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Korteweg-deVries solitons and helium films

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The possibility of propagating Korteweg-deVries solitons in a superfluid ^{4}He film and in a superfluid ^{4}He film overlayed by a ^{3}He film is discussed. Various dispersive and nonlinear contributions to the equation for the nonlinear modes of the films are analyzed. Conditions depending on the thickness of the films are obtained for the propagation of troughlike and bumplike solitons.

I. INTRODUCTION

A thin superfluid helium film (typical dimensions: $1 \times 1 \times 10^{-7}$ cm^3) at low temperatures behaves in many respects like a body of shallow water. The third sound mode of a thin helium film, the "shallow water wave" of the film, is described by hydrodynamic equations that are the same (except for details) as those which describe shallow water. Just as the hydrodynamic equations for shallow water admit the existence of nonlinear solitary wave modes, the Korteweg-deVries or KdV solitons, so also the hydrodynamic equations for a thin superfluid helium film admit the existence of soliton solutions. In the equations for the motion of these fluids there is a special balance between nonlinear terms and dispersive terms that results in the solitary wave modes, the solitons. The sources of nonlinearity and dispersion are different in a thin helium film and in shallow water. The structure of the hydrodynamic equations that describe a thin helium film gives rise to nonlinearity and dispersion that we call intrinsic nonlinearity and intrinsic dispersion. Huberman[1] pointed out that, since the force that holds a thin helium film in place, the van der Waals force, is not a linear force like gravity, the force that holds shallow water in place, there is an additional nonlinearity at work on a thin helium film. Furthermore, on a thin helium film there are forces, the surface tension and the quantum-mechanical bending that work to prevent distortion of the film, that are linear but dispersive.

Biswas and Warke[2] have discussed the nonlinear modes of thin helium films using the hydrodynamics of Rutledge *et al.*[3] Nakajima *et al.*[4] have described the nonlinear modes of thin helium films using two fluid hydrodynamics and standard nonlinear techniques. They have discussed the experimental conditions that should be appropriate for the search of solitons in both very thin and saturated ^{4}He films, predicting what should be the signature of a KdV soliton in a realistic experiment. The interest in the investigation of the nonlinear modes in superfluid films has been revived by the work of Kono *et al.*,[5] who suggest that they may have seen what would be the first experimental evidence for solitons in a thin ^{4}He film, although there are some problems with their interpretation of the data, as is pointed out below.

In this paper we employ the layered film model of Guyer and Miller[6,7] to describe the solitons that occur in ^{3}He-^{4}He mixture films. The possibility to control the thickness of the ^{3}He film adds a new degree of freedom to an experimental investigation of the system: by controlling this thickness the nonlinear force and the dispersion can be modified. As a consequence the qualitative and quantitative features of soliton propagation can be changed. An understanding of the behavior of the soliton can provide valuable information on the structure of the film. Soliton propagation on a well characterized film would make thin helium films valuable laboratories on which to conduct experimental soliton research.

In Sec. II we briefly review the hydrodynamic equations that describe thin helium films, we describe the basic physical process that leads to the solitons and we call attention to the various sources of nonlinearity and dispersion. In Sec. III we cast the hydrodynamic equation in a suitable dimensionless form so that we may examine the structure of the soliton (its size and shape) and identify the parameters that control the nonlinearity and dispersion. Thus, we are able to examine the evolution of the soliton characteristics as the parameters of the film are changed.

Finally, the effects of the population of the surface state of a superfluid ^{4}He film by a two-dimensional distribution of spin-polarized hydrogen atoms and related problems are briefly discussed.

