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Finite ensemble averages of the zero-temperature resistance and conductance of disordered one-dimensional systems

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Because of the unusual width of the probability distributions of the zero-temperature response functions for disordered systems their mathematical expectation value will not represent the results of a numerical or physical experiment. We show that it is possible to derive analytically a scaling law for the average resistance $\bar{\rho}$ and the average conductance $\bar{g}$ taken over a *finite* ensemble of m systems, which is of a different analytic form than $\langle\rho\rangle$ and $\langle g\rangle$, and which gives good qualitative and quantitative agreement with numerical results. The conditions under which this new scaling behavior might be observed experimentally are discussed. Our result also rigorously proves that $\ln\bar{\rho}/\langle\ln\rho\rangle = -\ln\bar{g}/\langle\ln\rho\rangle \to 1$ as $L\to\infty$; thus, as expected, it is the geometric means of the zero-temperature response functions which are the relevant quantities in this limit.

Despite the existence of long-standing arguments that all the eigenstates of a one-dimensional disordered system are localized,[1-3] until very recently there remained a number of puzzling disagreements between the results of various numerical studies on the average zero-temperature conductance and resistance of such systems, and also between analytic and numerical results. The explanation for these apparent conflicts, although implicit in the mathematical work of Furstenberg,[4] Tutubalin,[5] and O'Connor,[6] and the work of Landauer,[7] was first clearly stated by Anderson *et al.*[8] who showed that it is essentially the logarithm of the resistance which is statistically well behaved (in a sense to be defined below) for these systems. This implies that the distributions of the dimensionless resistance $\rho = R/T$ (Ref. 7), and conductance $g = 1/\rho$, will be poorly behaved in two different senses:

(1) $\mathrm{Var}(\rho)/\langle\rho\rangle^2 >> 1$,
(2) $\langle\rho\rangle$ much greater than the median value of ρ.

These properties have three consequences. First, $\langle g\rangle$ is not approximately equal to $\langle\rho\rangle^{-1}$ and more generally $\langle f(\rho)\rangle \neq f(\langle\rho\rangle)$. Second, $\langle\rho\rangle$ is not typical or representative of the distribution, although the geometric mean, $\exp(\langle\ln\rho\rangle)$ will be representative. These two points have been made by Anderson *et al.* (Ref. 8) and many others, and in themselves give a partial explanation of some of the discrepancies mentioned above. However, a third consequence of the two properties of the probability distribution $D(\rho)$ noted above has received little attention, and yet it turns out to be very important for the understanding of the relationship between analytic results, on the one hand, and numerical and experimental results on the other. If $D(\rho)$ is poorly behaved, then an analytic (infinite ensemble) average of ρ, denoted by $\langle\rho\rangle$, will in general differ significantly from a finite ensemble average, denoted by $\bar{\rho}$, obtained by summing a finite number m of observations distributed according to $D(\rho)$. If they do differ, it is clearly the latter quantity, $\bar{\rho}$, which is relevant to both laboratory and computer experiments.

For example, Fig. 1 illustrates the results of a computer calculation of $\ln\bar{\rho}$ (the squares) for the Anderson model for fixed disorder and various lengths as compared to an analytic calculation of $\ln\langle\rho\rangle$ (the solid line).[9] Clearly $\ln\bar{\rho}$ is not scaling linearly with L as is $\ln\langle\rho\rangle$. The purpose of this paper is to derive a scaling law for $\ln\bar{\rho}$ and $\ln\bar{g}$ which will explain both qualitatively and quantitatively the results of numerical calculations of $\ln\bar{g}$ and $\ln\bar{\rho}$ for disordered one-dimensional systems in the highly localized regime. We will show that such a scaling law can be derived on the basis of a single very general assumption, which has been found to hold in many different models of disordered 1D systems, that the distribution of $\ln\rho$, $H_L(\ln\rho)$, is well behaved with mean and variance growing as L, the length of the sample.

We begin by assuming that $\ln\rho$ is distributed with mathematical expectation $\langle\ln\rho\rangle = \gamma L$ and $\mathrm{Var}(\ln\rho) = \sigma^2 L$, thus the fractional variance of $\ln\rho$

