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Low-temperature magnon thermal conductivity of ferromagnetic insulators with impurities

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The spin-wave thermal conductivity at low temperatures has been calculated for ferromagnetic insulators containing magnetic defects. The temperature range is such that we neglect those magnon scattering processes which can redistribute the magnon distribution, such as the magnon-magnon interactions. The effect of the external magnetic field has also been studied, and we observe that within the validity of the energy relation $E(\vec{k}) = \alpha k^2$, the magnon conductivity has a T^2 dependence.

I. INTRODUCTION

Heat conduction in the magnetic insulators can be adequately described in terms of the energy transport by the elementary excitations such as phonons[1] and magnons.[2] The phonon conductivity can be elegantly calculated by Callaway's approach[3] and its modifications. Erdos[4] and Kazakov and Nageav[5] calculated the phonon conductivity at very low temperatures of the nonmetals, where the phonon-phonon scattering processes can be neglected by considering the boundary conditions in terms of the phonon distribution functions consistent with the experimental situations. Kumar and Joshi[6] successfully extended Kazakov and Nageav's calculations within the relaxation-time approximation to calculate the phonon conductivity of doped Ge systems for the temperature range 0.5 – 4.2 K. The expressions of Kumar and Joshi have been modified by Dubey[7] and Nava *et al.*[8] to explain the phonon conductivity of the different nonmetals. The spin-wave thermal conductivity or the magnon conductivity has been previously calculated by Sato.[9] Subsequently, Douthett and Friedberg,[10] Douglass,[11] Luthi,[12] and Callaway *et al.*[13,14] suggested different expressions to calculate the magnon conductivity. It is observed that in the specific-heat and the thermal conductivity calculations at the liquid-helium temperature range, the spin-wave contributions exceeds the phonon contribution. Callaway and Boyd[15] extended their previous calculations to analyze the magnon conductivity of the impure systems. They have also studied the effects of the external magnetic field on the magnon conductivity. Erdos[16] extended his previous calculations[4] to calculate the spin-wave thermal conductivity containing random impurity centers. Erdos observed that at very low temperatures in certain cases, the mean-free path due to the magnon-magnon scattering processes becomes very large at very low temperatures, and the phonon conductivity is small as compared to the magnon conductivity. For some systems, even when the Curie temperature is small compared to the Debye temperature, the magnon-magnon scattering still remains important. Bhandari and Verma[17] analyzed the thermal conductivity of yttrium ion garnet (YIG) by adding the phonon and the magnon conductivities $[K_T(T) = K_{\rm ph}(T) + K_m(T)]$. They observed that the magnon-magnon interaction and the spin-phonon interaction can be neglected without much error at very low temperatures. Understanding the importance of the magnon conductivity of the ferromagnetic insulators, ferri-, antiferri-, and antiferromagnetic insulators, we calculate the magnon conductivity by solving the Boltzmann equation as previously done by Kazakov and Nageav and Kumar and Joshi at very low temperature.

II. THEORY

We assume that the thickness of the crystal along the x axis direction is L and the two ends are in contact with the two thermostats of the magnon black bodies at temperatures $T(0)$ and $T(L)$ such that $T(0) > T(L)$. The corresponding magnon-distribution functions may be defined as

$$n(0,k,\theta) = \left[\exp\left[\frac{E(k)}{k_B T(0)}\right] - 1\right]^{-1} = \begin{cases} n^{+}(T,k,(0)) & (0 \le \theta < \pi/2) \\ 0 & (\pi/2 < \theta \le \pi), \end{cases} \quad (1)$$

$$n(L,k,\theta)=\left[\exp\left[\frac{E(k)}{k_BT(L)}\right]-1\right]^{-1}$$
$$=\begin{cases}0 & (0\le\theta<\pi/2)\\ n^-(T,k,(L)) & (\pi/2<\theta\le\pi).\end{cases} \quad (2)$$

