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Conduction in a strong field in two dimensions: The quantum Hall effect

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Several conditions under which a magnetic field can be considered to be strong in comparison with a random scattering potential are distinguished. These are the maximum magnitude of the scattering potential small compared with Landau-level separation, large spatial separation of otherwise arbitrary scattering centers compared with the quantum size of Landau states, and smoothness of the scattering potential on this scale. In all three cases, by somewhat different mechanisms, extended states exist and the ideal quantum Hall resistance is found. The case of scattering centers separated by a smoothly varying potential can also be solved. The actual experimental conditions are likely to involve aspects of all three conditions.

Ordinary conduction in two dimensions is interesting, in part because of the surprising and delicate result that all electronic states are localized in this case.[1] The same problem in the presence of a strong perpendicular magnetic field has also attracted considerable attention, largely because of the remarkable quantum Hall effect which is observed[2] but also because its relation with with the localization problem in the absence of the field has not been completely elucidated. We here treat the problem of two-dimensional electrons in a strong magnetic field from several standpoints which may contribute to the understanding of these matters.

We deal with the Hamiltonian $H_0+V(x,y)$ where $V(x,y)$ is a random scattering potential, specified in more detail later, and

$$H_0=\frac{\hbar^2}{2m}\left[-\frac{\partial^2}{\partial x^2}+\left(\frac{1}{i}\frac{\partial}{\partial y}+\frac{eBx}{\hbar c}\right)^2\right]-eEx \quad . \quad (1)$$

The magnetic field B is in the z direction and the electric field E is in the negative x direction. The geometry is that of Ref. 3. The question is then in what sense the magnetic field is strong. There are several possible limits that can be studied, each affording great simplification.

One approach has been taken by Ando and coworkers[4] who, without giving details, use large B to argue for the neglect of certain intermediate states appearing in the Kubo formula. An explicit criterion has been employed by Thouless[5] and Halperin.[6] This criterion is

$$\max_{x,y}|V(x,y)|<\hbar\omega_c/2 \quad .$$

Thouless points out that if this criterion is satisfied then perturbation theory for the single-electron Green's function, $G(1,2,E)$, will converge when E is between Landau levels and well away from the unperturbed energies. Halperin rediscovered and used this result to justify Laughlin's[7] method. His result is that given the Thouless condition, there are extended states with energies close to the unperturbed Landau levels and these states carry the correct Hall current when they are completely filled. The nature of these states remains unclear, however.

We next wish to do scattering theory so we consider a potential $V(x,y)$ which vanishes except in isolated regions, where it is completely arbitrary. There is some minimum distance d_0 between regions of support of V, which we term "scattering centers." (Since several scattering centers can be regarded as one, a more precise condition is that the region of vanishing V percolates throughout the system with minimum width d_0.) The magnetic field may be regarded as large when $l \ll d_0$, with $l^2=(n+1)\hbar c/eB$. Here n is the number of the Landau level in which we are interested. In the absence of the magnetic field, a wave-packet incident on one center scatters out in various directions, then impinges on a second center and so on. The result of the scattering from each center is stochastic so the electron undergoes a random walk and to first approximation can be regarded as diffusing away from its initial region of localization. Because of the properties of the random walk in two dimensions, there is destructive interference at large distance. There is good evidence that such a potential succeeds, in two dimensions, in localizing the electrons.[1]

Now consider the strong B case. The incident wave packet has well-defined energy, and is made up of states from a single Landau level, say the 0th. Such a wave packet, having no spread in energy (in the absence of the Hall electric field), will not move. This makes it very easy for a local potential, no matter how weak, to bind the electron, and indeed, all the electronic states in the system will, technically, be bound states. However, the binding energy will

be exponentially small for many of the states. To avoid this embarassment, we have included the Hall (electric) field in H_0 which gives to the wave packet the drift velocity $v = cE/B$ in the y direction. The incident wave packet can be taken as

$$\psi(xyt) \propto \exp\{-\tfrac{1}{4}[(x+p_0-v)^2+(y-y_0-vt)^2] + \tfrac{1}{2}i(x+p_0-v)(y-y_0-vt)\} \quad (2)$$

and is localized near $x = v - p_0$, $y = y_0$ at $t = 0$. We have adopted units, as in Ref. 3, in which l, $\omega_c (= eB/mc)$, and m are unity. If this wave packet is incident on a scattering region, we may find the results after the interaction by using standard methods. The total wave function is given by

$$\psi(xyt) = \psi_0(xyt) + \int dx'\,dy'\,dt'\,G_0(xyt,x'y't') \times V(x'y')\psi(x'y't') \quad . \quad (3)$$