II. BACKGROUND

The two fluid hydrodynamics of ^{4}He is derived for example by Khalatnikov.[8] When discussing a thin helium film at low temperatures this hydrodynamic formulation reduces to a continuity equation that relates the superfluid density ρ_s and the flow velocity $\vec{v}_s$, an Euler equation for $\vec{v}_s$, and a condition for irrotational flow, $\nabla \times \vec{v}_s = 0$. If in addition we take the fluid to be incompressible, which is an excellent approximation except for the thinnest films, the film is described by

$$\frac{\partial \eta}{\partial t} + \nabla \eta \cdot \nabla \phi = \phi_y \quad \text{(on the surface)} \ , \tag{1}$$

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\nabla \phi|^2 + \delta V(\eta) = 0 \quad \text{(on the surface)} \ , \tag{2}$$

and

$$\nabla^2 \phi = 0 \ , \tag{3}$$

where we have dropped the subscripts on ρ_s, $\vec{v}_s$ because $\rho_n = 0$, $\vec{v}_n = 0$. Here ϕ is the velocity potential, $\vec{v} = +\nabla \phi$, and η describes the departure of the surface of the film from its equilibrium position. The geometry of the film is shown in Fig. 1. The forces that work to restore the surface to its equilibrium position are contained in $\delta V(\eta)$, the difference between the potential energy of the film with the surface at $\eta(x,t)$ and with the surface at h. These forces are discussed in detail below. In addition to Eqs. (1)–(3) there is the condition that the fluid will not flow into the substrate

$$\phi_y|_{y=0} = 0 \ . \tag{4}$$

The nonlinear terms in Eqs. (1)–(4) are the terms $\nabla \eta \cdot \nabla \phi$ in Eq. (1), $|\nabla \phi|^2/2$ in Eq. (2) and possible nonlinear terms in $\delta V(\eta)$. Nonlinear terms in the equation of motion for the fluid that have their source in $\nabla \eta \cdot \nabla \phi$ or $|\nabla \phi|^2/2$ are termed intrinsic nonlinearities; they are due to the structure of the hydrodynamic equations. The nonlinear terms that have their source in $\delta V(\eta)$ are termed force nonlinearities.

Before going on to remark about the sources of dispersion in Eqs. (1)–(4) let us carry out a number of the standard manipulations.

(1) Use the dimensionless variables $u = x/l$, $z = y/h$, $w = \eta/a$, $\tau = c_3 t/l$, $\beta = h/l$, and $\alpha = a/h$ in terms of which all amplitudes, e.g., w and their derivatives w_u, θ_z, . . . , are of order 1. Equations (1) and (2) become

$$w_\tau + \alpha w_u \theta_u = \frac{1}{\beta^2}\theta_z \ , \tag{5}$$

and

$$\theta_\tau + \frac{\alpha}{2}\theta_u^2 + \frac{\alpha}{2\beta^2}\theta_z^2 + w + \alpha\lambda w^2 = 0 \ , \tag{6}$$

where $\theta = (h/c_3 a l)\phi$, $\lambda = |R_2 h/R_1|$; $c_3^2 = R_1 h$ is the adiabatic third sound velocity and $\delta V(\eta) = R_1 \eta + R_2 \eta^2 + \cdots$; see the Appendix. The parameters α and β defined above are a measure of the size of the disturbance in terms of the size of the film; see Fig. 2. The reason for using dimensionless variables is that each term in Eqs. (5) and (6) has a size determined by its coefficient.

(2) Expand $\theta(z,u,\tau)$ as a power series in z and functions of u and τ:

$$\theta(z,u,\tau) = \sum_n \theta_n z^n g_n(u,\tau) \tag{7}$$

and subject this form for θ to the conditions $\nabla^2\theta = 0$, and $\theta_z = 0$ at $z = 0$. Thus

$$\theta(z,u,\tau) = \sum_{n=0} \frac{(-1)^n}{(2n)!}\beta^{2n} z^{2n} \frac{d^{2n}}{du^{2n}} g(u,\tau) \ . \tag{8}$$

