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Electrostriction in liquid ^{4}He

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The two-fluid model of liquid ^{4}He is extended to include effects of applied electric fields. New terms that account for electrostriction in equations of motion for superfluid and normal fluid are derived with the aid of a variational principle subject to the assumption that the Clausius-Mossotti relation for the dielectric constant is applicable. Corresponding additions to conservation laws for momentum and energy are presented.

I. INTRODUCTION

An electric field applied to a collection of neutral atoms will induce dipole moments in them, subsequently interact with those moments, and thereby produce ponderomotive[1,2] forces on the dielectric material. Initially the effect of the forces is to cause relative motion of parts of the dielectric medium. Deformation of the dielectric in this way will give rise to elastic forces opposing the motion. The motion ceases when the elastic forces exactly balance the electric forces. The phenomenon of the production of stresses and strains in this way in an uncharged insulator is called *electrostriction*,[3] a term which has usually been reserved for static conditions in the past. The theory of electrostriction for chemically homogenous fluids obeying classical laws was established long ago. Inclusion of the effects of ponderomotive forces in the equations of motion of such fluids, describing nonstatic conditions in general, would involve only a simple step of adding more terms[2] to the divergence of the stress tensor, supplementing the pressure and viscous terms. A name is needed for such effects. Henceforth in this paper *electrostriction* will be used with a broader meaning than before, referring to any mechanical effects of ponderomotive forces even under nonstatic steady-state and time-varying conditions.

The theory of electrostriction in this generalized sense for a quantum liquid characterized by a two-fluid model appears not to have been given before. This paper treats the least complex system of that type, viz., liquid ^{4}He. Only relatively simple additions to existing theory are needed to incorporate electrostriction in the equations of motion and conservation laws in this case.

In general, the Lagrangian for the system of fluid plus electromagnetic fields is composed of three types of terms. The first describes the fluid even when the fields are absent. The second is associated with the energy of the electromagnetic fields including interaction of fields with bound charges and currents in the system. The third type accounts for interaction between fields and any free charges or currents that are present. Meservey[4] has given the formula for the Lagrangian including all three kinds of terms for a superconductor characterized by a two-fluid model. That formula is an extension of a Lagrangian written down earlier on phenomenological grounds by Zilsel[5] for liquid ^{4}He without electromagnetic fields present. Merservey's theory can be readily adapted to describe liquid ^{4}He interacting with electric fields alone, the problem that concerns us now.

In this instance terms of type three are absent. Furthermore, it is useful and instructive to consider the special case in which

$$\vec{E} = -\vec{\nabla}\phi , \tag{1}$$

where $\vec{E}$ is the macroscopic electric field and ϕ is the scalar potential. Recently it has been demonstrated that Zilsel's Langrangian for the two-fluid model of liquid ^{4}He alone follows from microscopic theory[6] and that for conditions of uniform flow this Lagrangian itself is a thermodynamic potential.[7] That is, under the stated conditions the Lagrangian is a Legendre transform of the internal energy. The primary independent variables of the potential were exhibited in the process of establishing that result. This makes it possible for one to circumvent[7] controversial issues[8-11] that clouded Zilsel's phenomenological treatment of the two-fluid equations for many years.

These observations and results provide a basis for deriving equations for the two-fluid model of liquid ^{4}He including electrostriction in the next section.

II. DERIVATION OF THE TWO-FLUID EQUATIONS FOR LIQUID ^{4}He WITH ELECTROSTRICTION

In the absence of applied fields the Lagrangian and its differential are given by the following formulas[7] for liquid ^{4}He when superfluid and normal fluid flow velocities are uniform throughout the liquid:

$$L=\vec{u}\cdot\vec{P}_0+\vec{v}_n\cdot\vec{P}\,'-E' , \tag{2a}$$

$$dL=pdV-TdS+\vec{P}\,'\cdot d\vec{v}_n-\mu dN+\vec{P}_0\cdot d\vec{u} , \tag{2b}$$

where E' is the internal energy of the flowing liquid, $\vec{P}_0$ is the correlated momentum, $\vec{u}$ is the velocity conjugate to $\vec{P}_0$, $\vec{P}\,'$ is the total momentum, $\vec{v}_n$ is the normal fluid velocity, V is the volume, p is the pressure, S is the entropy, T is the temperature, N is the number of ^{4}He atoms, and μ is the chemical potential per ^{4}He atom. Microscopic theory[6,7] provides explicit formulas for L, E', $\vec{P}_0$, $\vec{u}$, $\vec{P}\,'$, p, S, and μ. It should be noted that the correlated momentum $\vec{P}_0$ is related to superfluid velocity $\vec{v}_s$ by

