PHYSICAL REVIEW B VOLUME 25, NUMBER 5 1 MARCH 1982

Sound propagation in liquid helium near the lambda point: Thermodynamics

Richard A. Ferrell* and Jayanta K. Bhattacharjee†
Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742,
and Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
(Received 6 May 1981)

A new derivation, capable of straightforward finite-frequency generalization, is given of the velocity of sound in the Pippard-Buckingham-Fairbank approximation. The first correction to this approximation is expressed in a compact form and leads to a determination of the slope of the lambda line at low pressures. A crossover function for the critical specific heat is also obtained.

I. INTRODUCTION

Investigation of sound propagation near the lambda transition of liquid helium is a useful probe for both static and dynamic properties. In this work we study the thermodynamic sound velocity and the specific heat near the lambda point. Using the Pippard-Buckingham-Fairbank (PBF) relations,[1,2] it is found that superposed on a large noncritical part the sound velocity has a small critical part which is inversely proportional to the constant pressure specific heat C_P. Since C_P grows with the approach to the λ point,[3] the critical part gets smaller and the sound velocity dips. Precise measurements of the velocity v were made by Barmatz and Rudnick.[4] They plotted $v-v_m$, where v_m is the minimum velocity measured, against C_P^{-1}. The result is shown in Fig. 1. For $T>T_\lambda$, the lambda temperature, the data falls on a straight line as expected from the PBF approximation.[2] For $T<T_\lambda$, however, there is considerable deviation as one goes further away from the λ point. Ahlers[5] then considered the first correction to this linear approximation. Expressing his answer as a sum of several terms, he succeeded in bringing theory and experiment into full accord.

Our task here will be to provide a different derivation of the PBF result[2] and then consider the first correction to it using our new technique. The rederivation of the PBF equation is extremely important for the study of sound attenuation and dispersion. Our derivation of this thermodynamic relation makes the finite-frequency generalization[6,7] straightforward. The main points of the derivation are (i) the imposition of the isentropic condition of sound propagation at the outset, and (ii) the use of the variables $\Delta T = T - T_\lambda(P)$ and P instead of the conventional T and P, where P is the pressure. The first correction to the PBF approxi-

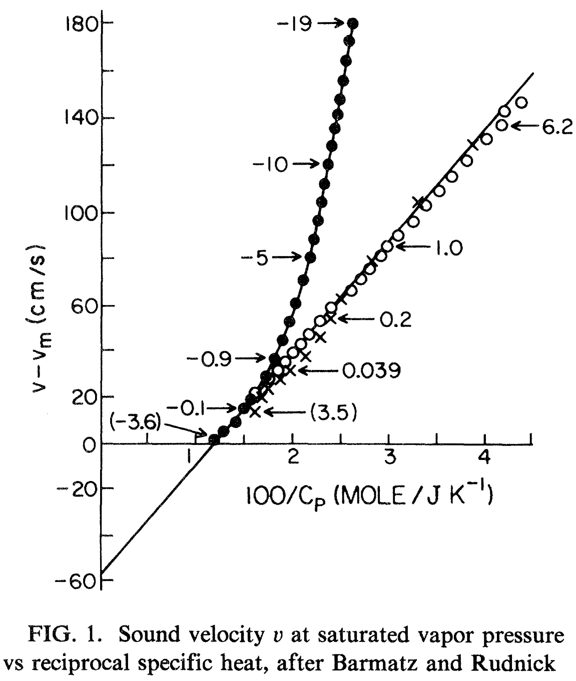

FIG. 1. Sound velocity v at saturated vapor pressure vs reciprocal specific heat, after Barmatz and Rudnick (Ref. 4). v_m is the minimum observed velocity. The temperatures indicated by the arrows are relative to the λ point and are measured in mK, except when in parentheses, in which case they are measured in μK. The straight line is that predicted by the Pippard-Buckingham-Fairbank relations (Ref. 2), while the curve includes the first-order correction (which can be neglected above the λ point).

mation[2] is then easily obtained. The result is simple, in contrast to Ahlers's[5] and allows a determination of the slope of the lambda line at low pressures. This determination is consistent with experiment.[8,9]

In Sec. II A we give the conventional derivation of the PBF equation. Section II B contains the new derivation, while in Sec. II C the first correction is computed. Finally in Sec. III, we present a crossover function for the critical specific heat. The crossover function becomes especially important at higher pressures[10] when the range of validity of the asymptotical form is smaller. Sec. IV constitutes a brief summary.