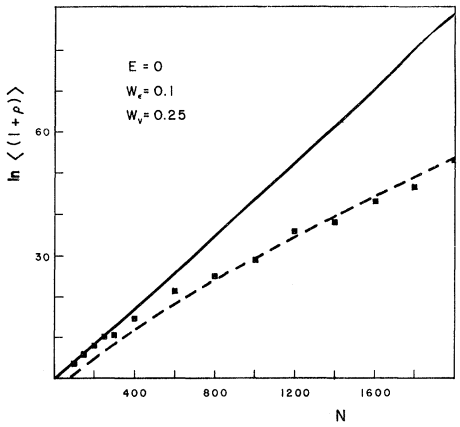

FIG. 1. Plot of $\ln(\langle 1+\rho\rangle)$ vs number of atoms for a 1D Anderson model with both diagonal and off-diagonal disorder at $E=0$ (from Ref. 9). The squares are the numerically computed (finite ensemble) average taken over an ensemble of 1000 systems. The solid curve is the analytic result of Eq. (41), Ref. 9. The dashed line is the analytic result for $\ln\bar{\rho}$ to be derived below [Eq. (3)].

goes to zero as $L\to\infty$. It is not hard to show that these properties of $H_L(\ln\rho)$ imply that ρ is *poorly* behaved in the two senses defined above as long as $\sigma\sqrt{L}>>1$. We note that it follows trivially that $\ln g$ has the distribution $H_L(-\ln\rho)$, with $\langle\ln g\rangle=-\gamma L$ and the same variance. Therefore, if H_L is symmetric around its mean, then each step in our subsequent argument also applies to $\ln\bar{g}$ with the simple replacement $\gamma\to-\gamma$.

We are interested in the behavior of the random variable $\bar{\rho}=1/m\sum_{i=1}^{m}\rho_i$, and in particular that of $\ln\bar{\rho}$ for $\sigma\sqrt{L}>>1$. It is very useful to define a new random variable, δ, so that

$$\ln\rho_i=\gamma L+\sigma\sqrt{L}\,\delta_i\ . \tag{1}$$

Thus δ has a distribution of the same form as $H_L(\ln\rho)$, but with mean zero and standard deviation unity, which we will denote as $H(\delta)$.

Then $\bar{\rho}=1/m\sum_{i=1}^{m}\exp(\gamma L+\sigma\sqrt{L}\,\delta_i)$, and since exponentiation spreads out the values of the ρ_i very broadly, the sum of the ρ_i tends to be dominated by its maximum value, call it $\rho_0\equiv\exp(\gamma L+\sigma\sqrt{L}\,\xi)$. One can see this clearly by considering, for fixed m

$$\lim_{L\to\infty}\frac{\bar{\rho}}{\rho_0/m}=\lim_{L\to\infty}\sum_{i=1}^{m}\exp[-\sqrt{L}\,\sigma(\xi-\delta_i)]\to 1\ . \tag{2}$$

Thus, we expect $\ln\bar{\rho}$ to behave like $\ln(\rho_0/m)$. In particular, the expected value $\langle\ln\bar{\rho}\rangle$ satisfies the scaling law

$$\langle\ln\bar{\rho}\rangle\simeq\langle\ln\rho_0-\ln(m)\rangle$$
$$=\gamma L+\sigma\langle\xi\rangle_m\sqrt{L}-\ln(m) \tag{3a}$$

and, similarly

$$\langle\ln\bar{g}\rangle=-\gamma L+\sigma\langle\xi\rangle_m\sqrt{L}-\ln(m)\ . \tag{3b}$$

Moreover, we expect observations of $\ln\bar{\rho}$ to be well represented by $\langle\ln\bar{\rho}\rangle$, since $(\ln\bar{\rho})^2\simeq(\ln\rho_0)^2$ implies $\mathrm{Var}(\ln\bar{\rho})\propto L$ so that the fractional variance of $\ln\bar{\rho}$ decreases as $1/L$.

Let us first examine the qualitative consequences of Eq. (3). First, it rigorously corroborates the arguments of Anderson *et al.* (Ref. 8) that at zero temperature as $L\to\infty$ the scale resistance, $\exp(\langle\ln\rho\rangle)=e^{\gamma L}$, and conductance $e^{-\gamma L}$ are the relevant quantities. From (3) we see that a finite ensemble average of ρ or g yields $\ln\bar{\rho}/\langle\ln\rho\rangle=\ln\bar{\rho}/\gamma L=-\ln\bar{g}/\gamma L\to 1$ as $L\to\infty$. Second, for large finite L, it predicts a correction to the scale resistance proportional to $\sqrt{L}$, which we shall see is important for explaining numerical results and may be experimentally observable under conditions to be discussed below. This $\sqrt{L}$ correction implies slower than linear growth of $\ln\bar{\rho}$ (i.e., second derivative negative) for finite L, which has been observed in the numerical studies of Andereck and Abrahams,[10] and Stone *et al.*,[9] whereas all known analytic calculations give $\ln\langle\rho\rangle\propto L$ for large finite L. In particular, in the special case of the Anderson model with purely off-diagonal disorder at $E=0$, it has been shown that the parameter $\gamma=0$.[9,11] Thus (3) implies $\ln\bar{\rho}\propto\sqrt{L}$ in this case as was found in the numerical studies of Economou and Soukoulis,[12] despite the fact that even in this special case the analytic result is $\ln\langle\rho\rangle\propto L$ (Ref. 11). Third, Eq. (3) implies that for $L\le\sigma^2\langle\xi\rangle_m^2/4\gamma^2$, $d\langle\ln\bar{g}\rangle/dL$ is positive and $\ln\bar{g}$ will typically increase with L, suggesting incorrectly the absence of localization.[13]