$$\vec{P}_0=Nm\vec{v}_s , \tag{3}$$

where m is the mass of a ^{4}He atom. The problem discussed in previous work[6,7] of determining whether $\vec{P}_0$, an extensive quantity, or $\vec{v}_s$, an intensive quantity, is the proper primary independent variable in L has been resolved in favor of $\vec{P}_0$, but the arguments will not be given here. It is useful to introduce Lagrangian densities l and $\bar{l}$ having the following properties:

$$l=\frac{M}{V}\frac{L}{M}=\rho\bar{l} , \tag{4}$$

$$dl=\bar{l}d\rho+\rho d\bar{l} , \tag{5}$$

$$d\bar{l}=-\frac{p}{\rho^2}d\rho-Tds+\vec{j}\cdot d\vec{v}_n+\vec{v}_s\cdot d\vec{u} , \tag{6}$$

where

$$s=\frac{S}{M} , \tag{7}$$

$$\vec{j}=\frac{\vec{P}\,'}{M} . \tag{8}$$

It has been established that the Lagrangian density l in Eq. (4) can be expressed in the form postulated by Zilsel,[5] viz.,

$$l=\rho[\tfrac{1}{2}(1-x)v_s^2+\tfrac{1}{2}xv_n^2-e] , \tag{9}$$

where

$$x=\frac{M_n}{M}=\frac{\rho_n}{\rho} , \tag{10}$$

M_n is the normal fluid mass, $M=Nm$ is the total mass of ^{4}He, and ρ_n and ρ are normal fluid density and total mass density, respectively. The condition that superfluid density ρ_s satisfies $\rho=\rho_n+\rho_s$ is implicit in Eq. (9). The intrinsic internal energy per unit mass is e. Formulas for ρ_n and e are available from earlier work.[6,7] That same work led to definition of intrinsic chemical potential per unit mass, z, and formulas for z and dz. It was found that

$$dz=\frac{1}{\rho}dp-sdT-\tfrac{1}{2}xd(\vec{v}_n-\vec{v}_s)^2 . \tag{11}$$

The equation of motion for superfluid is particularly simple when expressed with the aid of Eq. (11), as we shall see later. The formula for l was derived for conditions of thermodynamic equilibrium and uniform flow, but in applications here it will be assumed to hold also when those conditions are met only approximately and locally.

This Lagrangian density can be used in Eckart's variational principle[12,5,13] to derive equations of motion if dissipation is neglected. Conservation laws for mass and entropy, viz.,

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{j})=0 , \tag{12}$$

and

$$\frac{\partial(\rho s)}{\partial t}+\vec{\nabla}\cdot(\rho s\vec{v}_n)=0 , \tag{13}$$

where

$$\rho\vec{j}=(\rho_s\vec{v}_s+\rho_n\vec{v}_n)=\rho(\vec{u}+\vec{v}_n) , \tag{14}$$

are imposed as constraints with the aid of Lagrange multipliers α and β when the independent variables of the Lagrangian density are varied. When an electrical potential[4] ϕ, treated as an additional independent variable, and a gravitational potential[13] per unit mass $\bar{\phi}$ are present, Eckart's principle takes the form

$$0=\delta\int_{t_0}^{t_1}dt\int_V d^3r\left[\bar{l}-\rho\bar{\phi}+\frac{\epsilon}{2}(-\vec{\nabla}\phi)^2-\alpha\left[\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot[\rho(\vec{u}+\vec{v}_n)]\right]-\beta\left[\frac{\partial(\rho s)}{\partial t}+\vec{\nabla}\cdot(\rho s\vec{v}_n)\right]\right]. \tag{15}$$

The volume of integration is fixed in space without regard to any possible motion of the fluid, and variations in all relevant quantitites are specified to vanish on the boundaries of the space and time regions appearing in the integrals.

Before proceeding further it is necessary to comment on certain properties of the permittivity ϵ in the third term of Eq. (15). We shall assume that helium atoms have an electrical polarizability $\bar{\alpha}$ that is independent of density, temperature, and other conditions of the liquid and that it is a given constant. Furthermore, we will suppose that the Clausius-Mossotti relation[14] is applicable so that

$$\frac{\epsilon_r-1}{\epsilon_r+2}=\frac{\rho\bar{\alpha}}{3\epsilon_0 m}, \tag{16}$$

where ϵ_r is the dielectric constant. Then ϵ_r, and in turn $\epsilon=\epsilon_0\epsilon_r$, depends on the density ρ but not on any other primary independent variable of the Lagrangian density.