II. VELOCITY

A. Pippard-Buckingham-Fairbank relations

In this subsection, we use the conventional method to consider the critical temperature variation of the sound velocity v:

$$v^{-2}=mn\beta_S\ , \tag{2.1}$$

where m and n are the mass and density of the helium atoms, respectively. β_S is the isentropic compressibility and is related to the isothermal compressibility β_T by the thermodynamic identity

$$\beta_S=\beta_T-\frac{TV\alpha_P^2}{C_P}\ . \tag{2.2}$$

T, V, and C_P are the temperature, molar volume, and molar specific heat at constant pressure P, respectively. The thermal expansion coefficient is given by

$$V\alpha_P=\left[\frac{\partial V}{\partial T}\right]_P=-\left[\frac{\partial S}{\partial P}\right]_T \tag{2.3}$$

by virtue of a Maxwell relation. It is convenient to multiply the compressibilities by the molar volume, so that Eq. (2.2), upon substitution of Eq. (2.3), becomes

$$\left[\frac{\partial V}{\partial P}\right]_S-\left[\frac{\partial V}{\partial P}\right]_T=\frac{T}{C_P}\left[\frac{\partial S}{\partial P}\right]_T^2\ . \tag{2.4}$$

Near the lambda line, the entropy can be separated into its value at the lambda line plus its critical variation away from the lambda line,

$$S(T,P)=S_\lambda(P)+\Delta S(\Delta T,P)\ , \tag{2.5}$$

where

$$\Delta T=T-T_\lambda(P)\ . \tag{2.6}$$

$T_\lambda(P)$ is the temperature of the λ transition at pressure P. As the critical behavior depends upon ΔT, which is a function of both T and P, the following partial derivatives are needed:

$$\left[\frac{\partial \Delta T}{\partial P}\right]_T=-T'_\lambda\ , \tag{2.7a}$$

$$\left[\frac{\partial \Delta T}{\partial T}\right]_P=1\ . \tag{2.7b}$$

Following Pippard[1] and Buckingham and Fairbank,[2] we see that if a particular thermodynamic function depends upon pressure only through ΔT, Eqs. (2.7) can be combined into the equivalent equation

$$\left[\frac{\partial}{\partial P}\right]_T=-T'_\lambda\left[\frac{\partial}{\partial T}\right]_P\ . \tag{2.8}$$

Returning now to Eq. (2.4) we require the derivative

$$\begin{aligned}\left[\frac{\partial S}{\partial P}\right]_T&=S'_\lambda+\left[\frac{\partial \Delta S}{\partial P}\right]_T\simeq S'_\lambda+\left[\frac{\partial \Delta S}{\partial \Delta T}\right]_P\left[\frac{\partial \Delta T}{\partial P}\right]_T\\&=S'_\lambda-T'_\lambda\left[\frac{\partial \Delta S}{\partial T}\right]_P\\&=T'_\lambda\left[\left[\frac{dS}{dT}\right]_\lambda-\left[\frac{\partial S}{\partial T}\right]_P\right]\\&=T'_\lambda\left[\frac{C_\lambda}{T_\lambda}-\frac{C_P}{T}\right]\ ,\end{aligned} \tag{2.9}$$

where C_λ is the specific heat along the λ line. It is equal to the value of C_P at the temperature of maximum density (vanishing α_P). The first line of Eq. (2.9) uses Eq. (2.8) and neglects any pressure dependence other than that through ΔT. Substitution of Eq. (2.9) into (2.4) gives

$$\left[\frac{\partial V}{\partial P}\right]_S-\left[\frac{\partial V}{\partial P}\right]_T=T_\lambda'^2\left[\frac{C_P}{T}-2\frac{C_\lambda}{T_\lambda}+\frac{C_\lambda^2}{T_\lambda^2}\frac{T}{C_P}\right]\ . \tag{2.10}$$

In order to apply Eq. (2.10) to the calculation of the critical temperature dependence of the velocity of sound we need an expression for the isothermal compressibility. For this we break up the volume into its value at the lambda line plus the deviation away from the lambda line [as we did in Eq. (2.5) for the entropy]:

$$V(T,P)=V_\lambda(P)+\Delta V(T,P)\ . \tag{2.11}$$

The required pressure derivative is

$$\begin{aligned}\left[\frac{\partial V}{\partial P}\right]_T &= V'_\lambda+\left[\frac{\partial \Delta V}{\partial P}\right]_T\\ &\simeq V'_\lambda - T'_\lambda\left[\frac{\partial V}{\partial T}\right]_P\\ &= V'_\lambda + T'_\lambda\left[\frac{\partial S}{\partial P}\right]_T\\ &\simeq V'_\lambda + T'^{\,2}_\lambda\left[\frac{C_\lambda}{T_\lambda}-\frac{C_P}{T}\right]\ ,\end{aligned} \tag{2.12}$$

from Eqs. (2.8), (2.3), and (2.9). Substitution of (2.12) into Eq. (2.10) yields the desired isentropic derivative

$$\left[\frac{\partial V}{\partial P}\right]_S = V'_\lambda - T'^{\,2}_\lambda\frac{C_\lambda}{T_\lambda}+T'^{\,2}_\lambda\frac{C_\lambda^2}{T_\lambda^2}\frac{T}{C_P}\ . \tag{2.13}$$