Now let us examine the sources of dispersion in Eqs. (5) and (6). Putting the nonlinear terms equal to zero we obtain the linear equations

$$w_\tau = \frac{1}{\beta^2}\theta_z \tag{9}$$

and

$$\theta_\tau + w = 0 \ . \tag{10}$$

However, the use of Eq. (8) leads to

$$w_\tau + g_{uu} - \frac{\beta^2}{6} g_{uuuu} + \cdots = 0$$

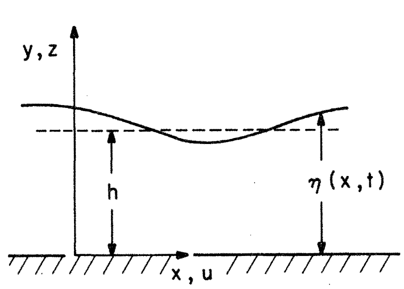

FIG. 1. Coordinates. The film profile is described by the height fluctuations $\eta(x,t) - h$.

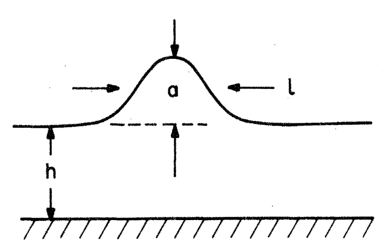

FIG. 2. Size of the soliton. The localized disturbance is characterized by the height a and the length l; the film is characterized by the thickness h. A soliton will propagate for $al^2 \approx h^3$.

and

$$g_\tau = w - \frac{\beta^2}{2} g_{\tau uu} + \cdots = 0 \ ,$$

where the terms g_{uuu} and $g_{\tau uu}$ are the lowest-order dispersive terms. Upon rearrangement

$$-g_{\tau\tau} + g_{uu} + \beta^2\left(\tfrac{1}{2} g_{\tau\tau uu} - \tfrac{1}{6} g_{uuuu}\right) + \cdots = 0 \ . \quad (11)$$

For $g \propto \exp i(Ku - \Omega\tau)$ the dispersion relation is ($\beta << 1$)

$$\Omega^2 = K^2\left(1 - \frac{K^2}{3} + \cdots\right) \ .$$

That is, for $kh << 1$, $K = kh$, and $\omega = c_3\Omega$,

$$\omega^2 = c_3^2 k^2 \left(1 - \tfrac{1}{3} k^2 h^2 + \cdots\right) \ . \quad (12)$$

Even for a linear force field, the nondispersive character of Eqs. (9) and (10) disappears when proper account is taken of the correlated motion of the fluid in the $x(u)$ and $y(z)$ directions. The dispersion that occurs in Eq. (11) has its source in the structure of the hydrodynamic equations and is termed intrinsic dispersion. Note that this "intrinsic" dispersion does not appear in Ref. 2 because Biswas and Warke constrain the velocity field to be parallel to the substrate.

There is additional dispersion present, even in the linear approximation to the hydrodynamic equations. This occurs if the force law has a surface tension term, which goes as

$$\sigma n_{xx} \rightarrow \sigma a_0^2 w_{uuu} \ , \quad (13)$$

where σ is the surface tension, or if the force law contains a quantum-mechanical bending energy,

$$\frac{\hbar^2}{m} \frac{d^2}{dx^2} w \rightarrow \frac{\hbar^2}{m a_0^2} w_{uuu} \ . \quad (14)$$

Here a_0 is a convenient microscopic length which we choose to be 3.6×10^{-8} cm, the average distance between atoms in bulk ^{4}He and a reasonable estimate of the thickness of a layer of atoms in a helium film.

Thus we have intrinsic and force law nonlinearities and intrinsic and force law dispersion. The force law nonlinearities and the force law dispersion are amenable to experimental manipulation with an attendant modification in the behavior of the soliton.

