^{m}\phi_i^2\right)^2 , \tag{2.4}$$

where $r(\vec{x})$ is a random function of position whose distribution function $p(\{r(\vec{x})\})$ is Gaussian:

$$p(\{r(\vec{x})\}) \sim \exp\left(-\int d^dx\,[r(\vec{x})-r]^2/16|u|\right) , \tag{2.5}$$

for some negative number u. We consider for simplicity the case $m=2$. The impurity-averaged, reduced free-energy density

$$f \equiv -\int d(\{r(\vec{x})\})p(\{r(\vec{x})\})\ln Z(\{r(\vec{x})\})/\Omega , \tag{2.6}$$

where Z and Ω are, respectively, the partition function and volume of the system, is finite for all $v>0$. To see this note that

$$\begin{aligned}\ln Z(\{r(\vec{x})\}) &< \ln \mathrm{Tr}\exp\left\{-\int d^dx\left[\frac{1}{2}r(\vec{x})\sum_i\phi_i^2 + v\left(\sum_i\phi_i^2\right)^2\right]\right\} \\ &< \int d^dx \ln(\pi^{3/2}v^{-1/2}e^{r^2/16v}) \\ &< \int d^dx\left[\frac{r^2(\vec{x})}{16v} + \frac{1}{2}\ln\frac{\pi^3}{v}\right] .\end{aligned} \tag{2.7}$$

Since

$$\int_{-\infty}^{\infty} d(\{r(\vec{x})\})\,r^2(\vec{x})\exp\left(-\frac{[r(\vec{x})-r]^2}{16|u|}\right)$$

exists, f is clearly finite for all $v>0$; the model defined by (2.4) and (2.5) is thus well defined.

Now imagine integrating out the random variables $\{r(\vec{x})\}$ via the replica method. The Gaussian integrals are simple: they yield the effective Hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}(\nabla\phi_i^\alpha)^2 + \frac{1}{2}r(\phi_i^\alpha)^2 + u\phi_i^{\alpha^2}\phi_j^{\beta^2} + v\phi_i^{\alpha^2}\phi_j^{\alpha^2} , \tag{2.8}$$

(2.8) where greek and roman indicies are, respectively, summed from 1 to n and 1 to m. Recall $u<0$. Suppose we check the stability of H_{eff} by breaking the replica symmetry: $\phi_i^\alpha = M\delta_{\alpha 1}\delta_{i1}$. The quartic terms of the Hamiltonian become $(u+v)M^4$ and we conclude that H_{eff} is unstable for $(u+v)<0$. We have, however, just demonstrated that the random system described by (2.6) is *stable* for all $v>0$, regardless of u. In other words, stability criteria obtained by breaking the replica symmetry in this model are spurious; the only real stability boundary, $v=0$, is obtained by choosing $\phi_i^\alpha = \delta_{i1}M$ for all α. Though similar arguments for Hamiltonians (2.3) and (2.1) are technically more complicated we believe the same principle applies: the only legitimate stability boundaries are obtained by preserving the replica symmetry. In particular, Hamiltonian (2.3) is stable in the $n=0$ limit so long as $v>0$. Since the condition $v=0$ is preserved[7(a)] under RG iteration, the sign of v cannot change. Starting with $v>0$, therefore, Hamiltonian (2.3) will not flow across a stability boundary to an unstable region for any small values of u and z. In this respect the problem differs from the cubic case where we argued that flow across a stability boundary must occur for some range of values of the coupling constants. It seems then that systems with hexagonal or higher anisotropy are more likely to undergo two transitions in preference to a single first-order transition than are cubic systems; we can do no more than speculate at this stage.