Because of the small value of the critical specific-heat index, it will be sufficiently accurate for the present purposes to set it equal to zero. The associated logarithmic behavior, which is studied in detail in Sec. III, causes C_P to diverge at the λ point. The last term in Eq. (2.13) consequently vanishes and Eq. (2.1) determines the limiting sound velocity u_λ by

$$\begin{aligned}u_\lambda^{-2} &= nm\beta_{S,\lambda} = -\frac{nm}{V_\lambda}\left[\frac{\partial V}{\partial P}\right]_{S,\lambda}\\ &= \frac{nm}{V_\lambda}\left[-V'_\lambda + T'^{\,2}_\lambda\frac{C_\lambda}{T_\lambda}\right]\ .\end{aligned} \tag{2.14}$$

u_λ, as determined by Eq. (2.14), can be regarded as a convenient extrapolation. The fact that a more accurate treatment of C_P would give a finite limiting value at the λ point will be of no consequence for the subsequent work.

Separating the velocity into its lambda-point value and its deviation from this value,

$$v=u_\lambda+u_1\ , \tag{2.15}$$

we see that the fractional increase in the sound velocity is given according to Eq. (2.13) by

$$\begin{aligned}\frac{u_1}{u_\lambda} &\simeq -\frac{T'^{\,2}_\lambda}{2V'_\lambda}\frac{C_\lambda^2}{T_\lambda^2}\frac{T}{C_P}\simeq -\frac{T'^{\,2}_\lambda C_\lambda}{2V'_\lambda T_\lambda}\frac{C_\lambda}{C_P}\\ &= -\frac{T'_\lambda}{2}\left[\frac{dS}{dV}\right]_\lambda\frac{C_\lambda}{C_P}\ .\end{aligned} \tag{2.16}$$

Here we have neglected the small second term of the right-hand member of Eq. (2.13), which is less than 1% of the first term. We have also approximated $T\simeq T_\lambda$. The coefficient of C_λ/C_P in Eq. (2.16) is a kind of dimensionless coupling constant

$$K_0=-\frac{T'_\lambda}{2}\left[\frac{dS}{dV}\right]_\lambda\ . \tag{2.17}$$

Thus, Eq. (2.16) takes on the simple form

$$\frac{u_1}{u_\lambda}=K_0\frac{C_\lambda}{C_P}\ . \tag{2.18}$$

Equation (2.18) is equivalent to the expression derived in Ref. 2 for the critical temperature variation of the sound velocity. The linearization employed here depends upon the smallness of K_0. A rough qualitative argument leads us to expect $K_0 \ll 1$. Because of the steepness of the λ line we can approximate $(dS/dV)_\lambda$ by $(\partial S/\partial V)_T$, which equals $(\partial P/\partial T)_V$ by Maxwell's relation. For this, the classical perfect-gas law gives the order-of-magnitude estimate P/T. This amounts to 15 atm/K at the top of the λ line. The steepness of the λ line corresponds to $-T'^{\,-1}_\lambda \simeq 10^2$ atm/K. Equation (2.17) therefore gives $K_0=O(7.5\times10^{-2})$, or one order of magnitude smaller than unity. Reference to the thermodynamic data shows that K_0 is even smaller than this qualitative estimate by one more order of magnitude. For saturated vapor pressure (SVP) we find $K_0=8.8\times10^{-3}$.

B. Isentropic condition

As seen above, the adiabatic constraint of constant entropy has a drastic effect on the critical behavior of the compressibility. The leading term in Eq. (2.12) for the isothermal compressibility is canceled by the leading term in Eq. (2.10), so that the isentropic compressibility has a much weaker critical behavior, inversely proportional to C_P. Subsequent applications of this paper will depend upon a frequency-dependent generalization of this weak critical behavior. It might naturally be argued that the cancellation might not be as complete at finite frequency, which might introduce some uncertainty into the generalization. For this reason we present in this subsection a derivation of Eq. (2.13) that does not involve any cancellation. This preferred derivation is based upon choosing as the pair of independent thermodynamic variables

$\Delta T, P$ instead of T, P.

The resulting improvement in the analysis is already apparent in the dependence of the entropy on the new pair of variables. For arbitrary variation of ΔT and P we find

$$\delta S = \delta S_\lambda + \delta \Delta S \simeq S'_\lambda \delta P + \frac{C_P}{T_\lambda} \delta \Delta T \,, \tag{2.19}$$

where T has been approximated by T_λ and the small effect of the explicit pressure dependence of ΔS has been neglected [i.e., $(\partial \Delta S/\partial P)_{\Delta T} \simeq 0$]. Imposing the isentropic condition

$$\delta S = 0 \tag{2.20}$$

requires that the variation of ΔT be related to that of the pressure by

$$\delta \Delta T = -\frac{T_\lambda S'_\lambda}{C_P} \delta P \,. \tag{2.21}$$

Equation (2.21) illustrates how this section differs in an essential way from the preceding. The numerator of the right-hand member of Eq. (2.21) is noncritical, so that as $T \to T_\lambda$, $C_P \to \infty$ and the fluctuation in ΔT vanishes. This shows the advantage of working with the variable ΔT rather than T. It is easy to see that in the latter case the coefficient of δP in Eq. (2.19) acquires an additional term proportional to C_P, so that $\delta T / \delta P$ remains finite as $T \to T_\lambda$. It is evident from Eq. (2.21) that in this limit the temperature variation corresponds to a displacement in the T-P plane which is parallel to the λ line, and therefore noncritical. By working with the variable ΔT, we automatically eliminate this parallel displacement from consideration and concentrate on the true critical displacement away from the λ line. From Eq. (2.21) we obtain the isentropic derivative

$$\left[\frac{\partial \Delta T}{\partial P}\right]_S = -\frac{T_\lambda S'_\lambda}{C_P} \,. \tag{2.22}$$

