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Critical properties of an altered Ising model

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A modified two-dimensional Ising model is studied. In this model the nearest-neighbor interactions are changed along an infinite line imbedded in the lattice. We show that the effect of this alteration can be represented by the action of a marginal operator on the critical Ising model. This marginal operator is seen to change the decays of correlation in the lattices near the altered line. We find that the magnetization index β is changed, while the index ν remains fixed. The expected crossover of the correlation functions to bulk behavior is also observed. We calculate the critical index for the disorder variable and predict the following relation between the index for the order and disorder variables: $\sqrt{2x_\sigma}+\sqrt{2x_\mu}=1$.

I. INTRODUCTION

In a recent paper, Bariev[1] demonstrated that a simple modification to the Ising model resulted in critical properties which were continuously dependent on a parameter in the Hamiltonian. Bariev, and later McCoy and Perk,[2] considered an Ising model where the interactions along one line have been changed. It was shown that the critical index η, measured along this altered line, was a continuous function of the new coupling, while ν remained fixed at the Ising value of 1.

Problems whose critical properties depend continuously on a parameter in the Hamiltonian have been extensively studied since Baxter solved the eight-vertex model in 1972.[3] The number of two-dimensional models known to have continuous critical behavior has since grown to include the Ashkin-Teller,[4] Q-state Potts[5] model and the Gaussian model with excitations.[6] Each of these models appears to be in the same universality class as the eight-vertex[7] model, hence along the critical line the index η remains fixed at the Ising value of $\frac{1}{4}$ while the index ν varies smoothly. It is clear that the eight-vertex models cannot be in the same universality class as the Bariev model except at the Ising point.

Much of what is understood about the critical properties of the eight-vertex class of problems is built upon the recognition of a marginal operator, the addition of which to the critical Hamiltonian serves to generate motion along the critical line. It is in this sense that the universality connection is made among the members of this class; each model is seen to have a marginal operator, the effect of which on the asymptotic behavior is the same for each model.

The purpose of this paper is twofold. First we show that the Bariev model can be formulated in terms of a marginal operator. The exact expression for the behavior of β, as calculated by Bariev, serves as a check of this formulation. Second, we derive the behavior of the two-spin correlation function as one moves away from the line of altered interactions. In this way we can examine the crossover to bulk Ising behavior of the two-point function.

In Sec. II we identify the marginal operator and examine its effect, to the first order, on the magnetization exponent β and the energy-density exponent ν. The change in ν is given by calculating the change in the energy-density exponent x_g, which is seen to be unchanged by the addition of the marginal operator. The change in β is found by calculating the variation in x_σ. It will be seen that the marginal operator alters the indices of operators which are odd under the symmetry operation $\sigma\rightarrow-\sigma$, while leaving the even operators unchanged. This is exactly the reverse of the effect of the eight-vertex marginal operator, the addition of which, to the critical Ising Hamiltonian, changes the even operators while leaving the odd operators fixed.

The presence of the Bariev marginal operator breaks the dual symmetry, hence the critical index for the order parameter x_σ is no longer equal to the critical index for the disorder variable x_μ. In the last part of Sec. II we derive the relation between x_σ and x_μ.

$$(2x_\sigma)^{1/2}+(2x_\mu)^{1/2}=1 \quad ,$$

which holds along the entire critical line.

In Sec. III the behavior of the two-point correlation functions are examined as one moves away from the altered line. In the case where one of the end points remains on the altered line there are three distinct behaviors which in the thermodynamic limit can be characterized by three critical indices. When both points are on the altered line, the decay of the spin-spin correlation is described by the critical index x_σ which varies as the altered interaction is changed. When one point of the two-point function lies on the altered line and the other lies perpendicular to the line, then a different behavior for x_σ is seen. Finally, when one point is moved infinitely far from the altered line the magnetization critical index crosses over to the bulk ($x_\sigma=\frac{1}{8}$) behavior for the isotropic Ising model.