C. Rare-earth systems with random impurities

1. *Ho, Tb, and Dy*

The rare-earth materials Ho, Tb, and Dy are hexagonal systems which exhibit spiral ordering associated with a reciprocal-lattice vector $\vec{k}$ in the z direction.[14] The magnetic moments lie in the basal plane, from which it follows that the system is described by an $m=4$ component order parameter[14]

$$\psi_{\pm k,p}=\phi_p \pm i\bar{\phi}_p,\quad p=x,y\ .$$

With definitions

$$\eta_1=(\phi_x+\bar{\phi}_y)/\sqrt{2},\quad \eta_2=(\phi_x-\bar{\phi}_y)/\sqrt{2}\ ,$$
$$\bar{\eta}_2=(\bar{\phi}_x+\phi_y)/\sqrt{2},\quad \bar{\eta}_1=(\bar{\phi}_x-\phi_y)/\sqrt{2}\ ,$$

one can write the GLW Hamiltonian describing the transition in these systems in the absence of randomness as[13]

$$H_4=\left[\frac{1}{2}\sum_{i=1}^{2}[(\nabla\eta_i)^2+(\nabla\bar{\eta}_i)^2+r(\eta_i^2+\bar{\eta}_i^2)]+v_1\left(\sum_{i=1}^{2}(\eta_i^2-\bar{\eta}_i^2)\right)^2+v_2\sum_{i=1}^{2}(\eta_i^2+\bar{\eta}_i^2)^2\right]\ . \tag{2.9}$$

As in Sec. I, disorder is treated by introducing random quadratic coupling terms (both diagonal and off-diagonal) among the four components $\eta_i,\bar{\eta}_i$ and performing the random averages with the replica procedure. This gives rise to the Hamiltonian

$$\mathcal{H}_4=\left[\frac{1}{2}\sum_{i,\alpha}[(\nabla\vec{\psi}_i^{\alpha})^2+r(\vec{\psi}_i^{\alpha})^2]+v_1\sum_{\alpha}\left[\sum_i(\vec{\psi}_i^{\alpha})^2\right]^2+v_2\sum_{\alpha}\sum_i[(\vec{\psi}_i^{\alpha})^2]^2+w_1\left[\sum_{i,\alpha}(\vec{\psi}_i^{\alpha})^2\right]^2\right.$$
$$\left.+w_2\sum_{\alpha,\beta}\left[\sum_i\vec{\psi}_i^{\alpha}\cdot\vec{\psi}_i^{\beta}\right]^2+w_3\sum_i\left[\sum_{\alpha}(\vec{\psi}_i^{\alpha})^2\right]^2+w_4\sum_i\left[\sum_{\alpha\beta}\vec{\psi}_i^{\alpha}\cdot\vec{\psi}_i^{\beta}\right]^2\right]\ , \tag{2.10}$$

where $\vec{\psi}_i\equiv(\eta_i,\bar{\eta}_i)$ for $i=1,2$, and α and β are to be summed from 1 to n.

We have not yet analyzed the fixed points of $\mathcal{H}_4$ in $4-\epsilon$ dimensions though it seems extremely unlikely that there is a stable, accessible fixed point since $\mathcal{H}_4$ is invariant under the rotation

$$\eta_i\to\eta_i'=\eta_i\cos\theta+\bar{\eta}_i\sin\theta\ ,$$
$$\bar{\eta}_i\to\bar{\eta}_i'=-\eta_i\sin\theta+\bar{\eta}_i\cos\theta\ , \tag{2.11}$$

for any angle θ and either value of i. One can therefore prove, just as in the ordinary RAM [Eq. (1.2)], that there can be no magnetic long-range order at any finite temperature: $\langle\eta_i\rangle=\langle\bar{\eta}_i\rangle=0$ for $i=1$ and 2. Let us, therefore, in analogy with the RAM, *assume* the absence of an accessible stable fixed point of H_4 and speculate briefly on possible scenarios for its critical behavior consistent with this assumption. An important element of this speculation is that, while nonzero values of $\langle\vec{\psi}_i\rangle$ are forbidden, other order parameters (which are quadratic in $\vec{\psi}_i^{\alpha}$) may become nonzero at low temperatures. A simple Landau-theory treatment of H_4 shows that, roughly speaking, the nature of the ordering is determined by the parameter v_2. For $v_2<0$ it is energetically favorable to break the symmetry between the 1 and 2 subspaces, and therefore one expects $(\langle\vec{\psi}_1^{\alpha^2}\rangle-\langle\vec{\psi}_2^{\alpha^2}\rangle)\neq0$. Since $\vec{\psi}_1^{\alpha}$ and $\vec{\psi}_2^{\alpha}$ describe right- and left-handed magnetic spirals, respectively, the order parameter $(\langle\vec{\psi}_1^{\alpha^2}\rangle-\langle\vec{\psi}_2^{\alpha^2}\rangle)$ breaks invariance under space inversion. On the other hand for $v_2>0$ one expects $\langle\vec{\psi}_1^{\alpha}\cdot\vec{\psi}_2^{\alpha}\rangle=\langle(\phi_x^{\alpha^2}+\bar{\phi}_x^{\alpha^2})-(\phi_y^{\alpha^2}+\bar{\phi}_y^{\alpha^2})\rangle\neq0$. This order parameter is associated with a transversely polarized spin-density wave. Note that the Hamiltonian (2.11) does not single out a preferred axis for the polarization vector. This axis is determined by the sixth-order terms in the LGW model.