θ is the angle between the wave vector $\vec{k}$ and the x axis. The spin-wave energy can be written as

$$E(\vec{k})=\alpha k^2 . \quad (3)$$

The kinetic equation, suitable for the situation under study, can be written as

$$n(x,k,\theta)v_j\cos\theta=-\frac{n(x,k,\theta)-n^0(x,k)}{\tau_{kj}} , \quad (4)$$

where

$$n^0(x,k)=\tfrac{1}{2}\int_0^\pi d\theta\,\sin\theta n(x,k,\theta) . \quad (5)$$

v_j is the magnon velocity with the polarization j. τ_{kj} is the relaxation time of the magnon scattering processes which are mainly elastic in nature. In the present work, we neglect those processes which redistribute the magnon distribution such as the magnon-magnon scattering processes. Assuming only defect scattering, the relaxation time can be defined as

$$\tau_{kj}^{-1}=A_mk^4v_j$$
$$=\frac{V_0^2v_j}{4\pi^2}\left[\left[\frac{J'}{J}\right]^2\left[1-\frac{S'}{S}\right]^2+\left[1+\frac{J'}{J}-2\frac{J'}{J}\frac{S'}{S}\right]\right] . \quad (6)$$

Here V_0 is the atomic volume, S is the spin of the atom which constitutes the magnetic defects with spin S' and is coupled to the nearest neighbors with the exchange integrals J', and J refers to the host. As the defect scattering does not redistribute the magnons, so the distribution relaxes to the isotropic function $n^0(x,k)$. Following Kazakov and Nageav,[5] and Kumar,[18] we get

$$n(x,k,\theta)=\begin{cases}n^++\dfrac{1}{2}\dfrac{x}{v_j\tau_{kj}}\dfrac{n^--n^+}{\cos\theta} & (0\le\theta<\pi/2) \quad (7)\\[2ex] n^-+\dfrac{1}{2}\dfrac{L-x}{v_j\tau_{kj}}\dfrac{n^--n^+}{\cos\theta} & (\pi/2<\theta\le\pi). \quad (8)\end{cases}$$

The heat flux of the magnon gas in the x direction can be expressed as

$$Q_x(\vec{k})=\Omega\int d^3k\; n^0(x,k)E(\vec{k})v_{jx}(\vec{k}) . \quad (9)$$

The magnon velocity $v_j(\vec{k})$ can be obtained as

$$v_j(\vec{k})=\frac{\partial\omega_{\vec{k}j}}{\partial k} . \quad (10)$$

If we define

$$E(\vec{k})=\alpha k^2=\hbar\omega_{\vec{k}j} , \quad (11)$$

we find that the magnon velocity comes out to be

$$v_j(\vec{k})=\frac{2\alpha}{\hbar}k . \quad (12)$$

The x component of the magnon velocity comes out to be

$$v_{jx}(\vec{k})=\frac{2\alpha}{\hbar}k\cos\theta . \quad (13)$$

The heat flux for the magnon gas is obtained as

$$Q_x(\vec{k})=\Omega\int d^3k\, n^0(x,k)E(\vec{k})\left[\frac{2\alpha}{\hbar}\right]k\cos\theta . \quad (14)$$

Equation (14) can be further simplified as

$$Q_x(k)=Q_x^1+Q_x^2=\Omega\int_0^\infty 4\pi k^2dk\,\tfrac{1}{2}\int_0^\pi d\theta\,\sin\theta\,\cos\theta(\alpha k^2)\left[\frac{2\alpha}{\hbar}\right]kn(x,k,\theta)$$
$$=\frac{4\alpha^2\pi\Omega}{\hbar}\int_0^\infty k^5dk\left[\int_0^{\pi/2}d\theta\,\sin\theta\,\cos\theta n(x,k,\theta)+\int_{\pi/2}^\pi d\theta\,\sin\theta\,\cos\theta n(x,k,\theta)\right] . \quad (15)$$