II. CORRELATIONS ALONG THE ALTERED LINE

A. Marginal operator

In this section we will examine how the Bariev model can be described in terms of a marginal operator acting on a critical Ising Hamiltonian. Consider an Ising model with nearest-neighbor interactions where the interactions along one particular line assume the value of K'. This Hamiltonian can be written as a sum over nearest-neighbor interactions

$$\mathcal{H}=K\sum_{\langle rr'\rangle}\sigma_r\sigma_{r'}+\kappa\sum_{\langle rr'\rangle}{}'\sigma_r\sigma_{r'},\quad \kappa=K'-K\ ,\qquad (2.1)$$

where the prime on the sum indicates that the points lie along a line.

Bariev, and later McCoy and Perk, showed that the critical properties of this model are dependent on the strength of the altered line κ, and found the magnetization critical index $x_\sigma(x_\sigma=\beta\nu)$ as a function of κ

$$x_\sigma(\kappa)=\frac{1}{2}\left[\frac{1}{\pi}\arccos\tanh 2\kappa\right]^2\ ,\qquad (2.2)$$

where $K=J/(k_BT)$. It has been suggested by Fisher[8] that the operator $\int dr\,\mathcal{E}(r)$, the line integral of the energy density, is a marginal operator at $\kappa=0$. We will show that this operator is a marginal operator for all κ, and calculate, to first order in κ, $x_\sigma(\kappa)$ and $x_\mu(\kappa)$.

To see how the marginal operator acts to change the critical index, consider the spin-spin correlation function

$$\langle\sigma(0)\sigma(r)\rangle_{\kappa=0}\equiv\sum_{\{\sigma\}}\sigma_0\sigma_r e^{-KH_0}\Big/\sum_{\{\sigma\}}e^{-KH_0}\ .\qquad (2.3)$$

If a term κM is added to the Ising Hamiltonian the two-spin function becomes

$$\langle\sigma_0\sigma_r\rangle_M=\sum_{\{\sigma\}}\sigma_0\sigma_r e^{\kappa M}e^{-KH_0}\Big/\sum_{\{\sigma\}}e^{-KH_0}e^{\kappa M}\ .\qquad (2.4)$$

Equation (2.4) can be rewritten in terms of pure Ising correlations

$$\langle\sigma_0\sigma_r\rangle_M=\frac{\langle\sigma_0\sigma_r e^{\kappa M}\rangle_0}{\langle e^{\kappa M}\rangle_0}\qquad (2.5a)$$

or

$$\langle\sigma_0\sigma_r\rangle_M=\langle\sigma_0\sigma_r\rangle_0 f[\kappa M]\ .\qquad (2.5b)$$

The critical point is characterized by algebraic decay of correlations, therefore

$$\langle\sigma_0\sigma_r\rangle_M\sim\frac{1}{r^{2x_\sigma}}f[\kappa M]\ .\qquad (2.6)$$

The factor $f[\kappa M]$ can thus provide a modification of the algebraic decay as well as a change in the scaling function of the correlation function.

The function f can be expanded in a power series for small κ, the expansion to first order being given by

$$f[\kappa M]\cong\exp\left[\kappa\left(\frac{\langle\sigma_0\sigma_r M\rangle}{\langle\sigma_0\sigma_r\rangle}-\langle M\rangle\right)\right]\ .\qquad (2.7)$$

If the argument of the exponential contains a term proportional to $\ln r$, then the critical index x_σ will be changed.

The marginal operator can be written as the integral over a density, i.e.,

$$M\equiv\int d^\rho\vec{r}\,\mathcal{M}(\vec{r})\ ,\qquad (2.8)$$

where ρ is the dimensionality of the marginal interaction. The argument of the exponential in Eq. (2.7) will then have a term

$$\int d^\rho\vec{r}\,'\langle\mathcal{M}(\vec{r}\,')\sigma_0\sigma_r\rangle\ .\qquad (2.9)$$

In this context, the condition under which Eq. (2.9) will be proportional to $\langle\sigma_0\sigma_r\rangle\ln r$ is then $\mathcal{M}(r)\sim r^{-\rho}$.