The Euler variational equations generated by Eq. (15) are

$$\delta\rho:\ \bar{l}-\frac{p}{\rho}-\bar{\phi}+\tfrac{1}{2}(-\vec{\nabla}\phi)^2\frac{\partial\epsilon}{\partial\rho}+\frac{\partial\alpha}{\partial t}+(\vec{u}+\vec{v}_n)\cdot\vec{\nabla}\alpha+s\frac{\partial\beta}{\partial t}+s\vec{v}_n\cdot\vec{\nabla}\beta=0, \tag{17}$$

$$\delta s:\ -T+\frac{\partial\beta}{\partial t}+\vec{v}_n\cdot\vec{\nabla}\beta=0, \tag{18}$$

$$\delta v_n:\ \vec{u}+\vec{v}_n+\vec{\nabla}\alpha+s\vec{\nabla}\beta=0, \tag{19}$$

$$\delta\vec{u}:\ \vec{v}_s+\vec{\nabla}\alpha=0, \tag{20}$$

$$\delta\phi:\ \vec{\nabla}\cdot[\epsilon(-\vec{\nabla}\phi)]=0. \tag{21}$$

Equation (21) is just Maxwell's equation $\vec{\nabla}\cdot\vec{D}=0$ in a form applicable when the free charge density is zero.

Taking the curl of Eq. (20), one finds

$$\vec{\nabla}\times\vec{v}_s=0, \tag{22}$$

a condition derived here and in Zilsel's theory, but treated as a postulate in Landau's phenomenological theory.[15] It may be worth noting that according to Eq. (19)

$$\vec{\nabla}\times\left[\frac{1}{s}(\vec{u}+\vec{v}_n-\vec{v}_s)\right]=0.$$

This restriction on flows is also implicit in the earlier treatments of Zilsel,[5] London,[13] and Jackson[7] without electric fields present.

Elimination of Langrange multipliers α and β from Eqs. (17)–(20) leads to simple equations of motion for superfluid and normal fluid. One can infer from earlier work[16] that

$$\bar{l}=\frac{p}{\rho}-Ts-z+\tfrac{1}{2}v_s^2-xv_s^2+x\vec{v}_n\cdot\vec{v}_s. \tag{23}$$

Combining Eqs. (1), (23), (18), (19), (20), and (14) with (17) and then simplifying, one finds

$$\frac{\partial\alpha}{\partial t}=z+\bar{\phi}-\tfrac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}+\tfrac{1}{2}v_s^2. \tag{24}$$

Equation (24) is essentially one of Josephson's equations,[17] but now it refers to liquid ^{4}He instead of superconductors. Taking the negative gradient of Eq. (24) and using Eq. (20), one arrives at the equation of motion for the superfluid. It is written below in useful alternative forms:

$$\frac{\partial}{\partial t}v_s=-\vec{\nabla}\left[z+\bar{\phi}-\tfrac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}+\tfrac{1}{2}v_s^2\right], \tag{25}$$

$$\frac{D_s}{Dt}\vec{v}_s=-\vec{\nabla}\left[z+\bar{\phi}-\tfrac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}\right] \tag{26a}$$

$$=-\frac{1}{\rho}\vec{\nabla}p+s\vec{\nabla}T+\tfrac{1}{2}x\vec{\nabla}(\vec{v}_n-\vec{v}_s)^2-\vec{\nabla}\bar{\phi}+\vec{\nabla}\left[\tfrac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}\right]. \tag{26b}$$

The convective derivative following the motion of the superfluid in Eqs. (26a) and (26b) is defined by

$$\frac{D_s}{Dt}=\frac{\partial}{\partial t}+\vec{v}_s\cdot\vec{\nabla}. \tag{27}$$

Combining Eq. (11) with (26a) yields Eq. (26b).