To obtain the compressibility we need an expression for V. This we can get from the Gibb's function $G = G_\lambda(P) + \Delta G$. ΔG is related to the entropy (assumed known throughout the $\Delta T, P$ plane) by

$$S = -\left[\frac{\partial G}{\partial T}\right]_P = -\left[\frac{\partial \Delta G}{\partial \Delta T}\right]_P \,. \tag{2.23}$$

Integration gives

$$\Delta G(\Delta T, P) = -\langle S \rangle \Delta T = -S_\lambda \Delta T - \langle \Delta S \rangle \Delta T \simeq -S_\lambda(P) \Delta T \,, \tag{2.24}$$

where the higher-order term involving the average of ΔS over the interval ΔT can be dropped. The remaining derivative that we need is therefore

$$\left[\frac{\partial \Delta G}{\partial P}\right]_{\Delta T} = -S'_\lambda \Delta T \,, \tag{2.25}$$

so that

$$V = \left[\frac{\partial G}{\partial P}\right]_T = G'_\lambda + \left[\frac{\partial \Delta G}{\partial P}\right]_T = G'_\lambda + \left[\frac{\partial \Delta T}{\partial P}\right]_T \left[\frac{\partial \Delta G}{\partial \Delta T}\right]_P + \left[\frac{\partial \Delta G}{\partial P}\right]_{\Delta T} = G'_\lambda + T'_\lambda S - S'_\lambda \Delta T \,, \tag{2.26}$$

by substitution of Eqs. (2.23) and (2.25). In the isentropic derivative of Eq. (2.26), we impose Eq. (2.20) so that only the second term contributes by virtue of the pressure dependence of T'_λ. In other words, we keep S equal to a constant, which for the present purposes can be approximated by S_λ. The variation of S'_λ in the last term is a higher-order contribution that we also neglect for the moment. Thus, differentiating Eq. (2.26), we obtain

$$\left[\frac{\partial V}{\partial P}\right]_S \simeq G''_\lambda + T''_\lambda S_\lambda - S'_\lambda \left[\frac{\partial \Delta T}{\partial P}\right]_S \,. \tag{2.27}$$

Returning to Eq. (2.26) we note

$$V_\lambda = G'_\lambda + T'_\lambda S_\lambda \,. \tag{2.28}$$

Substituting Eq. (2.21) and the derivative of Eq. (2.28) into Eq. (2.27) gives

$$\left[\frac{\partial V}{\partial P}\right]_S = V'_\lambda - T'_\lambda S'_\lambda + \frac{T_\lambda S'^2_\lambda}{C_P} \,, \tag{2.29}$$

in complete agreement with Eq. (2.13). The present derivation of this result has not depended on any cancellation and is therefore more suited to generalization at finite frequency.

C. First-order correction

The preceding derivation of the sound velocity is valid only in the immediate vicinity of the λ point. This restriction is due to the simplifying approximations introduced in going from Eq. (2.26) to (2.29). Equation (2.26) is accurate to first order in ΔT. The same accuracy can be achieved for Eq. (2.29) by keeping all of the first-order terms which result from differentiating Eq. (2.26). Two such terms come into question. One of these is $-S_\lambda'' \Delta T$ from the last term. The second term, $T_\lambda''(S-S_\lambda)=T_\lambda'' \Delta S$, gives a larger correction because of the logarithmically diverging specific heat. As both of these terms vanish at the λ point, u_λ is unaffected by them. But Eq. (2.15) has to be amended to read

$$v=u_\lambda+u_1+\Delta u \ , \tag{2.30}$$

where the additional fractional increase in the sound velocity is

$$\frac{\Delta u}{u_\lambda}=-\frac{1}{2V_\lambda'}(T_\lambda'' \Delta S-S_\lambda'' \Delta T) \ . \tag{2.31}$$

We have [as in Eq. (2.16)] again neglected the small term $-T_\lambda' S_\lambda'$ in the denominator. Equation (2.31) is equivalent to some results of Ahlers. Ahlers's[5] derivation is, however, more complicated than that given here because he splits up the second term of the right-hand member of Eq. (2.26) as $T_\lambda' S_\lambda + T_\lambda' \Delta S$. In the ensuing differentiation both S_λ and ΔS contribute. But all such terms must in the final analysis cancel because of the isentropic "clamping" condition, Eq. (2.20).