In the case of the Bariev model, the marginal operator is a line integral of the energy-density operator, thus $\rho=1$, with the marginal density equal to the energy density along the altered line. The energy-density operator in the Ising model is known to scale as $1/r$, hence this operator satisfies the requirement for the marginal operator, $M(r)\sim 1/r^\rho$.

B. Effect of the marginal operator

In Sec. II A we saw that the line integral of the energy density of the two-dimensional Ising model was a candidate for a marginal operator. We saw that a detailed knowledge of the correlation between the marginal operator and other operators was necessary to calculate the behavior of the critical indices in the presence of a marginal perturbation. In our case, we need to know the correlation between the energy-density operator and other operators in the Ising model. There exists a body of exact results due to Kadanoff[9] for the Ising correlations in the special case where all operators lie along one line. If we restrict our consideration to the effect of the marginal operator only along the altered line, then the expression for the Ising correlations will be sufficient to calculate, to any order, the change in critical indices. In this section we will show that the energy-density operator effects a change in the critical index of operators which are "odd" with respect to the symmetry operation $\sigma \to -\sigma$. We will also show that critical indices of even operators, such as energy density, remain unchanged in the presence of the marginal operators. Thus the application of the above marginal operators changes the critical index η ($\eta = 2x_\sigma$) while the index $\nu[\nu = 1/(2 - x_{\mathcal{E}})]$ remains fixed.

Before we calculate the effect of the marginal operator, we will review the form which the Ising correlations assume when all points lie along a line. Kadanoff and Ceva have shown that the operators which are used to formulate the two-dimensional (2D) Ising model form an operator algebra. The operators within this algebra obey certain short-distance product laws much akin to normal angular-momentum operators. For example, the product of two operators O_α and O_β labeled by α and β yields a third operator O_γ,

$$O_\alpha(\vec{r}_1) O_\beta(\vec{r}_2) = O_\gamma\left(\frac{\vec{r}_1 + \vec{r}_2}{2}\right) , \tag{2.10}$$

where

$$\gamma = \alpha + (-1)^{2\alpha}\beta \ .$$

The set of operators which describe the Ising model can be identified. For example, the order and disorder variables have the identifications

$$O_{1/2}(\vec{r}) = \sigma(\vec{r}), \quad O_{-1/2}(\vec{r}) = \mu(\vec{r}) \ . \tag{2.11}$$

The correlation of a product of these operators can be calculated for the case when all operators lie on a common line with r_i to the right of r_{i+1}. This average is given by

$$\left\langle \prod_{i=1}^{N} O_{\gamma_i}(\vec{r}_i) \right\rangle = \begin{cases} 0, & \Gamma \neq 0 \\ \prod\limits_{1 \leq i < j \leq N} [c\,|\vec{r}_{ij}|]^{\gamma_i \gamma_j p_i p_j} \end{cases} \tag{2.12}$$

with $p_i = (-1)^{2\Gamma_{i-1}}$, where the "quantum number" Γ is defined recursively as

$$\Gamma_0 = 0, \quad \Gamma_{i+1} = \Gamma_i + (-1)^{2\Gamma_i}\gamma_{i+1} \ . \tag{2.13}$$

We note that Eqs. (4.9) and (4.10) in Ref. 9 are incorrect as they stand; the corrected formulas are given by Eqs. (2.12) and (2.13).

For example, using Eqs. (2.12) and (2.13), the spin-spin correlation function ($\sigma = D_{1/2}$) will be given by

$$\langle O_{1/2}(r_1) O_{1/2}(r_2) \rangle = c\,|r_{12}|^{-1/4} \tag{2.14}$$

which is the exact known result.