Substituting the value of $n(x,k,\theta)$ from Eqs. (7) and (8) in Eq. (15), we get integrals of the form

$$I_1=\int_0^{\pi/2}d\theta\,\sin\theta\,\cos\theta\left[n^+-\frac{1}{2}\frac{x}{v_j\cos\theta}n^+A_mk^4\right] ,$$
$$I_2=\int_{\pi/2}^\pi d\theta\,\sin\theta\,\cos\theta\left[n^-+\frac{1}{2}\frac{L-x}{v_j\cos\theta}n^-A_mk^4\right] . \quad (16)$$

Finally, we get

$$Q_x^1=\frac{(2\alpha)^2\pi\Omega}{\hbar}\frac{1}{2}\int_0^\infty\left[\frac{k_BT(0)}{\alpha}\right]^3 t^2dt\int_0^{\pi/2}d\theta\,\sin\theta\,\cos\theta\left[n^+-\frac{1}{2}\frac{L}{2\cos\theta}n^+A_M\left[\frac{k_BT(0)}{\alpha}\right]^2 t^2\right], \quad (17a)$$

$$Q_x^2=\frac{(2\alpha)^2\pi\Omega}{\hbar}\frac{1}{2}\int_0^\infty\left[\frac{k_BT(L)}{\alpha}\right]^3 t^2dt\int_{\pi/2}^{\pi}d\theta\,\sin\theta\,\cos\theta\left[n^-+\frac{1}{2}\frac{L}{2\cos\theta}n^-A_m\left[\frac{k_BT(L)}{\alpha}\right]^2 t^2\right]. \quad (17b)$$

Equations (17a) and (17b) can be integrated to simplify them further as

$$\begin{aligned}Q_x^1&=\frac{(2\alpha)^2\pi\Omega}{2\hbar}\left[\left[\frac{k_BT(0)}{\alpha}\right]^3\int_0^\infty\frac{t^2dt}{e^t-1}-\tfrac{1}{2}LA_m\left[\frac{k_BT(0)}{\alpha}\right]^5\int_0^\infty\frac{t^4dt}{e^t-1}\right],\\ Q_x^2&=\frac{(2\alpha)^2\pi\Omega}{2\hbar}\left[-\left[\frac{k_BT(L)}{\alpha}\right]^3\int_0^\infty\frac{t^2dt}{e^t-1}+\tfrac{1}{2}LA_m\left[\frac{k_BT(L)}{\alpha}\right]^5\int_0^\infty\frac{t^4dt}{e^t-1}\right].\end{aligned} \quad (18)$$

Substituting

$$\mathscr{I}_n=\int_0^\infty\frac{t^ndt}{e^t-1}, \quad (19)$$

and

$$\begin{aligned}T(0)+T(L)&=2T,\\ T(0)-T(L)&=\Delta T.\end{aligned} \quad (20)$$

Assuming $\nabla T << T$, we get

$$Q_x=\frac{(2\alpha)^2\pi\Omega}{2\hbar}\mathscr{I}_2\left[\frac{k_BT}{\alpha}\right]^3 3\frac{\Delta T}{T}\times\left[1-\tfrac{1}{2}LA_m\left[\frac{k_BT}{\alpha}\right]^2\frac{5}{3}\frac{\mathscr{I}_4}{\mathscr{I}_2}\right]. \quad (21)$$

The thermal conductivity can be defined as

$$Q_x=K(T)\nabla T, \quad (22)$$

where $\nabla T=(\Delta T)/L$. We therefore obtain the magnon conductivity as

$$K_m(T)=3\frac{(2\alpha)^2\pi\Omega k_BL}{2\hbar\alpha}\times\left[\frac{k_BT}{\alpha}\right]^2\mathscr{I}_2\left[1-\tfrac{1}{2}LA_m\left[\frac{k_BT}{\alpha}\right]^2\frac{\mathscr{I}_4}{\mathscr{I}_2}\right]. \quad (23)$$