We close this section with several remarks. (1) Nonlinear contributions to the van der Waals potential occur only if the height of the film is allowed to vary. For a purely compressional film, for which the height of the film is fixed, the variations in density appear linearly in the van der Waals potential. Thus, we believe that the force nonlinearity occurring in Ref. 2 should not be there. (2) The KdV equation takes into account nonlinearities only up to the lowest order. The initial conditions in the experiment in Ref. 5 seem to have generated modes whose effective shape (a/h and h/l) cannot be correctly described by the KdV equation. This can be seen from the relation $|v - c_3|/c_3 \sim a/h$, where v is the phase velocity measured in the lab frame. From Fig. 1 in Ref. 5 it is clear that some of the components of the traveling disturbance have velocities of order $c_3/3$, for which $a/h \approx 1$. It would seem that the interpretation of this experiment in terms of solitons is suspect.

III. KORTEWEG-deVRIES SOLITONS IN MIXTURE FILMS

Consider a phase separated ^{3}He-^{4}He film at $T = 0$ K, Fig. 3. If the ^{4}He film and the ^{3}He blanket are reasonably thin, the ^{3}He blanket is clamped (there are no flows of the ^{3}He parallel to the surface) and the presence of the ^{3}He acts only to modify the force law that describes the energy expended to distort the upper surface of the superfluid, the ^{4}He surface. For $\delta V(\eta)$ we take

$$\delta V(\eta) = R_1 \eta + R_2 \eta^2 + D \eta_{xx} \ , \quad (15)$$

where R_1 and R_2 are given by Eqs. (A3) and (A4) in the Appendix and

$$D = \frac{\sigma}{m_4 n_4} + \frac{\hbar^2}{2m_4} \ ; \quad (16)$$

σ is the surface tension (discussed below) and n_4 is the bulk number density for ^{4}He, $n_4^{-1} = a_0^3 = (3.6 \times 10^{-8}\ \text{cm})^3$. The potential energy in Eq. (15) contains a linear force term, $R_1\eta$, that will give the velocity of third sound, a nonlinear force term, $R_2\eta^2$, that will modify the amplitude of the leading nonlinear term in the hydrodynamics, and two dispersive force terms due to surface tension and the quantum-mechanical bending. We can keep track of the consequences of R_2 and D in the hydrodynamics by writing

$$\delta V(\eta) = R_1 a (w + \lambda w^2) + \cdots + \frac{\delta a}{l^2} w_{uu} \ , \quad (17)$$

where $\lambda = |R_2 h_1 / R_1|$ and $\delta = (3\sigma a_0^2 / m_4 c_3^2)$

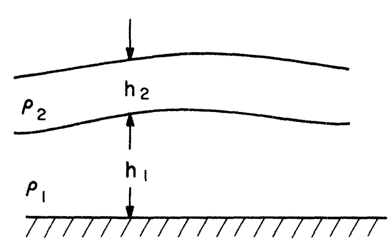

FIG. 3. Mixture films. For a ^{3}He-^{4}He mixture film h_1 is the height of the ^{4}He of density ρ_1 and h_2 is the height of the ^{3}He of density ρ_2, $\rho_2/\rho_1 \simeq 0.70$. The ^{3}He film remains at constant thickness and rides on top of the ^{4}He film.

$+(3\hbar^2/2m_4^2 a_0^2 c_3^2)$; λ and δ are a measure of the size of the force nonlinearity and the force dispersion, respectively. When $\delta V(\eta)$ is used in Eq. (2) and the standard treatment of Eqs. (1) and (2) outlined above is carried through, one finds to third order in smallness (β^3)

$$w_\tau + w_u + \alpha(\tfrac{3}{2}-\lambda) w w_u + \frac{\beta^2}{6}(1-\delta) w_{uuu} = 0 \ , \quad (18)$$

the Korteweg-deVries equation in standard form with modification of the amplitude of the nonlinear term and the dispersion term. Before examining the influence of λ and δ on the solution to Eq. (18) several remarks are in order.