G_0 is the retarded Green's function for the Hamiltonian H_0. The second term above is nonvanishing only for times after the wave packet reaches the scattering center. (We consider only times such that just one scattering center of the many making up V enters the integral.) It is convenient to look at the component of (3) for definite energy E_0 which may be taken as $E_0 = vp_0$. Then

$$\psi(xyE_0) = \psi_0(xyE_0) + \int dx'dy'\,G_0(xy,x'y',E_0) \times V(x'y')\psi(x'y'E_0) \quad . \quad (4)$$

Here G_0 is explicitly

$$G_0(xy,x'y',E) = \sum_n \int \frac{dp}{2\pi} \frac{\psi_{np}(xy)\psi_{np}(x'y')^*}{E-n-pv+i\delta} \quad , \quad (5)$$

where $\psi_{np}(xy) = e^{ipy}\phi_n(x-v+p)$. ϕ_n is the harmonic oscillator wave function, and δ is a positive infinitesimal. By shifting the p integration to the steepest descent path, it is seen that G_0 is exponentially small for large $|x-x'|$ as well as for large negative $y-y'$. For large positive $y-y'$ it will be small except for the contribution of the pole at $pv = E - n$, namely,

$$G_0(xy,x'y',E_0) \sim -i\psi_{0p_0}(xy)\psi_{0p_0}(x'y')/v \quad . \quad (6)$$

This will again be small unless $x \sim x' \sim -p_0$. (Note that $y-y'$ large means, in standard units, $y-y' >> l$. This may hold without requiring that y be far away from the region of scattering compared with the linear dimension of the scattering center, as is usually assumed in scattering theory.) The fact that only the pole contributes is just the statement that far from the scatterer, thus long after the scattering, the final energy must be equal to the initial energy. This asymptotic behavior is radically different from that of the Green's function in the absence of the field, namely, $G_0(\vec{r}, \vec{r}', E) \sim e^{ikR}/\sqrt{R}$ where k is $\sqrt{E}$ and $R = |\vec{r} - \vec{r}'|$. The scattered wave in this familiar case is into any direction, but this degeneracy of the energy is completely lifted by the crossed fields.

The scattered wave therefore has the form $e^{ipy}\phi_n(x-v+p)[\exp(i\delta_p)-1]$ for strong B. The form of the coefficient follows from unitarity. The propagation of a wave packet perpendicular to a strong magnetic field with crossed electric field is thus super one dimensional. It will over long distances continue in its initial line of propagation. Moreover, it does not admit time reversal and there is propagation only in the positive, $\vec{E} \times \vec{B}$ direction. Further, a barrier of finite extent is not capable of stopping the packet, as there is always a way around in the second dimension.

The multiple-scattering analysis now becomes trivial. There will be localized states, perhaps many of them, in the neighborhood of the localized regions of support of the potential. However, between these regions, there are states which are extended. The density of electrons in these states is the same as in the case $V = 0$, and if all of these states are filled, one can calculate the total Hall current crossing a line which remains entirely in a potential free region from the lower edge of the sample to the upper. This gives the same total Hall current as if $V = 0$ everywhere.

The current carried by each state may be formally analyzed as well. Let $\delta(p)$ now mean the total phase shift accumulated as a wave of energy vp propagates from one end of the sample, $y = 0$, to the other, at $y = L$. Demanding periodicity gives discrete momenta satisfying $p_\alpha L + \delta(p_\alpha) = 2\pi n_\alpha$ where the n_α are successive integer values. As usual, $\delta(p)$ will decrease by 2π for each bound state. There are thus, over the total range of the p_α, N_B fewer solutions for p_α than in the absence of scattering with N_B the total number of bound, i.e., localized states. The solutions for p_α and thus for $E_\alpha = vp_\alpha$ are nicely ordered, and Laughlin's technique[7] is easily applied to see explicitly that the Hall current takes its ideal value. The eigenstates carry an excess current just compensating that not carried by the localized states, as in Ref. 3.