Derivation of the equation of motion for the normal fluid from Eqs. (17)–(21) is nontrival, but it involves only steps that differ little from those already in the literature[5,13,18] provided that Eqs. (23), (14), and (11) are first taken into account. The final result is

$$\frac{D_n}{Dt}\vec{v}_n=-\frac{1}{\rho}\vec{\nabla}p-\frac{1-x}{x}s\vec{\nabla}T-\frac{1-x}{2}\vec{\nabla}(\vec{v}_n-\vec{v}_s)^2-(\vec{v}_n-\vec{v}_s)\frac{\Gamma}{\rho x}-\vec{\nabla}\bar{\phi}+\vec{\nabla}\left[\frac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}\right], \quad (28)$$

where the convective derivative following the motion of the normal fluid is defined by

$$\frac{D_n}{Dt}=\frac{\partial}{\partial t}+\vec{v}_n\cdot\vec{\nabla}\ . \quad (29)$$

In Eq. (28) Γ is a source density of normal fluid (rate of production of normal fluid per unit volume) and a sink density of superfluid. It satisfies the following continuity equations for the two fluids:

$$\frac{\partial\rho_n}{\partial t}+\vec{\nabla}\cdot(\rho_n\vec{v}_n)=\Gamma\ , \quad (30a)$$

$$\frac{\partial\rho_s}{\partial t}+\vec{\nabla}\cdot(\rho_s\vec{v}_s)=-\Gamma\ . \quad (30b)$$

Gravitational and electrostrictive terms enter the equations of motion for superfluid and normal fluid only in the combination $\bar{\phi}-\frac{1}{2}E^2(\partial\epsilon/\partial\rho)$. This observation makes it possible to write down immediately the conservation laws including both effects from laws derived earlier for gravitational forces alone.[13] Momentum conservation is now expressed by

$$\frac{\partial}{\partial t}(\rho_n\vec{v}_n+\rho_s\vec{v}_s)=-\vec{\nabla}\cdot(\rho_n\vec{v}_n\vec{v}_n+\rho_s\vec{v}_s\vec{v}_s)-\vec{\nabla}p -\rho\vec{\nabla}\bar{\phi}+\rho\vec{\nabla}\left[\frac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}\right]. \quad (31)$$

Energy conservation is expressed by

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_n v_n^2+\tfrac{1}{2}\rho_s v_s^2+\rho e) =-\nabla\cdot\{\tfrac{1}{2}\rho_n v_n^2\vec{v}_n+\tfrac{1}{2}\rho_s v_s^2\vec{v}_s+(\rho e+p)\vec{w} +\rho(\vec{v}_n-\vec{w})[Ts+\tfrac{1}{2}x(\vec{v}_n-\vec{v}_s)^2]\} -\rho\vec{w}\cdot\vec{\nabla}\bar{\phi}+\rho\vec{w}\cdot\vec{\nabla}\left[\frac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}\right]. \quad (32)$$

In Eq. (32) $\vec{w}$ is the local center-of-mass velocity that satisfies

$$\rho\vec{w}=\rho_n\vec{v}_n+\rho_s\vec{v}_s\ , \quad (33)$$

and e is the intrinsic internal energy per unit mass, the same as in Eq. (9).

The electrical term in Eq. (31) can be written as

$$\rho\vec{\nabla}\left[\frac{1}{2}E^2\frac{\partial\epsilon}{\partial\rho}\right]=\vec{\nabla}\left[\frac{1}{2}E^2\rho\frac{\partial\epsilon}{\partial\rho}\right]-\frac{1}{2}E^2\vec{\nabla}\epsilon\ , \quad (34)$$

where

$$\vec{\nabla}\epsilon=\frac{\partial\epsilon}{\partial\rho}\vec{\nabla}\rho\ . \quad (35)$$

When Eqs. (34) and (35) are taken into account, one can readily see that Eq. (31) is consistent with the expression for ponderomotive force given by Landau and Lifshitz[2] and by Stratton[19] for classical fluids. It is also noteworthy that the Clausius-Mossotti relation, Eq. (16) implies

$$\frac{\partial\epsilon_r}{\partial\rho}=\frac{1}{\rho}\frac{(\epsilon_r-1)(\epsilon_r+2)}{3}\ . \quad (36)$$

Therefore, when $v_s=v_n=0$ and $\bar{\phi}=0$, one finds from Eqs. (31) and (34)–(36) that the pressure gradient satisfies

$$\vec{\nabla}p=-\frac{\epsilon_0E^2}{2}\vec{\nabla}\epsilon_r+\frac{\epsilon_0}{6}\vec{\nabla}[E^2(\epsilon_r-1)(\epsilon_r+2)]\ . \quad (37)$$

Equation (37) agrees with a result found by Neidhardt and Fajans[20] for liquid ^{4}He under equilibrium conditions.

The theory of electrostriction developed here for liquid ^{4}He can be extended to dilute solutions of ^{3}He in ^{4}He. That theory and also possible applications of it is to cryogenic devices are planned to be reported in future papers.

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