In addition to calculating the correlation between magnetization operators, Kadanoff and Ceva were able, via the operator product expansion, to calculate the correlation between the energy-density operator and any other product of operators in the Ising model. We know that the product of two nearby fluctuating variables σ_r and $\sigma_{r+\delta}$ can be expressed as an expansion in terms of the other operators in the model. For example the expansion of the product of two-spin variables is given by

$$\sigma_r \sigma_{r+\delta} = c\,\delta^{-1/4} + b\,\delta^{3/4} \mathcal{E}(r) + \dots , \tag{2.15}$$

where $\mathcal{E}(r)$ represents the Ising energy-density operator $\sigma_r \sigma_{r+1} - \langle \sigma_r \sigma_{r+1} \rangle$, and b and c are known constants. The correlation of products of operators which involve energy-density operators can thus be found by allowing two magnetization operators in Eq. (2.12) to approach one another, thus generating an energy-density operator. In this way we see that the correlation between $\mathcal{E}(r)$ and any product of operators $X = \prod_{i=1}^{N} O_{\gamma_i}(r_i)$ is

$$\langle X \mathcal{E}(r) \rangle = \frac{\langle X \rangle}{b} \sum_{i=1}^{N} \frac{\gamma_i p_i g_R}{(r_i - R)} , \tag{2.16}$$

where $g_R = (-1)^{2\Gamma_R}$ and Γ_R is the value of p_i for the operator X just before the energy-density operator.

A similar line of logic gives the correlation between two energy-density operators and X to be

$$\langle X \mathcal{E}(r_1)\, \mathcal{E}(r_2) \rangle = \frac{\langle X \mathcal{E}(r_1) \rangle \langle X \mathcal{E}(r_2) \rangle}{\langle X \rangle} + \langle X \rangle \langle \mathcal{E}(r_1) \mathcal{E}(r_2) \rangle \ . \tag{2.17}$$

To calculate the change in the magnetization index x_σ we need to calculate the integral over the altered line of energy-spin-spin correlation, i.e., from Eq. (2.9) we have

$$\frac{\partial}{\partial \kappa} \langle \sigma_{r_1} \sigma_{r_2} \rangle |_{\kappa=0} = \int_{-\infty}^{\infty} dx_3 \langle \mathcal{E}(x_3) \sigma(x_1) \sigma(x_2) \rangle , \tag{2.18}$$

where x_1, x_2, and x_3 all lie on the same line, say $y = 0$. Using the above form of the correlation functions given by Eq. (2.16), the integrand in Eq. (2.18) is given by

$$\langle \sigma_{x_1} \sigma_{x_2} \mathcal{E}_{x_3} \rangle = \frac{1}{2\pi} \frac{x_{12}}{x_{13} x_{23}} \langle \sigma_{x_1} \sigma_{x_2} \rangle \ . \tag{2.19}$$

The integral Eq. (2.18) will carry us into difficulties, because the integrand becomes singular as $x_3 \to x_1, x_2$. This problem can be avoided by the use of a cutoff. At the critical point we are dealing with diverging length scales, hence we expect the critical behavior to be unaffected by such a short-range cutoff. With this cutoff, the integral (2.18) becomes straightforward

$$\frac{\partial}{\partial\kappa}\langle\sigma_1\sigma_2\rangle\Big|_{\kappa=0} = \left[\int_{-\infty}^{x_1-\Omega} + \int_{x_1+\Omega}^{x_2-\Omega} + \int_{x_2+\Omega}^{\infty}\right]\frac{dx_3}{2\pi}\frac{x_{12}}{x_{13}x_{23}}\langle\sigma_1\sigma_2\rangle \;, \tag{2.20a}$$

or

$$\frac{\partial}{\partial\kappa}\langle\sigma_0\sigma_r\rangle\Big|_{\kappa=0} = \left[\frac{2}{\pi}\ln(r/\Omega)\right]\langle\sigma_0\sigma_r\rangle \;. \tag{2.20b}$$

We see that the integral of the energy density-spin-spin correlation function is proportional to $\ln r$. From Eq. (2.5) the variation in x_σ as κ is changed is

$$\frac{\partial x_\sigma}{\partial\kappa}\Big|_{\kappa=0} = -\frac{1}{\pi} \;, \qquad x_\sigma(\kappa) = \frac{1}{8} - \frac{1}{\pi}\kappa + O(\kappa^2) \;, \tag{2.21}$$

which agrees with McCoy and Perk's result to first order in κ.