(1) Equation (18) is the nonlinear equation for right-hand going disturbances. The corresponding left-hand equation is

$$w_\tau - w_u - \alpha(\tfrac{3}{2}-\lambda) w w_u - \frac{\beta^2}{6}(1-\delta) w_{uuu} = 0 \ . \quad (19)$$

(2) The size of the soliton is determined by the competition of the nonlinear and dispersive terms in Eq. (18). Thus since w, w_u, . . . are of order 1 the size of the soliton obeys ($\alpha \ll 1, \beta^2 \ll 1$)

$$\alpha \approx \beta^2$$

or

$$al^2 \approx h^3 \ .$$

This condition puts a severe limit on the detection of individual solitons in thin helium films. When this condition is met the nonlinear flow that would lead to distortion cancels against an equal and opposite distortion due to dispersion and a soliton results.

(3) As λ and δ vary the signs of the nonlinear and dispersion terms can change. For $\lambda > 3/2$, $\delta = 0$, the soliton is a trough instead of a bump. In Table I we list the four possibilities given by the sign of the nonlinear and dispersive terms. In this table we use the notation (sign of nonlinearity, sign of dispersion), e.g., $\lambda = 0$, $\delta = 0$ would be represented by $(+,+)$, etc.

TABLE I. Characteristics of the soliton. The soliton solution to Eq. (18) is $w = A\,\mathrm{sech}^2(u - vt)$ with the amplitude A and velocity v having the properties listed.

	$\lambda < \frac{3}{2}$	$\lambda > \frac{3}{2}$
$\delta < 1$	$(+,+)$ $A = +1$ $v > c$	$(-,+)$ $A = -1$ $v > c$
$\delta > 1$	$(+,-)$ $A = -1$ $v < c$	$(-,-)$ $A = +1$ $v < c$

Let us begin the discussion of Eq. (18) by examining the consequences of a modification of the dispersion. The intrinsic dispersion is responsible for the factor 1 in the w_{uuu} term; the force dispersion is made up of two parts, the surface tension term

$$S = \frac{3\sigma a_0^2}{mc_3^2} \ ,$$

and the quantum-mechanical bending term[3]

$$Q = \frac{3}{2}\frac{(\hbar^2/ma_0^2)}{mc_3^2} \ ,$$

where c_3^2 is given by Eq. (A5). Both Q and S vary as the film thickness is varied. However $Q \ll S$ under all conditions and we can neglect it. How does S vary as the structure of the film is varied? For $h_2 = 0$, no ^{3}He on the ^{4}He, and reasonably thick ^{4}He films, we would expect $\sigma = \sigma_{4B}$, σ_{4B} being the surface tension of bulk ^{4}He. We use the value σ_{4B} for all films $(h_1, 0)$ regardless of the ^{4}He film thickness. If at fixed h_1 we add ^{3}He to a ^{4}He film the primary effect of the first small amounts of ^{3}He will be to reduce the surface tension of the film because of the spreading pressure of the ^{3}He in the surface state. Thus we write

$$\sigma(h_2) = \sigma_{AB} + (\sigma_{\rm sat} - \sigma_{AB}) h_2, \quad 0 \le h_2 \le 1 \ ,$$

$$\sigma(h_2) = \sigma_{\rm sat}, \quad 1 \le h_2 \ .$$

If $h_2 \ll 1$, it would be more accurate to calculate the reduction in the surface tension by considering the pressure corresponding to a two-dimensional Fermi gas,[9] instead of using our simple interpolation formula. In our equations $\sigma_{\rm sat}$ stands for the surface tension corresponding to a saturated film. From Edwards and Saam[9] we find $\sigma_{\rm sat}/\sigma_{4B} \approx 0.4$ so that the variation of S by a factor of about 2 is possible upon loading the ^{4}He film with ^{3}He. Upon calculating S as a function of h_1 and h_2/h_1 for several values of A (A is the van der Waals constant that determines the third sound velocity) we find that for $\sigma = \sigma_{AB}$ or $\sigma_{\rm sat}$ the value of S is of order 1 for very thin films, typically of the order of 1 layer. S becomes much larger than 1 as the film thickness increases. Because of the uncertainties in the size of σ for thin films we are reluctant to draw a strong conclusion. It would be prudent to remark that the dispersion term in the KdV equation for thin helium mixture films is dominated by the surface tension and is negative, $\delta > 1$, for all except the thinnest films. For these very thin films the dispersion is dominated by the intrinsic term and is positive. One important outcome of soliton experiments on thin films may be a determination of the crossover and a measure of this surface tension.