Because of finite-temperature resolution and the effect of gravity, the sharp dip of the sound velocity at the λ point is observable only to a limited extent. In any given experiment u_λ is not directly measurable, but only some minimum velocity v_m. It is therefore convenient to rewrite Eq. (2.30) as

$$\begin{aligned} v-v_m &= u_\lambda - v_m + u_1 + \Delta u \\ &= u_\lambda - v_m + u_\lambda \left[\frac{u_1}{u_\lambda} + \frac{\Delta u}{u_\lambda} \right] \\ &= u_\lambda - v_m + u_\lambda \left[K_0 \frac{C_\lambda}{C_P} + \frac{\Delta u}{u_\lambda} \right] , \end{aligned} \tag{2.32}$$

where we have substituted Eq. (2.18). It still remains to substitute Eq. (2.31) for the last term. In the latter step we note that Eq. (2.31) makes a negligible contribution to Eq. (2.32) for $T>T_\lambda$. This is mainly due to the fact that for a given value of $|\Delta T|$, C_P is much smaller for $T>T_\lambda$ than for $T<T_\lambda$. The last term in parentheses is consequently reduced while the first term is increased. To a good approximation the last term can be dropped entirely for $T>T_\lambda$, giving the plot of $v-v_m$ vs C_P^{-1} shown by the straight line in Fig. 1 and based on the parameters $u_\lambda=218$ m/sec, $C_\lambda=24.0$ J/mole K, and $K_0=8.8\times10^{-3}$. The data of Barmatz and Rudnick[4] for $T>T_\lambda$ are shown by the crosses and the hollow circles (only a portion of the data points are shown). The values of ΔT indicated by the arrows are given in mK except for the lowest one on the figure which is enclosed in parentheses and which is given in μK. The vertical intercept $u_\lambda - v_m = -56$ cm/sec has been adjusted to give a best fit to the data.

Below the λ point the last term of Eq. (2.32) cannot be neglected, as can be seen by the strong deviation of the Barmatz-Rudnick data from the straight line of Fig. 1. A portion of these points is shown by the solid circles. In attempting to account for this deviation by substituting Eq. (2.31) into Eq. (2.32), we are frustrated by the fact that the SVP value of T_λ'' is not accurately enough known for this purpose. We therefore adopt the point of view that the Barmatz-Rudnick measurements provide the most accurate determination of this parameter. The choice $T_\lambda''=-7.69\times10^{-4}$ K/atm^{-2} gives the good fit shown by the curve in Fig. 1.

The value of T_λ'' found above can be compared with Ahlers's[8,9] measurements of the pressure dependence of the slope of the λ line, as shown in Fig. 2. These data are consistent with those of

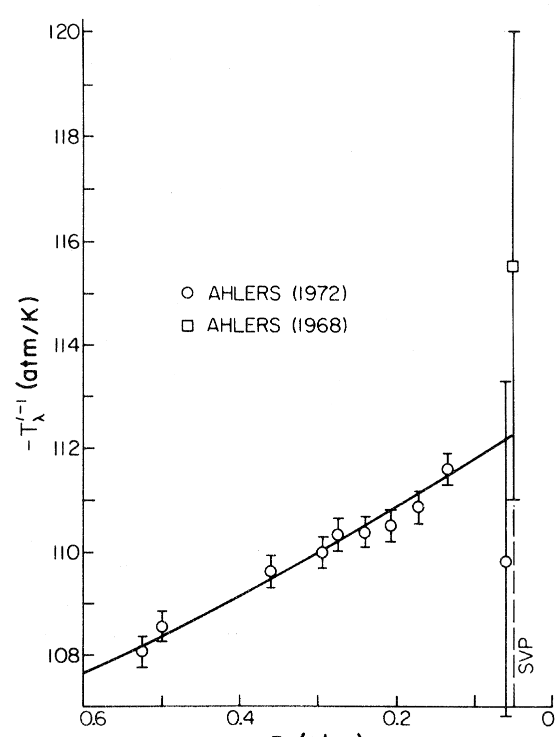

FIG. 2. Pressure dependence of the slope of the λ line. The slope of the curve at its upper end is fixed by the value of the second derivative T_λ'' found at $P_\lambda=0.05$ atm (saturated vapor pressure) from the first-order correction of Fig. 1.

Elwell and Meyer[11] and of Kierstead[12] at higher pressures. The curve has been constructed to fit Ahlers's data and to have a slope at its right-hand end which corresponds to the above value of T_λ''. The modest upwards curvature which is evident in Fig. 2 is necessary in order for the latter "boundary condition" to be satisfied. The intercept at $P_\lambda=0.05$ atm (SVP) yields $T_\lambda'^{-1}=-112.3$ atm/K.