Upon encountering a scattering center, an electron reappears with a time *advance* $T_A = -d\delta/dE$. The wave packet during the scattering detours from the path x constant along the edge of the region $V(x,y) \neq 0$. There is an area $A(E)$ between the paths with and without scattering, which is avoided by the extended wave functions (and depends on the incident energy). It may be shown that $dA/dE = T_A$. [For $v \to 0$, we note that the packet speed where $V \neq 0$ is much greater than v, so $T_A = [y_{\max}(x) - y_{\min}(x)]/v$ where $y_{\max(\min)}$ is the maximum (minimum) value of y for which $V(x,y) \neq 0$ at fixed x. The above relation follows since $E = vx$.] Using the connection

between phase shift δ and the number of bound states gives $A = 2\pi l^2 N_B$. The arguments above have literally correct meaning only if the spatial scale of the scattering center is large compared with l. It may, however, be extended readily to the case discussed later in this paper. The results discussed up to this point thus justify the assumptions made by Tsui and Allen[8] who postulate that a certain area is associated with localized states and is insulating, while the remaining area has the ideal Hall conductivity. They show that if the conducting area percolates throughout the system then the system as a whole will have the ideal Hall resistance. Our result states that the area avoided by extended states, or occupied by localized states, is $2\pi l^2$ for each localized state and is associated with the area for which $V \neq 0$.

Another possible strong-field condition is that the potential be smooth on the scale of the magnetic length l. The condition is $\nabla V(x,y) << \hbar\omega_c/l$. This situation has been considered by Iordansky.[9] We give a treatment which will enable us to combine this case with that discussed previously. A similar smoothness condition which of course does not involve Planck's constant is well known in magnetohydrodynamics, and is used as the basis of the theory of adiabatic invariants and of guiding center motion.[10] A very similar development can be carried over to the extreme quantum case. The most important adiabatic invariant is the magnetic moment, or equivalently the perpendicular kinetic energy. In the two-dimensional world under study, there is no parallel kinetic energy, so that the potential energy $V(x,y)$ becomes an adiabatic invariant also. The guiding center motion, and in consequence the motion of wave packets, is thus along the lines of constant $V(x,y)$.

We consider the path-integral representation of the propagator in the presence of U.[11] ($U = V - eEx$.) We write this using units in which $\hbar$, m, and e/c are unity. The field B will be an explicit large parameter. Then

$$G(\vec{r}, \vec{r}', t-t') = \int D\vec{r}(\tau) \exp\left\{ i \int d\tau \left[\tfrac{1}{2}\dot{r}^2 - B x\dot{y} - U(\vec{r}(\tau)) \right] \right\} .$$

The integral in the exponent is along some particular path $\vec{r}(\tau)$ with the end conditions $\vec{r}(t) = \vec{r}$, $\vec{r}(t') = \vec{r}'$. Just as the semiclassical approximation is developed by thinking of $\hbar$ as small and making the stationary phase approximation in the above integral, the quantum guiding center approximation consists of taking B as large in the sense adopted here and making the stationary phase approximation. The semiclassical approximation expands about the classical path. In our case, we shall expand about the guiding center path whose center coordinates satisfy the equations

$$B\dot{X} = \frac{\partial U(X,Y)}{\partial Y}, \quad B\dot{Y} = \frac{-\partial U}{\partial X} . \tag{7}$$

This minimizes the sum of the last two terms in the action above. We put $x = X + u$, $y = Y + v$. We expect u, v to be of order l, that is, of order $1/\sqrt{B}$. U may then be expanded in powers of u, v. On the other hand, we expect the time dependence of u, v to be on the scale of ω_c; that is, we estimate $\dot{u}, \dot{v} \sim \sqrt{B}$. The leading terms in the Lagrangian, of order B, are $(\dot{u}^2 + \dot{v}^2)/2 - Bu\dot{v}$. Terms linear in u, v arise only from the kinetic energy, as follows from the definition of the guiding center. Integration by parts casts this contribution into the form $-\ddot{X}u - \ddot{Y}v$. In order to neglect these terms we make the condition that the acceleration of the guiding center be small. The second derivatives of U must not be too large if the quadratic terms in u, v are to be small. Thus, $\nabla V(x,y) << \hbar\omega_c/l$ is somewhat stronger than strictly necessary.