If we set

$$K_0(T)=3\frac{(2\alpha)^2\pi\Omega k_BL}{2\hbar\alpha}\left[\frac{k_BT}{\alpha}\right]^2\mathscr{I}_2, \quad (24)$$

we can write Eq. (23) in the form

$$K_m(T)=K_0(T)\left[1-\tfrac{5}{6}LA_m\left[\frac{k_BT}{\alpha}\right]^2\frac{\mathscr{I}_4}{\mathscr{I}_2}\right]. \quad (25)$$

Equation (25) is almost similar to the magnon conductivity expression as obtained by Erdos. Callaway's expression for the magnon conductivity can be written as

$$K_m(T)=\frac{2\pi\Omega k_B(k_BT)^2}{\hbar\alpha}\int_0^\infty\frac{t^3e^tdt}{(e^t-1)^2}. \quad (26)$$

At very low temperatures, one can take the limits of the integration t_m as ∞. The integration can be shown as

$$\int_0^\infty\frac{e^tt^3dt}{(e^t-1)^2}=3\int_0^\infty\frac{t^2dt}{(e^t-1)}. \quad (27)$$

Thus we find that the magnon conductivity expressed by Eq. (25) is similar to that suggested by Callaway when we neglect the expansions in the spin-wave energy.

The temperature dependence of the magnon conductivity is clearly T^2 dependent as observed previously. Recent calculations have suggested that the effective temperature dependence of the magnon conductivity can approximately be T^2. The author feels that this appears because the validity of the spin-wave approximation depends entirely upon the linearization procedure. The spin-wave energy is obtained after the solutions of the equations of motion, using different decoupling schemes. Therefore, when the spin deviation is assumed to be smaller, the validity is qualitatively close to the reality at low temperatures only where one can assume a few excited magnons.

The presence of the magnetic defects reduces the magnon conductivity from $K_0(T)$ to $K_m(T)$; the relative change in the thermal conductivity due to magnons scattering by defects can, therefore, be obtained as:

$$\frac{\Delta K}{K}=\frac{K_0(T)-K_m(T)}{K_0(T)}=\tfrac{5}{6}LA_m\left[\frac{k_BT}{\alpha}\right]^2\frac{\mathscr{I}_4}{\mathscr{I}_2}. \quad (28)$$

III. EFFECT OF THE MAGNETIC FIELD

Within random phase approximation, the spin-wave energy can be in general expressed as[10]

$$\epsilon(\vec{k})=E(\vec{k})+g\mu_BH(1+\phi_k\sin^2\theta_k)^{1/2}, \quad (29)$$

where H is the external magnetic field, θ_k is the angle between the wave vector k and the external magnetic field, and ϕ_k can be given by

$$\phi_k=4g\mu_BH/[E(\vec{k})+g\mu_BH]. \quad (30)$$

If we ignore the dependence of the spin-wave energy on the direction on the propagation, Eq. (29) reduces to

$$\epsilon(\vec{k})=E(\vec{k})+g\mu_B(H+\tfrac{4}{3}\pi M). \quad (29')$$

Equation (11) now modifies to

$$K_BTt=\alpha k^2+g\mu_B\bar{H}=\hbar\omega_{\vec{k}j},$$

with

$$\bar{H}=H+\tfrac{4}{3}\pi M.$$

We can obtain the value of k^2 as

$$k^2=\frac{k_BT}{\alpha}(t-h), \quad (31)$$

where

$$h=g\mu_BH/k_BT).$$

Using Eq. (31), Eqs. (23) and (24) modify to

$$K_0(T,h)=3\frac{(2\alpha)^2\pi\Omega k_BL}{2\hbar\alpha}\left[\frac{k_BT}{\alpha}\right]^2\int_h^\infty\frac{(t-h)t\,dt}{e^t-1}, \quad (32)$$

$$K_m(T,h)=K_0(T,h)\left\{1-\tfrac{5}{6}LA_m\left[\frac{k_BT}{\alpha}\right]^2\left[\int_h^\infty\frac{(t-h)^3t\,dt}{e^t-1}\left[\int_h^\infty\frac{(t-h)t\,dt}{e^t-1}\right]^{-1}\right]\right\}. \quad (33)$$