How does the nonlinear term in Eq. (18) vary with

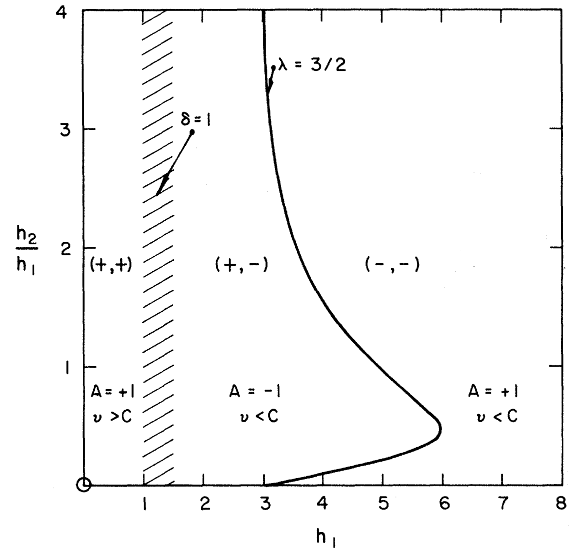

FIG. 4. Dispersion and nonlinearity. The signs of the dispersive and nonlinear terms in Eq. (18) are controlled by the preparation of the mixture film (h_1,h_2). The parentheses are (sign of $\frac{3}{2}-\lambda$, sign of $1-\delta$). The solitons are given by $w=A\,\mathrm{sech}(u-vt)$ with $v=1+\alpha A/2$ and $A\alpha(3/2-\lambda)=2\beta^2(1-\delta)$. The amplitude A and velocity v as functions of h_1 and h_2/h_1 are shown.

the structure of the film? In Fig. 4 we plot the locus of points h_1 and h_2/h_1 at which $\lambda=3/2$. For values of h_1, h_2/h_1 to the left of this locus the sign of the nonlinear term is dominated by the intrinsic nonlinearity, and for values of h_1, h_2/h_1 to the right of this locus the sign of the nonlinear term is dominated by the force nonlinearity. On Fig. 4 we show with the notation (sign of force nonlinearity, sign of dispersive nonlinearity) the sign of the two terms which compete to give the soliton. For $(+,+)$ and $(-,-)$ the soliton is a bump; for $(+,-)$ or $(-,+)$ the soliton is a trough. Thus we see that in a variety of ways it is possible to change the conditions of an experiment so that the solitons will evolve from trough to bump, etc. For example we have at $h_1=5$ (bump, $0\leq h_2\lesssim 1.25$), (trough, $1.25\lesssim h_2\leq 5$), and (bump, $5\leq h_2$). At $h_2/h_1=1$: (bump, $0\leq h_1\lesssim 1-2$), (trough, $1-2\lesssim h_1\leq 5$), and (bump, $h_1>5$).

Not only is the nature of the soliton that propagates determined by h_1 and h_2 but so is its size. The size of the soliton is related to λ and δ through

$$\alpha(\tfrac{3}{2}-\lambda)\approx\frac{\beta^2}{6}(1-\delta)$$

where λ and δ depend upon h_1 and h_2/h_1. If individual solitons can be studied then λ and δ can be learned, etc. With present day detection equipment it would be possible to observe single solitons in saturated films,[4] but only aggregates of solitons in the thinner films, which would make it more difficult to draw quantitative conclusions for the films. The inverse scattering method can be used to describe the envelope of a soliton aggregate.[4]