To calculate the change in ν, the change in the energy-density critical index must be calculated. It is possible to show that $x_{\mathcal{E}}$ remains unchanged to all orders in κ via a symmetry argument. The critical index $x_{\mathcal{E}}(\kappa)$ is defined in terms of the two-point function $\langle\mathcal{E}(r_1)\mathcal{E}(r_2)\rangle_\kappa$. An argument identical to that which led to Eq. (2.5a) implies that $\langle\mathcal{E}_{r_1}\mathcal{E}_{r_2}\rangle_\kappa$ can be written in terms of isotropic ($\kappa=0$) Ising correlations

$$\langle\mathcal{E}_{r_1}\mathcal{E}_{r_2}\rangle_\kappa = \frac{\langle\mathcal{E}_{r_1}\mathcal{E}_{r_2}e^{\kappa M}\rangle_0}{\langle e^{\kappa M}\rangle_0}, \quad M = \int\mathcal{E}(r')dr' \;. \tag{2.22}$$

Because r_1, r_2, and r' all lie along a line, the numerator of Eq. (2.22) can be reduced via Eq. (2.17) to

$$\langle\mathcal{E}_{r_1}\mathcal{E}_{r_2}\rangle_\kappa = \frac{\langle\mathcal{E}_{r_1}e^{\kappa M}\rangle_0\langle\mathcal{E}_{r_2}e^{\kappa M}\rangle}{\langle e^{\kappa M}\rangle_0^2} + \langle\mathcal{E}_{r_1}\mathcal{E}_{r_2}\rangle_0 \tag{2.23a}$$

or

$$\langle\mathcal{E}_{r_1}\mathcal{E}_{r_2}\rangle_\kappa = \langle\mathcal{E}_{r_1}\rangle_\kappa\langle\mathcal{E}_{r_2}\rangle_\kappa + \langle\mathcal{E}_{r_1}\mathcal{E}_{r_2}\rangle_0 \;. \tag{2.23b}$$

In the case of the unaltered Ising model the average of the energy-density $\langle\mathcal{E}_r\rangle_0$ vanishes. This is not the case for $\kappa\neq 0$. However a detailed knowledge of $\langle\mathcal{E}_r\rangle_\kappa$ is not necessary to show that the critical index $x_{\mathcal{E}}(\kappa)$ is independent of κ. We are interested in the part of Eq. (2.23b) which scales as $|r_1-r_2|$ to some power. The first term on the right-hand side of Eq. (2.23b) is the product of a function of r_1 and a function of r_2 and thus cannot scale as $|r_1-r_2|$ to a power. The second term does scale in the proper manner, with a critical exponent $x_{\mathcal{E}}(\kappa=0)$, hence $x_{\mathcal{E}}(\kappa)=x_{\mathcal{E}}(0)$, to all orders in κ. This property is required of a marginal operator, whose critical index must remain unchanged along the entire critical line.

C. Disorder variables

In the operator formulation of the Ising model the disorder variable μ is an essential element of the operator algebra. This disorder variable is the dual of the order variable σ, hence $\langle\mu\rangle=0$ in the ordered phase and $\langle\mu\rangle=1$ at infinite temperatures. In the normal Ising model the dual transformation is an exact symmetry of the model, hence at the critical point μ and σ have the same properties, i.e., their critical exponents are identical.

An interesting feature of the Bariev model is that the exact dual symmetry is broken, thus the critical index x_σ, as measured along the altered line, will not be equal to x_μ. To see this, consider the two-spin correlation function $\langle\sigma_0\sigma_r\rangle_\kappa$ expressed in terms of pure Ising correlations

$$\langle\sigma_0\sigma_r\rangle_\kappa = \left\langle\sigma_0\sigma_r\exp\left[\kappa\int\mathcal{E}(r)dr\right]\right\rangle_0 \Big/ \left\langle\exp\left[\kappa\int\mathcal{E}(r)dr\right]\right\rangle_0 \;. \tag{2.24}$$