We conclude that (if indeed $\mathcal{H}_4$ has no accessible, stable fixed point) two basic scenarios for the critical behavior of H_4 are possible: (1) a single, first-order transition into a phase with either $\langle\vec{\psi}_1^{\alpha^2}\rangle\neq\langle\vec{\psi}_2^{\alpha^2}\rangle$ or $\langle\vec{\psi}_1^{\alpha}\cdot\vec{\psi}_2^{\alpha}\rangle\neq0$; (2) two successive transitions, one into a symmetric infinite-susceptibility phase and the second into the broken-symmetry phase.

2. Er and Tm

Er and Tm are described[14] by the $(m=2)$-component order parameter $\psi_{\pm k,z}=\phi_z \pm i\bar{\phi}_z$. In terms of

$$\eta_3 \equiv (\phi_z+\bar{\phi}_z)/\sqrt{2}\ ,\quad \bar{\eta}_3 \equiv (\phi_z-\bar{\phi}_z)/\sqrt{2}\ ,$$

the GLW Hamiltonian for these systems is[14]

$$H_2=\int d^dx\,(\tfrac{1}{2}\{[(\nabla\eta_3)^2+(\nabla\bar{\eta}_3)^2]+r(\eta_3^2+\bar{\eta}_3^2)\} + u_2(\eta_3^2+\bar{\eta}_3^2)^2)\ . \qquad (2.12)$$

H_2 is an isotropic XY-model Hamiltonian; randomness therefore produces from it the RAM [Eq. (1.2)].

III. MULTICRITICAL BEHAVIOR

Consider a disordered magnetic system associated with an m-component order parameter. The Hamiltonian which corresponds to this system is given by (1.2) with the appropriate higher-order anisotropic terms. We now apply a symmetry breaking field g (e.g., uniaxial stress) which introduces the following coupling term into the Hamiltonian[9]:

$$-g\left[\frac{1}{m_1}\sum_{\mu=1}^{m_1} S_\mu^2-\frac{1}{m-m_1}\sum_{\mu=m_1+1}^{m} S_\mu^2\right]\ . \qquad (3.1)$$

The point $(T=T_c,\ g=0)$ where T_c is the transition temperature of the system at $g=0$, is a multicritical point in the (g,T) plane. The critical behavior of this symmetric multicritical point is thus determined by the model (1.2) and not by the random-exchange model.[15] The model (1.2) with cubic anisotropy or with sixth- or higher-order anisotropies does not have an accessible fixed point and the previous discussion of the phase transition (see Secs. II A and II B) should apply to the multicritical point.