It is possible to carry out the functional Gaussian integral exactly,[11] keeping all terms up to quadratic in u and v. However, the coefficients of the small terms are dependent on time through their dependence on the guiding center coordinates, so we just neglect them. The remaining functional integral is the propagator for free electrons in a magnetic field whose Fourier transform has been displayed in Eq. (5). Thus the propagator has the form

$$G(\vec{r}, \vec{r}', t-t') = \exp\left\{ i \int d\tau \left[\tfrac{1}{2}(\dot{X}^2 + \dot{Y}^2) - B\dot{X}Y - U(X,Y) \right] \right\} G_0(u\,v\,t, u'v't') . \tag{8}$$

We now take up the initial conditions to be imposed on the equations for the guiding center. We choose the initial conditions $X(t') = x'$, $Y(t') = y'$, which implies that $u(t') = u'$ and $v(t') = v'$ vanish. If the final positions $X(t), Y(t)$ are far from x,y, respectively, then $u(t) = u = x - X(t)$ and $v(t) = v = y - Y(t)$ will have to be large, and G_0 will be very small. We may therefore assume that $X(t), Y(t)$ are rather close to x,y, respectively. For large separation $|\vec{r} - \vec{r}'|$ the Green's function of Eq. (8) will only be nonvanishing if the two points are close to the same guiding center trajectory. We may therefore treat an additional set of local but otherwise arbitrary scattering potentials, and we may again do scattering theory with much the same conclusions.

We have seen that the problems of electron prop-

agation in random potentials satisfying two different strong-field conditions can each be solved. The first condition is that scattering centers be well isolated on the scale of the magnetic length, and the second that the potential be sufficiently smooth that the acceleration of the guiding center be small. These two conditions can be combined into the single weaker condition that regions of arbitrary scattering be separated by distances large compared with the magnetic length in which the potential is smoothly varying.

All of the conduction will take place, in the limit of zero Hall field, at the energy for which the contour lines of the potential extend across the sample. This type of potential thus has a band of extended states which is arbitrarily narrow, and indeed, disappears as the electric field vanishes. This may well approximate the situation in the best samples at very low temperature and high fields, where extremely sharp steps are observed in the Hall resistance and where very little parallel resistance is found, even at fields where the Fermi level lies in the region of extended states.[12] In these samples the scattering centers probably consist of occasional impurity atoms and interface defects. Dopant atoms lying close to but not in the inversion layer will contribute to a random but smoothly fluctuating potential.

At higher temperature there will be a phonon contribution which in addition to providing some of the inelastic effects will give an effective potential which is pervasive and rapidly varying but which may still satisfy the Thouless condition. Such a potential no doubt gives rise to a band of extended states whose width increases with $\max|V(x,y)|$. No proof that this is the correct dependence of the width has so far been given, however. In any case, the picture that the width of the region of extended states is very narrow at low temperature and increases with temperature qualitatively fits the experimental observations.

A serious shortcoming of all the theories so far presented is that electron interactions have been neglected. Recent work for small magnetic fields has shown that there are indeed surprising interaction effects in that case. Further, at sufficiently strong fields and low enough electron density the Coulomb effects should, on dimensional grounds, begin to play a major role, but it is difficult to estimate reliably the fields and densities needed.

One relatively elementary and innocuous consequence of interaction is screening. This should have the result that as the density increases the self-consistent potential becomes smoother. This may well have something to do with the failure to observe a quantum Hall step after the lowest spin and valley component of the 0th Landau level has been filled in the case of Si samples.[13] It would be interesting to know whether all the states in this level are localized or whether they are all extended. Either of these possibilities could account for the observation. The two cases can be distinguished by the temperature dependence of the resistance, and both are possible theoretically, as far as we know.

Finally, it will be of interest to understand the case of intermediate field strengths in order to make connection with low-field results, where at high electron energy the states are extended, and at low energy they are localized. Much further work remains to be done on the nature of the mobility edge between localized and extended states in the magnetic case if indeed such a concept is appropriate as we have assumed.

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