Equation (3) has no adjustable parameters since the quantities γ, σ, and $\langle\xi\rangle_m$ are all determined by the distribution of $\ln\rho$ and the ensemble size. The parameter γ may be shown to be equal to twice the average inverse localization length, and for the Anderson model with diagonal disorder this is known from a perturbative calculation[14] which gives $\gamma\approx 0.084(W/V)^2$, where $2W$ is the width of the rectangular distribution of site energies. In addition, many models of disordered 1D systems satisfy the relation $\gamma=\frac{1}{2}\sigma^2$ for weak disorder, including the Anderson model (Ref. 9). Thus in many cases Eq. (3) may be calculated *entirely* analytically, given the value of $\langle\xi\rangle_m$. Therefore, we should be able to get meaningful quantitative

comparisons of (3) with numerical studies by taking either the analytically or numerically determined values of σ and γ, and by calculating $\langle\xi\rangle_m$. $\langle\xi\rangle_m$ is the expected value of the maximum of a set of m independent random variables with the standard distribution $H(\delta)$. We shall find that in fact $\langle\xi\rangle_m$ is both relatively insensitive to the exact nature of $H(\delta)$ and is a slowly varying function of m.

To obtain the probability density for ξ, $P(\xi)$, we note that since $\xi=\max(\delta_1,\ldots,\delta_m)$ it follows that $\text{Prob}(\xi\le z)=\prod_{i=1}^{m}\text{Prob}(\delta_i\le z)$. Therefore,

$$P(\xi)d\xi=m[H(\xi)]^{m-1}dH\ . \tag{4}$$

It turns out to be easier to get a good approximation for $\langle\xi\rangle_m$ in general by considering it as a function of the random variable H, so that

$$\langle\xi\rangle_m=m\int_0^1\xi(H)H^{m-1}dH\ . \tag{5}$$

The reason that this is convenient is that $\langle H\rangle=m/(m+1)$ and $\text{Var}(H)=m/(m+1)^2\times(m+2)$, so that for m large, $\text{Var}(H)/\langle H\rangle^2\simeq 1/m^2$ and $\langle\xi(H)\rangle$ is close to $\xi(\langle H\rangle)$. We can then Taylor expand $\xi(H)$ around $\langle H\rangle$ in (5) to give

$$\langle\xi\rangle=m\int_0^1 H^{m-1}dH\xi(\langle H\rangle)+\left.\frac{\partial\xi}{\partial H}\right|_{\langle H\rangle}(H-\langle H\rangle)+\frac{1}{2}\left.\frac{\partial^2\xi}{\partial H^2}\right|_{\langle H\rangle}(H-\langle H\rangle)^2+\cdots\ , \tag{6}$$

$$\langle\xi\rangle_m\simeq\xi(\langle H\rangle)+\frac{1}{2m^2}\xi''(\langle H\rangle)\ . \tag{7}$$

Equation (6) is an entirely general approximation for $\langle\xi\rangle$ which is expected to be good for $m>>1$, and whose accuracy can be made arbitrarily good by including higher terms in the Taylor series. All one needs to evaluate (6) is an expression for the inverse of the standardized distribution function H of $\ln\rho$.

A particularly important case is when $\ln\rho$ is normally distributed. There exists a great deal of recent numerical and analytic work which shows that $\ln\rho$ is approximately normally distributed in the highly localized regime for many simple models of disordered one-dimensional systems.[8,10,11,15–17] In this case H is the standard error function which is approximated by

$$H(\xi)\simeq 1-(2\pi\xi^2)^{-1/2}\exp(-\xi^2/2)$$

for $\xi>>1$. Then

$$\xi=\left[\ln\left[\frac{1}{2\pi(1-H)^2}\right]-\ln(\xi^2)\right]^{1/2}$$

and

$$\xi\simeq\left\{\ln\left[\frac{1}{2\pi(1-H)^2}\right]-\ln\left[\ln\left[\frac{1}{2\pi(1-H)^2}\right]\right]\right\}^{1/2}\ . \tag{8}$$