At the point $\kappa=0$, the Kramers-Wannier (KW) dual transformation acts on the Ising operators in the following manner:

$$\sigma\leftrightarrow\mu \;, \quad \mathcal{E}_v\leftrightarrow-\mathcal{E}_h \;, \quad \mathcal{E}_h\leftrightarrow-\mathcal{E}_v \;, \tag{2.25}$$

where $\mathcal{E}_h$ and $\mathcal{E}_v$ are the energy-density operators in the horizontal and vertical directions. Defining the energy density as the sum of the horizontal and vertical operators $\mathcal{E}(r)=[\mathcal{E}_h(r)+\mathcal{E}_v(r)]$ the action of the dual transformation is

$$\sigma(\vec{r})\leftrightarrow\mu(\vec{r}) \;, \quad \mathcal{E}(\vec{r})\leftrightarrow-\mathcal{E}(\vec{r}) \;. \tag{2.26}$$

With these relations and the above definition of the energy density, the KW transformation acting on the two-spin function yields

$$\langle\mu_0\mu_r\rangle_\kappa = \frac{\left\langle\mu_0\mu_r\exp-\kappa\int\mathcal{E}(r)dr\right\rangle_0}{\left\langle\exp-\kappa\int\mathcal{E}(r)dr\right\rangle_0} \;. \tag{2.27}$$

At the critical point of the pure Ising model the correlation of a product of disorder variables is equal to the correlation of the product of order variables, hence

$$\langle\mu_0\mu_r\rangle_\kappa = \frac{\left\langle\sigma_0\sigma_r\exp-\kappa\int\mathcal{E}(r)dr\right\rangle_0}{\left\langle\exp-\kappa\int\mathcal{E}(r)dr\right\rangle_0} = \langle\sigma_0\sigma_r\rangle_{-\kappa} \;. \tag{2.28}$$

Therefore $x_\mu(\kappa)=x_\sigma(-\kappa)$ for all κ. The knowledge of $x_\sigma(\kappa)$ [Eq. (2.2)] and the above symmetry relation are sufficient to determine the form of $x_\mu(\kappa)$.

It is interesting to note that the form of $x_\sigma(\kappa)$ and $x_\mu(\kappa)$ imply that the indices are related by

$$\sqrt{2x_\sigma}+\sqrt{2x_\mu}=1 \qquad (2.29)$$

for all values of κ. This relation has been verified by a different argument in a recent paper of Kadanoff.[10]

III. CORRELATION OFF THE ALTERED LINE

In Sec. II we explored the effect of adding a linear marginal operator to the critical Ising model. The calculation was carried out for the special case where both points of the two-point function were imbedded in the line of altered couplings. The effect of the marginal operator was calculated to the first order in an expansion about the critical Ising model and has seen to agree with the work of McCoy and Perk. In this section we will calculate the effect on the two-point function of moving one or both of the points off the altered lines. These calculations will show three regions of quantitatively different behavior. When the distance between the altered line and one point of the two-point function (denoted by y) is much less than the separation between the two points (r), then one sees a smooth angular dependence in the correlation function. As $r\to\infty$ but y remains finite, the angular dependence degenerates into two behaviors. If $\vec{r}$ is parallel to the altered line $\vec{l}$, the decay of the spin-spin correlation function is characterized by an exponent $x_\parallel$, which is equal to the exponent of McCoy and Perk. When $\vec{r}$ lies perpendicular to $\vec{l}$, the two-point function is characterized by $x_\perp$ which is equal to $(\frac{1}{2})x_\parallel$, to first order in κ. Finally for $|r|\leq y$ there is a crossover to the bulk critical exponent $x_\sigma=\frac{1}{8}$.