Equation (32) can be further simplified as

$$K_0(T,h)=K_0(T)\left[1-h\frac{\mathscr{I}_1}{\mathscr{I}_2}\right]. \quad (32')$$

Doing some simple mathematics, Eq. (33) can also be reduced to

$$K_m(T,h)=K_0(T)+K_0(T)[a_1h+a_2h^2+a_3h^3+a_4h^4], \quad (33')$$

where

$$a_1=-\left[\frac{\mathscr{I}_1}{\mathscr{I}_2}-3b\frac{\mathscr{I}_3}{\mathscr{I}_2}\right],$$

$$a_2=-3b\left[1-\frac{\mathscr{I}_1\mathscr{I}_3}{\mathscr{I}_2^2}+\frac{\mathscr{I}_3\mathscr{I}_4}{\mathscr{I}_2^2}-\frac{\mathscr{I}_1\mathscr{I}_4^2}{\mathscr{I}_2^3}\right],$$

$$a_3=-b\left[\frac{\mathscr{I}_1}{\mathscr{I}_2}-3\frac{\mathscr{I}_4}{\mathscr{I}_2}+3\frac{\mathscr{I}_1\mathscr{I}_2\mathscr{I}_3}{\mathscr{I}_2^3}\right], \quad (34)$$

$$a_4=b\left[2\frac{\mathscr{I}_1\mathscr{I}_4}{\mathscr{I}_2^2}+\frac{\mathscr{I}_1^2}{\mathscr{I}_2}\right],$$

$$b=\tfrac{5}{6}LA_m\left[\frac{k_BT}{\alpha}\right]^2.$$

Comparing Eq. (33′) with those obtained by the previous[17,14] workers, we find Eq. (33′) is almost similar to the expressions of McCollum *et al.* except for the fact that we limit our calculations to first few terms. Otherwise we can obtain a series dependence on the higher powers of h as suggested by the exponentials in the results of McCollum *et al.* Equation (33′) gives the effect of the magnetic field on the magnon conductivity in the presence of defects.

IV. DISCUSSION

The magnon conductivity used to be analyzed most widely by Callaway's approach for the different magnetic substances. Walton *et al.*[19] calculated the magnon conductivity by expressing Callaway's results as

$$K_m(T)=B'T^2\int_0^\infty\frac{t^3\mathrm{csch}^2(\tfrac{1}{2}t)dt}{1+b(T)t^2}, \quad (35)$$

$$K_m(T,h)=B'T^2\int_0^\infty\frac{t^2(t-h)\mathrm{csch}^2(\tfrac{1}{2}t)dt}{1+b(T)(t-h)^2}, \quad (36)$$

where $t=(\hbar\omega/K_B T)$ and $b(t)=gT^2$. For very low temperatures, Eqs. (35) and (36) can be reduced to

$$K_m(T)=B'T^2\int_0^\infty dt\, t^3 \mathrm{csch}^2(\tfrac{1}{2}t) \times[1-b(T)t^2]\,, \qquad (35')$$

$$K_m(T,h)=B'T^2\int_0^\infty dt\, t^2(t-h)\mathrm{csch}^2(\tfrac{1}{2}t) \times[1-b(T)(t-h)^2]\,. \qquad (36')$$

Comparing Eqs. (25) and (27) we observe that the reduced Eqs. (35′) and (36′) are almost similar.

In the present work, we find that in the presence of the defect scattering alone, the magnon conductivity analysis is simple in its computation,[20] and also the temperature dependences are effectively the same as obtained experimentally and by previous workers. It is clearly observed that the reduction in the magnon conductivity is proportional to T^4 as previously calculated by Callaway.[13]

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