Evaluating (8) at $\langle H\rangle=m/(m+1)$ gives the first term in (7) and higher terms may be obtained in a straightforward fashion. For $m=1000$, $\xi(\langle H\rangle)=3.08$ and the next term, $(1/2m^2)\xi''(\langle H\rangle)=0.16$, so our series approximation for $\langle\xi\rangle$ seems nicely converged. For a bounded distribution H the calculation of $\langle\xi\rangle$ is trivial for m large since the probability that the maximum of $\langle\delta\rangle$ is very near the upper bound becomes very high, and one can take $\langle\xi\rangle$ equal to the upper bound. For a rectangular distribution with unit variance the upper bound is $\sqrt{3}$ and the exact result is $\langle\xi\rangle=\sqrt{3}(m-1)/(m+1)$. Thus, as noted above, we see that for reasonable size ensembles ($m\simeq 1000-10000$), $\langle\xi\rangle$ only varies by a factor of 2 or 3 between these two very different distributions, and in both cases $\langle\xi\rangle$ is a slowly varying function of m.

Having calculated $\langle\xi\rangle_m$ we are in a position to apply Eq. (3) to various numerical results. Here we assume that $\ln\rho$ is normally distributed and take $\langle\xi\rangle_m$ from Eqs. (7) and (8). Figures 2(a) and 2(b) show that Eq. (3) does indeed give good quantitative agreement with numerical data. In Fig. 2 the circles are the results of the computer experiment, while the thick curve is the analytic result for $\ln\bar{\rho}$ and $\ln\bar{g}$ of Eq. (3). The thin solid line in 2(a) is the analytic result for $\ln\langle\rho\rangle$ derived in Ref. 9, which, as usual, predicts much too large values of $\ln\bar{\rho}$. The agreement is excellent in view of the facts that: (i) Eq. (3) is applied with no parameters left free; (ii) our computer experiment only constitutes one observation of $\ln\bar{\rho}$ and thus should have some small random deviation from the exact $\langle\ln\bar{\rho}\rangle$. In plotting Eq. (3), we have used the average value of the numerically obtained values of σ and γ for each length, which are $\gamma=0.0381$, $\sigma=0.272$. Using the perturbative calculation for γ, and the relation $\gamma=\frac{1}{2}\sigma^2$ yields $\gamma=0.041$, $\sigma=0.29$ which gives a reasonable, but less good, fit to the data. This is to be expected, since the approximation of weak disorder is beginning to break down for $W/V=1/\sqrt{2}$. Equation (3) also gives a good fit for the data in Fig. 1 using the numerically obtained values $\gamma=0.0163$, $\sigma=0.196$.

Figure 2 suggests that Eq. (3) and the assumptions on which it is based provide a useful descrip-

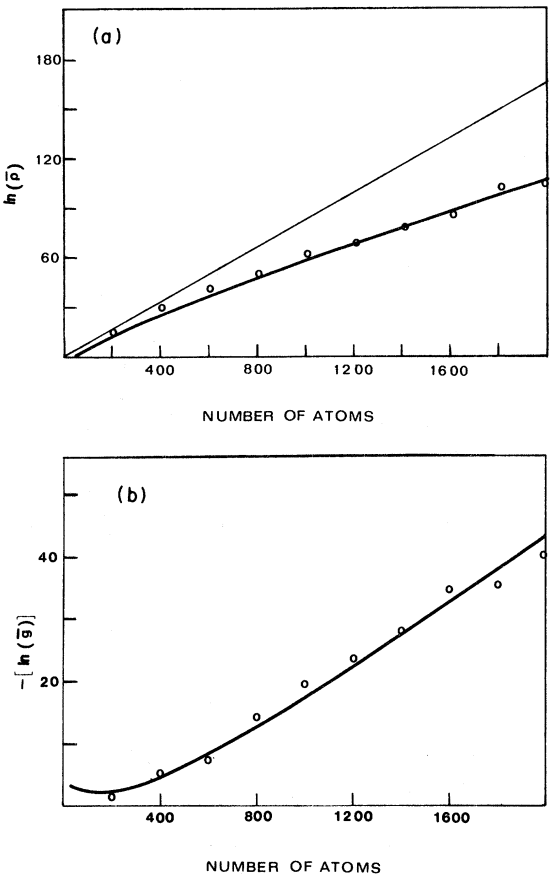

FIG. 2. (a) Plot of $\ln(\bar{\rho})$ vs number of atoms for Anderson model with purely diagonal disorder of half-width $W/V=1/\sqrt{2}$. Circles are the numerically computed average over 1000 systems; the thick curve is calculated from Eq. (3a). The values of σ and γ are obtained numerically. The thin solid line is the analytic result $\ln(\langle\rho\rangle)$ from Ref. 9. (b) Plot of $\ln(\bar{g})$ vs number of atoms for the same system parameters as (a). Again, the solid curve is the analytic result of (3b) and the circles the numerical values.