III. SPECIFIC HEAT

As is evident in Eq. (2.18), the specific heat plays a key role in determining the critical variation of the sound velocity. This section is devoted to some of the important features of the specific heat for $T>T_\lambda$. Within the critical region the specific heat is well described by the logarithmic expression

$$C_P=A\ln(t_0/t)=\tfrac{3}{4}AL_{\rm TH}\ , \tag{3.1}$$

where $t=\Delta T/T_\lambda$, $t_0=0.25$, $A=5.33$ J/K mole, and

$$L_{\rm TH}\equiv\ln(t_0/t)^{4/3}\ . \tag{3.2}$$

Substitution of Eq. (3.1) into Eq. (2.18) gives

$$\frac{u_1}{u_\lambda}=\frac{C_1}{L_{\rm TH}}\ , \tag{3.3}$$

where the "coupling constant" in this alternative form is

$$C_1=\frac{4}{3}\frac{K_0C_\lambda}{A}\ . \tag{3.4}$$

At saturated vapor pressure (SVP) we find $C_1^{\rm SVP}=5.3\times10^{-2}$. Figure 3 shows the variation of C_1 along the λ line. C_1 at first drops with rising pressure, and then exhibits a minimum approximately 25% below $C_1^{\rm SVP}$, at about halfway up the λ line. The subsequent rise higher up on the λ line almost restores C_1 to its full $C_1^{\rm SVP}$ value.

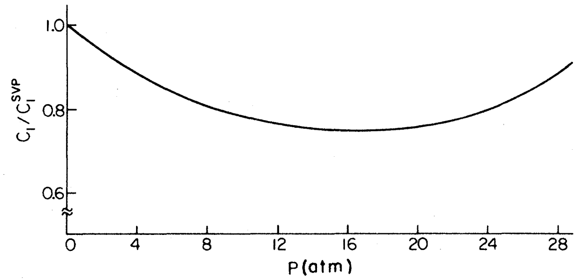

FIG. 3. Pressure dependence of the coupling constant C_1 relative to its value at saturated vapor pressure.

In our subsequent work we will need an expression for C_P valid outside the limited range of Eq. (3.1). A complete theory of C_P including the crossover from critical to noncritical behavior is not available at the present time. Therefore, we are forced to resort to a simple model in which, added to a noncritical background contribution $B(t)$ to $L_{\rm TH}$, we have a critical contribution $L_{\rm TH}^c$ resulting from fluctuations in the order parameter. We calculate the latter in lowest order in the ϵ expansion, i.e., for four-dimensional space. This approximation corresponds to a single-loop Feynman graph involving the Ornstein-Zernike order-parameter correlation function

$$g(p,\kappa)=\frac{1}{p^2+\kappa^2}\ , \tag{3.5}$$

where p is the wave number of a fluctuation and κ^{-1} is the correlation length. The integration over all fluctuations is restricted by a Debye cutoff p_D. In carrying out the integration, it is convenient to replace the integration variable p by the thermodynamic stiffness

$$\gamma'=g^{-1}=p^2+\kappa^2\ . \tag{3.6}$$

Thus $2pdp=d\gamma'$ and the minimum value of γ' is

$$\gamma=g^{-1}\,|_{p=0}=\kappa^2\ . \tag{3.7}$$

We represent the cutoff by

$$\gamma_D=p_D^2\ , \tag{3.8}$$

so that the upper limit of γ' is $\gamma_D+\gamma$. In this notation the single-loop integral becomes

$$\begin{aligned}L_{\rm TH}^c&=\frac{1}{\pi^2}\int d^4p\,g^2(p,\kappa)=2\int_0^{p_D}\frac{p^3dp}{(p^2+\kappa^2)^2}\\&=\int_\gamma^{\gamma_D+\gamma}\frac{d\gamma'}{\gamma'}\left[1-\frac{\gamma}{\gamma'}\right]\\&=\ln\left[1+\frac{\gamma_D}{\gamma}\right]-\left[1+\frac{\gamma}{\gamma_D}\right]^{-1}\\&=\ln\left[1+\frac{p_D^2}{\kappa^2}\right]-\left[1+\frac{\kappa^2}{p_D^2}\right]^{-1}\ ,\end{aligned} \tag{3.9}$$

where the last line follows from substitution of Eqs. (3.6)—(3.8). The temperature dependence of $L_{\rm TH}^c$ is now brought in by

$$\kappa=\kappa_0t^\nu\ , \tag{3.10}$$

where $\kappa_0=0.7\times10^8$ cm^{-1} and the critical exponent is $\nu \simeq \frac{2}{3}$ because of the smallness of α. It is convenient to define a parameter t_D corresponding to the Debye cutoff by

$$p_D = \kappa_0 t_D^{2/3} , \tag{3.11}$$

so that

$$\frac{p_D^2}{\kappa^2} = \left[\frac{t_D}{t} \right]^{4/3} . \tag{3.12}$$

Substituting Eq. (3.12) into (3.9) we find

$$\begin{aligned} L_{\mathrm{TH}} &= L_{\mathrm{TH}}^c + B(t) \\ &= \ln \left[1 + \left[\frac{t_D}{t} \right]^{4/3} \right] - \left[1 + \left[\frac{t}{t_D} \right]^{4/3} \right]^{-1} + B(t) \\ &\simeq \ln \left[\frac{t_D}{t} \right]^{4/3} + B_0 - 1 , \end{aligned} \tag{3.13}$$

where $B_0 \equiv B(0)$. The last line holds in the critical region $t << t_D$. Comparison with Eq. (3.2) imposes on t_D and B_0 the constraint

$$t_D = t_0 \exp[-\tfrac{3}{4}(B_0 - 1)] . \tag{3.14}$$

Once B_0 is chosen, t_D is fixed by the known value of t_0.