The possibility of investigating the properties of spin-polarized hydrogen (H↓) by propagating a third sound wave in a ^{4}He film in which the hydrogen occupies its surface state has been suggested elsewhere.[6] The binding energy of the surface state of H↓ on ^{4}He has been recently measured to be about 1 K.[10] If we have the H↓ on a saturated film of ^{4}He, the effects of the H↓ layer will appear mostly as a reduction in the surface tension, which can be estimated at least for low H↓ densities from the pressure in a two-dimensional gas of hard-core bosons.[11] Note that the smallness of the hydrogen mass contributes to make this pressure relevant. The presence of the H↓ will thus show up as a decrease in the amount of "negative" dispersion: the soliton will be thinner.

Finally, let us mention that it is also possible to study the nonlinear modes in a system of two layered films where both films are superfluid. There are two relevant cases: (a) superfluid ^{3}He on superfluid ^{4}He: both fluids can be treated as incompressible to a good approximation. Two solutions for the third sound velocity are obtained[6] and this originates two sets of coefficients for the KdV equation. (b) Superfluid H↓ on superfluid ^{4}He: the two-dimensional H↓ film has been predicted to undergo a Kosterlitz-Thouless transition into the superfluid state.[12] The resulting superfluid H↓ could in principle be thought of as forming a purely compressible film laying on the incompressible ^{4}He.

ACKNOWLEDGMENT

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APPENDIX

The addition of ^{3}He to a ^{4}He film causes essentially two effects: the change in the surface tension decreases the corresponding "negative" dispersion term, and the increase in the amount of material modifies the effective external field felt by the film.

The van der Waals potential for a film of unperturbed height h_1+h_2 and instantaneous height $z_s(x,t)=h_1+h_2+\eta(x,t)$ (h_1 and h_2 are the equilibrium thicknesses of the ^{4}He and ^{3}He films, respectively), can be written

$$V_{\mathrm{VDW}}(x,t)=A\left[\frac{1}{(D+h_1)^3}-\frac{1}{(D+h_1+\eta)^3}\right. \\ \left.+\frac{\rho_2}{\rho_1}\left[\frac{1}{(D+h_1+\eta)^3}-\frac{1}{(D+h_1)^3}+\frac{1}{(D+h_1+h_2)^3}-\frac{1}{(D+h_1+h_2+\eta)^3}\right]\right] , \tag{A1}$$

where ρ_2/ρ_1 is the ratio between the number densities of the ^{4}He and ^{3}He films, D is the thickness of the solid ^{4}He layer that separates the substrate from the liquid helium ($D\approx 1.2$ atomic layers), and A is the van der Waals force constant which depends on the substrate but in general satisfies 15 (layers)^3 $K \lesssim A \lesssim 45$ (layers)$^3 K$.

Expanding V_{VDW} up to second order in η, we can write

$$V_{\mathrm{VDW}}(x,t)=R_1\eta(x,t)+R_2\eta^2(x,t) , \tag{A2}$$

where

$$R_1=3A\left[\frac{1}{(D+h_1)^4}+\frac{\rho_2}{\rho_1}\left[\frac{1}{(D+h_1+h_2)^4}-\frac{1}{(D+h_1)^4}\right]\right] , \tag{A3}$$

and

$$R_2=6A\left[-\frac{1}{(D+h_1)^5}+\frac{\rho_2}{\rho_1}\left[\frac{1}{(D+h_1)^5}-\frac{1}{(D+h_1+h_2)^5}\right]\right] . \tag{A4}$$

The third sound velocity turns out to be

$$c_3^2=3Ah_1\left[\frac{1-\rho_2/\rho_1}{(D+h_1)^4}+\frac{\rho_2/\rho_1}{(D+h_1+h_2)^4}\right]=R_1h . \tag{A5}$$

When $h_2=0$ one recovers the usual formula for the adiabatic third sound velocity in thin ^{4}He films.

If we have H↓ instead of ^{3}He on ^{4}He, the only difference is that we will have two different van der Waals constants. The extension of our equations to this case is immediate.

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