Consider now a nonsymmetric multicritical point, which is obtained by mixing two systems associated with order parameters with m_1 and $(m-m_1)$ components, respectively.[15] Here, the analysis is more complicated. As an example we examine the multicritical behavior of an Ising-Ising-like system. The Hamiltonian takes the form

$$\mathcal{H}=-\sum_{l=1}^{2}\sum_{\langle ij\rangle}(J_l+\Delta J_{ij}^l)S_{li}S_{lj}-\sum_i D_i S_{1i}S_{2i}\ , \qquad (3.2)$$

where S_1 and S_2 are single-component order parameters. By considering n replicas of this Hamiltonian and averaging over the random couplings ΔJ_{ij}^l and D_i, we obtain the following LGW Hamiltonian:

$$\mathcal{H}_{\rm LGW}=\tfrac{1}{2}\sum_{\mu=1}^{2} r_\mu\sum_{\alpha=1}^{n}S_{\mu\alpha}^2+\tfrac{1}{2}\sum_{\mu=1}^{2}\sum_{\alpha=1}^{n}(\nabla S_{\mu\alpha})^2+\sum_{\mu,\nu=1}^{2}\sum_{\alpha,\beta=1}^{n}(u_{\mu\nu}+v_{\mu\nu}\delta_{\alpha\beta})S_{\mu\alpha}^2S_{\nu\beta}^2+w\sum_{\alpha,\beta=1}^{n}S_{1\alpha}S_{2\alpha}S_{1\beta}S_{2\beta}\ , \qquad (3.3)$$

where r_μ, $u_{\mu\nu}$, $v_{\mu\nu}$, and w are coupling constants. For $r_1=r_2$, $u_{11}=u_{22}$, and $v_{11}=v_{22}$ the model coincides with the cubic model (2.1) which is the appropriate model for symmetric Ising-Ising-like multicritical points. It would be interesting to study the critical behavior of the model (3.3) and compare it with that of the symmetric Ising-Ising multicritical point.

This approach may also be used to study nonsymmetric multicritical points which are not Ising-Ising-like. However, in these cases, one should be careful to examine *all* possible terms which couple the two order parameters, allowed by the symmetry of the problem. In certain cases the symmetry allows coupling terms which are not energy-energylike, $|\vec{S}_1|^2|\vec{S}_2|^2$. This may affect the phase diagram quite drastically, as was demonstrated for the case of[16] $Fe_{1-x}Co_xCl_2$. We would like to point out that coupling terms which are not energy-energylike are not uncommon in non-Ising-Ising systems. As an example, we consider the LGW model associated with binary rare-earth alloys (such as Tb-Er, Tb-Tm, Ho-Er, etc.). The $(m_1=4)$-component helimagnets (Ho, Dy, and Tb) are described by the Hamiltonian H_4 [see Eq. (2.9)] while the $(m_2=2)$-component spin-density-wave systems (Er and Tm) are described by the Hamiltonian H_2 [see Eq. (2.12)].

We now consider the coupling terms between the two order parameters. They take the form

$$\mathcal{H}_{\rm int}=-w_1\psi_{kz}\psi_{-kz}(\psi_{kx}\psi_{-kx}+\psi_{ky}\psi_{-ky}) - w_2[\psi_{kz}^2(\psi_{-kx}^2+\psi_{-ky}^2)+\psi_{-kz}^2(\psi_{kx}^2+\psi_{ky}^2)]\ , \qquad (3.4)$$

where the first term is energy-energylike, while the second term, also allowed by the symmetry of the problem, is *not* energy-energylike. In terms of η_i and $\bar{\eta}_i$, $\mathcal{H}_{\rm int}$ takes the form

$$\mathcal{H}_{\rm int}=-w_1(\eta_3^2+\bar{\eta}_3^2)\sum_{i=1}^{2}(\eta_i^2+\bar{\eta}_i^2) - w_2[2\eta_3\bar{\eta}_3(\eta_1\eta_2-\bar{\eta}_1\bar{\eta}_2) - (\eta_3^2-\bar{\eta}_3^2)(\eta_1\bar{\eta}_2+\eta_2\bar{\eta}_1)]\ . \qquad (3.5)$$

In order to study the critical behavior of the binary alloy one has to introduce random coupling terms (diagonal as well as off-diagonal) between the components η_i and $\bar{\eta}_i$, and consider n replicas of the Hamiltonian

$$\mathcal{H} = H_4 + H_2 + H_{\text{int}} \ . \qquad (3.6)$$

This is a rather complicated Hamiltonian, and it has not been analyzed in the present study.

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