The specific-heat data[3] for saturated vapor pressure (not significantly different from C_P) are reproduced in Fig. 4. We have found that Eq. (3.13) gives the good fit shown by the solid curve when we choose for $B(t)$ the linear background function

$$B(t) = B_0 + B_1 t , \tag{3.15}$$

with $B_0 = 1.46$ and $B_1 = 3.03$. Equation (3.15) is shown in Fig. 4 by the dashed curve. Although there is no rigorous theory with which Eq. (3.15) can be compared, the free roton gas provides a rough guide. The reasonableness of the simple linear form of Eq. (3.15) is supported by the fact that Landau's formula[13] for the specific heat of the roton gas has a point of inflexion in the vicinity of the λ point and that it varies more or less linearly in that region. Furthermore, the intercept of the tangent at the point of inflexion is -0.43, close to the value $t = -B_0/B_1 = -0.48$ found here.

In closing this section it is useful to discuss some limiting forms of Eq. (3.9). For $\kappa >> p_D$,

$$L_{\mathrm{TH}}^c \simeq 1/2 \frac{p_D^4}{\kappa^4} , \tag{3.16}$$

the numerator being essentially the volume of phase space available for the fluctuations of the order parameter (in this lowest-order ϵ expansion model). Equation (3.16) is obtained more directly by approximating the strength of the fluctuations by

$$g(p,\kappa) \simeq \kappa^{-2} , \tag{3.17}$$

and taking the factor g^2 outside the integral. As the temperature is lowered the correlation length κ^{-1} increases and the fluctuational contribution to the specific heat as expressed by Eq. (3.16) rises rapidly. But in the crossover region $\kappa \simeq p_D$, Eq. (3.17) is no longer a satisfactory approximation. Now the more correct Ornstein-Zernike formula, which limits the fluctuation strength by p as well as by κ, has to be used in the integration, leading to Eq. (3.9). This results in a slower rise than indicated by Eq. (3.16). Further lowering of the temperature brings ultimately the much weaker true critical variation

$$L_{\mathrm{TH}}^c = \ln \frac{p_D^2}{\kappa^2} - 1 \tag{3.18}$$

for $\kappa << p_D$, corresponding to a scaling behavior with critical exponent $\alpha = 0$. Equation (3.9) is thus a crossover function which smoothly connects the two extreme cases of Eqs. (3.16) and (3.18). It is based on the simple physical idea of phase-space limitation, as quantitatively represented by the Debye cutoff p_D. This is similar to the approach that has been applied successfully to the crossover behavior of the fluctuation-enhanced viscosity in a

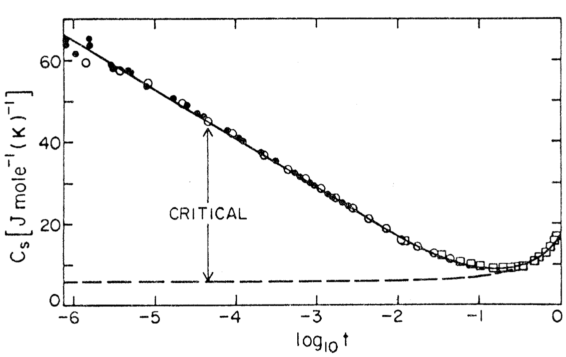

FIG. 4. Specific heat at saturated vapor pressure vs logarithm of reduced temperature $t = (T/T_\lambda) - 1$. The dashed line shows the noncritical background, with linear temperature dependence. The surplus specific heat above background is identified as the critical portion. The best fit, shown by the solid curve, determines the upper cutoff parameter of the theory.

classical fluid.[14]

This type of approach is adequate for describing the general overall trend, but cannot be expected to yield a precise crossover function, accurate in every respect. An important shortcoming of Eq. (3.9) is revealed by the Taylor's series for $\kappa << p_D$. Separating off the scaling form by subtracting Eq. (3.18) yields the corrections to scaling

$$L_{\mathrm{TH}}^{c}-\left[\ln\frac{p_D^2}{\kappa^2}-1\right]=2\frac{\kappa^2}{p_D^2}-\frac{3}{2}\frac{\kappa^4}{p_D^4}+\cdots . \tag{3.19}$$