The starting point for this section will be the expression for the spin-spin energy-density correlation function calculated by Bander and Richardson.[11] This correlation function is not limited by the requirement that all three points lie along a line, thus we can investigate the effect of the marginal operator on an operator not lying on the altered line. From Ref. 11 we have an expression of the three-point function in terms of the two-point functions

$$\langle \mathcal{E}(r_1)\sigma(r_2)\sigma(r_3)\rangle=\frac{1}{2\pi}\frac{|\vec{r}_{23}|}{|\vec{r}_{12}|\,|\vec{r}_{13}|}\times\langle\sigma(\vec{r}_2)\sigma(\vec{r}_3)\rangle \ . \qquad (3.1)$$

As in Sec. II we need to calculate the integral of $\langle\mathcal{E}(r_1)\sigma(r_2)\sigma(r_3)\rangle$ over the line $\vec{l}$ to find the first-order change produced by the marginal operator $\int d\vec{r}\cdot\vec{l}E(\vec{r})$. Writing Eq. (3.1) in terms of components, for example $\vec{r}_{12}=(x_1-x_2, y_1-y_2)$, the right-hand side of Eq. (2.18) becomes

$$\int d\vec{r}_1\cdot\vec{l}\langle E(\vec{r}_1)\sigma(\vec{r}_2)\sigma(\vec{r}_3)\rangle = \int_{-\infty}^{\infty}dx_1\frac{|\vec{r}_{23}|}{[(x_{12}^2+y_{12}^2)(x_{13}^2+y_{13}^2)]^{1/2}} \ , \qquad (3.2)$$

where $y_1-y_2\neq 0$, $y_1-y_3\neq 0$.

This integral is recognized as a standard form of the elliptical integral of the first kind [see Integraltapel Bestimute Integrale 223.21 (Ref. 12)]. Equation (3.2) then becomes

$$\frac{1}{\pi}\frac{1}{(|y_{12}|\,|y_{13}|k_1)^{1/2}}K(k) \ , \qquad (3.3a)$$

$$k\equiv\frac{k_1^2-1}{k_1^2} \ , \qquad (3.3b)$$

$$k_1\equiv A+\sqrt{A^2-1} \ , \qquad (3.3c)$$

$$A\equiv\frac{r_{12}^2+y_{12}^2+y_{13}^2}{2|y_{12}|\,|y_{13}|} \ , \qquad (3.3d)$$

where $K(k)$ is the complete elliptic integral of the first kind. With the altered line lying at $y_1=0$ the expression for A reduces to

$$A=1+\frac{1}{2}\frac{r_{23}^2}{y_2y_3} \ . \qquad (3.4)$$

Equation (3.3) governs the change in the spin-spin function for the spins located anywhere except directly on the line of altered interactions. This expression should then assume the known limit as the two points lie infinitismally close to the altered line, as well as exhibit the crossover to the bulk behavior as the two points move infinitely far from the line.

Consider the limit $r_{23}^2/y_2y_3\to\infty$, i.e., one or both points lie near the altered line with respect to the separation between the points. In this case Eqs. (3.3b), (3.3c), and (3.3d) become

$$k\simeq 1-\left(\frac{y_2y_3}{r_{23}^2}\right)^2 \ , \qquad (3.5a)$$

$$k_1\simeq 2A=\frac{r_{23}^2}{y_2y_3} \ , \qquad (3.5b)$$

$$A\simeq\frac{1}{2}\frac{r_{23}^2}{y_2y_3} \ . \qquad (3.5c)$$

We can make use of the form of $K(k)$

$$K(k)=\left(\frac{2}{\pi}\ln\frac{4}{k'}\right)K(k')-\sum_{\nu=1}^{\infty}c_\nu k'^{2\nu}, \quad k'=(1-k^2)^{1/2} \qquad (3.6)$$

to write an expansion for Eq. (3.3a) in this case $y_2y_3/r_{23}^2 \to 0$. Defining $\epsilon = y_2y_3/r_{23}^2$, we see from Eqs. (3.6) and (3.5) that $k' = \epsilon$. To lowest order in ϵ, Eq. (3.6) becomes

$$K\left[1-\frac{\epsilon^2}{2}\right] = \frac{2}{\pi}K(\epsilon)\ln\frac{4}{\epsilon} - \epsilon^2 c_1 + \cdots \tag{3.7a}$$

with the small ϵ expansion of K being given by

$$K(\epsilon) = \frac{\pi}{2}\left[1+\frac{\epsilon}{4}+O(\epsilon^2)\right] . \tag{3.7b}$$