tion of computer experiments: Whether it will contribute to a useful description of actual experiments which measure the resistance of very thin metallic wires is a great deal more problematic. Several different experimental groups have found evidence for one-dimensional localized behavior by measuring the low-temperature resistance of such wires.[18] While the observations are in qualitative agreement with theory, the quantitative agreement is still rather poor. In real experiments at finite temperature, inelastic scattering provides alternative transport mechanisms with much higher conductivity than that due to quantum tunneling which is studied in the zero-temperature scaling theories based on the Landauer formula for the dc resistance. Thouless has analyzed in detail the role of these inelastic processes in various temperature regimes.[19,20] In another paper we have shown that the resistance due to these processes is not subject to the exponentially large fluctuations which make it necessary to distinguish $\ln\bar{\rho}$ from $\ln(\langle\rho\rangle)$, except possibly in the exponential hopping regime.[21] Since this regime is not accessible in metallic wires using present experimental techniques (Refs. 20 and 21), we conclude that zero-temperature statistical fluctuations are not playing an important role in the present set of experiments. Nonetheless, if it becomes possible to probe the regime where the actual resistance depends exponentially on temperature then one must ask whether statistical fluctuations lead to a difference between $\ln\bar{\rho}$ and $\ln\langle\rho\rangle$.

To analyze the statistics of the response functions at finite temperatures, it is useful to regard the studies of the zero-temperature resistance, based on the Landauer formula as studies of the inverse localization length α, using the relation $\alpha=(1/2L)\ln\rho$ (Ref. 21). Then the assumption that $\ln\rho$ is normally distributed implies that α is also, with

$$P(\alpha)=\frac{1}{\sqrt{\pi\gamma/L}}\exp\left[\frac{(\alpha-\gamma/2)^2}{\gamma/L}\right], \tag{9}$$

where we have used the relation $\gamma=\frac{1}{2}\sigma^2$. Kurkijarvi[22] has shown that for a one-dimensional system in the exponential hopping regime, the Mott variable range hopping does not occur, but instead the hopping resistance satisfies $\rho_h\propto e^{T_0/T}$, where $kT_0=\alpha/4An_f$, A is the cross-sectional area and n_f is the density of states per unit volume at the Fermi level. This may be rewritten in the form

$$\rho_h\propto e^{\alpha d(T)}, \tag{10}$$

where $d(T)=(4An_fk_BT)^{-1}$ has the dimensions of length, and α is normally distributed according to Eq. (9). Then exactly analogous calculations of $\langle\rho_h\rangle$ and $\bar{\rho}_h$ as for $\langle\rho\rangle$ and $\bar{\rho}$ yield

$$\ln\langle\rho_h\rangle=\frac{\gamma d}{2}\left[1+\frac{d}{2L}\right], \tag{11a}$$

$$\ln(\bar{\rho}_h)=\frac{\gamma d}{2}\left[1+\langle\xi\rangle\left[\frac{2}{\gamma L}\right]^{1/2}\right]-\ln(\gamma L), \tag{11b}$$

where we have assumed the hopping resistance of the sample is roughly the sum of the resistances due to γL segments of length $1/\gamma$ (the localization length). The interesting point which emerges is that as long as $1/\gamma << d(T) << L$, then $\ln\langle\rho_h\rangle \approx \ln(\bar{\rho}_h) \approx \gamma d/2$; thus, we see that statistical fluctuations only give a relatively small correction to the finite temperature resistance even in the exponential hopping regime, as long as $d(T) << L$. Note also that $\gamma d/2$ is equal to $\langle \ln\rho_h \rangle$ and is the finite temperature analog of the logarithm of the scale resistance, so again we find this to be the relevant quantity. Finally, if $d(T)$ becomes greater than L, then the conductivity due to tunneling becomes greater than that due to hopping and all the results of the zero-temperature scaling theories should apply. Then we would expect the resistance averaged over many samples to satisfy Eq. (3a).

Unfortunately, even for a truly one-dimensional wire of length 100 μ ($Ak_f^2 \to 1$), $d(T)$ becomes greater than L only at about 10^{-1} K, so, as we emphasize in Ref. 21, the observation of these effects are well beyond the capabilities of present experimental techniques.

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