Equation (3.19) would predict the leading correction to have the temperature dependence t^{Δ_1} with $\Delta_1=2\nu=1.33$, whereas the renormalization group yields[15] $\Delta_1=0.5$. This discrepancy in the value of Δ_1 indicates that Eq. (3.13) would not be appropriate for close comparison with the first deviation from scaling, as the true correction to scaling should be more singular.[16] Such a fine detail will not show up, however, in a plot such as Fig. 4 of the specific heat versus temperature over the entire temperature range. Therefore we believe that Eq. (3.13), although an approximation, is nevertheless a sufficiently accurate crossover function to be useful for describing the general trend of the "quenching" of the critical specific heat as the temperature is raised. The determination of a more accurate crossover function will require the extension of the renormalization-group calculation to take into account in a realistic way both the Debye cutoff and the molecular-field-type precritical rise.[17–19]

IV. SUMMARY

We have derived the velocity of first sound near the λ transition of ^{4}He by imposing the isentropic condition at the outset. The derivation has also been simplified by considering the variables $T-T_\lambda(P)$ and P instead of the conventional T and P. The conventional derivation of the PBF result,[2] Eq. (2.13), requires cancellation between $(\partial V/\partial P)_T$ and parts of $TV\alpha_P^2/C_P$. While this cancellation can be proven in the thermodynamic limit treated in this paper, it is not guaranteed in the finite-frequency version of Eq. (2.13), which is required for studying the dispersion and attenuation.[6,7] Consequently we have provided a derivation of Eq. (2.13) in Sec. II B that does not depend on any cancellation and lends itself to frequency-dependent generalization. This method of deriving the sound velocity is relevant not only for the λ transition but also for the liquid-gas transition[20,21] and the binary-liquid consolute-point transition.[22,23]

The technique of obtaining the sound velocity presented in Sec. II B has been used to obtain the first correction to the PBF result in Sec. II C. The result is very simple. The calculation is more direct and concise than that of Ahlers.[5] The large-scale cancellation of the various terms in his calculation is taken into account *ab initio* by clamping the entropy. Also we obtained an approximate crossover function for the specific heat. The crossover effect is important at higher pressure where the range of validity of the asymptotic logarithmic form is smaller.

ACKNOWLEDGMENTS

This work was done at the Institute for Theoretical Physics, University of California, Santa Barbara. We acknowledge support from the National Science Foundation by Grant Nos. PHY 77-27084 to the Institute and DMR 79-01172, 79-00980, and 79-10819 to the University of Maryland.

---

*Permanent address: Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742.

†Present address: Institut fuer Festkoerperforschung der Kernforschungsanlage Juelich, D-5170 Juelich, West Germany.

[1]A. B. Pippard, Philos. Mag. 1, 473 (1956).

[2]M. J. Buckingham and W. M. Fairbank, in *Progress in Low Temperature Physics,* edited by C. J. Gorter (North-Holland, Amsterdam, 1961), Vol. III, p. 80.

[3]G. Ahlers, Phys. Rev. A 3, 696 (1971).

[4]M. Barmatz and I. Rudnick, Phys. Rev. 170, 224 (1968).

[5]G. Ahlers, Phys. Rev. 182, 352 (1969).

[6]R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. Lett. 44, 403 (1980).

[7]R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. B 23, 2434 (1981).

[8]G. Ahlers, J. Low Temp. Phys. 7, 361 (1962).

[9]G. Ahlers, Phys. Rev. 171, 275 (1968).

[10]R. A. Ferrell and J. K. Bhattacharjee (unpublished).

[11]D. L. Elwell and H. Meyer, Phys. Rev. 164, 245

(1967).

[12]H. A. Kierstead, Phys. Rev. 138, A1594 (1965).

[13]L. D. Landau, J. Phys. (Moscow) 11, 91 (1947).

[14]J. K. Bhattacharjee, R. A. Ferrell, R. S. Basu, and J. V. Sengers, Phys. Rev. A 24, 1469 (1981).

[15]J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. Lett. 39, 95 (1977); Phys. Rev. B 21, 3976 (1980).

[16]For a review of the comparison of experiment with theory see J. V. Sengers, *Proceedings of the 1980 Cargese Summer Institute on Phase Transitions,* edited by M. Levy, J. C. Le Guillou, and J. Zinn-Justin (Plenum, New York, 1981), p. 95.

[17]R. A. Ferrell, in *Proceedings of the Conference on Fluctuations in Superconductors, Asilomar, 1968,* edited by W. S. Goree and F. Chilton (Stanford Research Institute, Menlo Park, California, 1968), pp. 239–249.

[18]R. A. Ferrell, *Contemporary Physics,* (International Atomic Energy Agency Vienna, 1969), pp. 129–156.

[19]R. A. Ferrell, J. Low Temp. Phys. 1, 241 (1969).

[20]J. K. Bhattacharjee and R. A. Ferrell, Phys. Lett. A (in press).

[21]R. A. Ferrell and J. K. Bhattacharjee, Phys. Lett. 86A, 109 (1981).

[22]R. A. Ferrell and J. K. Bhattacharjee, Phys. Rev. B 24, 4095 (1981).

[23]J. K. Bhattacharjee and R. A. Ferrell, Phys. Rev. A 24, 1643 (1981).