With these expressions Eq. (3.2) takes the form

$$\int d\vec{1}\langle \mathcal{E}(\vec{r}_1)\sigma(\vec{r}_2)\sigma(\vec{r}_3)\rangle = \frac{-1}{2\pi}\left[2\ln 4\frac{y_2y_3}{r_{23}^2}\right]\left[1+\frac{1}{4}\frac{y_2y_3}{r_{23}^2}\right]\langle\sigma(r_2)\sigma(r_3)\rangle . \tag{3.8}$$

Equation (3.8) is our basic formula for determining the behavior of that critical index x_σ. If we allow the two end points to approach the altered line ($y_2 = y_3 = \Omega << r_{23}$ the asymptotic form of Eq. (3.8) becomes

$$\int d\vec{1}\langle \mathcal{E}(\vec{r}_1)\sigma(\vec{r}_2)\sigma(\vec{r}_3)\rangle = \frac{2}{\pi}\left[\ln\frac{|\vec{r}_{23}|}{\Omega}\right]\langle\sigma_0\sigma_{r_{23}}\rangle \tag{3.9}$$

which agrees with Eq. (2.20b), and implies that the decay of the correlation is characterized by a critical index $x_\sigma = \frac{1}{8} - (1/\pi)\kappa + \cdots$.

Given Eq. (3.8) we can also consider the case where one end point remains fixed on the line and the other is moved away from the line. For example, if $\vec{r}_{23}$ is perpendicular to $\vec{1}$ then $y_{23} = |\vec{r}_{23}|$ and Eq. (3.8) becomes

$$\int d\vec{1}\langle E(\vec{r}_1)\sigma(\vec{r}_2)\sigma(\vec{r}_3)\rangle = \frac{1}{\pi}\left[\ln\frac{|\vec{r}_{23}|}{\Omega}\right]\langle\sigma_0\sigma_{r_{23}}\rangle . \tag{3.10}$$

Hence the first-order change in x_σ is exactly $\frac{1}{2}$ that of the case where both end points are fixed on the altered line. The question of interest is, how does the critical index cross over from $x(\kappa)_\parallel$ to $x(\kappa)_\perp$. Set

$$y_2 = \Omega \ , \qquad y_3 = \Omega + r_{23}\sin\theta \ , \tag{3.11}$$

where θ is the angle between $\vec{1}$ and $\vec{r}_{23}$. The argument of the log term in Eq. (3.8) becomes

$$\frac{r_{23}^2}{y_2y_3} = \frac{r_{23}^2}{\Omega(\Omega + r_{23}\sin\theta)} . \tag{3.12}$$

The two limits considered in Eqs. (3.9) and (3.10) can then be recovered with $\theta \cong 0$ and $\pi/2$, respectively

$$\frac{r_{23}^2}{y_2y_3} \simeq \begin{cases} \dfrac{r_{23}}{\Omega}\left[1-\dfrac{\Omega}{r_{23}}\sin\theta+\cdots\right], & \theta \simeq \dfrac{\pi}{2} \quad (3.13a) \\ \dfrac{r_{23}^2}{\Omega^2}\left[1-\dfrac{r_{23}\sin\theta}{\Omega}+\cdots\right], & \theta \simeq 0 \ . \quad (3.13b) \end{cases}$$

The higher-order terms in the expansion in Eq. (3.13) represent nonuniversal changes in the scaling function and do not effect the critical indices.

The expected crossover to the bulk behavior can also be seen from Eq. (3.3). In the limit y_2 and/or $y_3 >> |\vec{r}_{23}|$ Eqs. (3.3c) and (3.3d) become

$$k_1 \simeq 1 + \frac{r_{23}}{\sqrt{y_2y_3}} \ , \tag{3.14a}$$

$$k \simeq 2\frac{r_{23}}{\sqrt{y_2y_3}} \tag{3.14b}$$

Using the small k expansion in Eq. (3.3a) we see that there is no $\ln r_{23}$ piece to Eq. (3.2) in this limit, hence the critical index remains at the bulk value $x_\sigma = \frac{1}{8}$.

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