Exponent behavior at a dissipative phase transition of a driven Josephson junction

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Static and dynamic critical exponents and a set of spinodal exponents are calculated within a mean-field approximation for the case of a driven Josephson junction undergoing a nonequilibrium phase transition. These universal exponents obey the exponent relations obtained from scaling-for-equilibrium phase transitions. The exponents are directly related to experimental observables such as the junction voltage, its noise bandwidth, and the Josephson radiation linewidth.

I. INTRODUCTION

Response functions near the critical point of an equilibrium second-order phase transition exhibit exponent behavior as a function of scaled variables. The working out of a full set of exponents from models of real systems is desirable, since these exponents are independent of detailed sample parameters, and obey simple algebraic relations useful for checking independent measurements.

It is well known that systems driven far from equilibrium can undergo dissipative transitions analogous to first- and second-order phase transitions.[1] Exponent behavior has been seen in particular nonequilibrium measurements, such as the spontaneous convective velocity at a Benard instability[2] and the relaxation time at an optical bistability threshold.[3] One might ask whether, by analogy to equilibrium transitions, a full range of nonequilibrium exponents can be obtained, and whether these obey the scaling relations. Recently, Bishop and Trullinger[4] have considered the noise voltage across a Josephson junction near its current threshold, and showed that, as the temperature approaches zero, mean-field exponents analogous to those of a ferromagnet near its critical point, can be extracted.

We have elsewhere shown[5] that a Josephson junction and resistance, driven by external coherent radiation or a battery, exhibits a first-order dissipative phase transition. The transition, involving the self-consistent dc voltage across the junction, is analogous to the equilibrium phase transition of a van der Waals gas, with a first-order line ending at a second-order critical point. In this work we show that, within this model, a set of mean-field critical exponents can be obtained that obey the equilibrium scaling relations. Furthermore, at the spinodal curve[6] or limit of metastability, another set of nonequilibrium "spinodal" exponents can be obtained, that differ from the second-order critical exponents, but independently satisfy the exponent relations. We also show that the (dynamic) exponent characterizing the relaxation time may be directly related (at transition) to the broadening of the linewidth of the Josephson radiation, and the narrowing of the voltage noise spectrum.

II. EXPONENTS

Consider a Josephson junction with maximum tunneling current I_J and capacitance C, with external resistance R across it. The oxide layer of the junction forms a cavity of resonance frequency ω_c and quality factor Q. A magnetic field is required for the current-photon coupling. (The size of the junction is assumed less than the Josephson length.) External coherent radiation, with photon number N, and/or a battery V_b inserted across the junction-resistance combination, can drive the junction into a nonequilibrium state[5], with dissipation occurring due to the resistor and the leakage of photons. The scaled voltage $f \equiv 2eV_c/\hbar\omega_c$ has been shown to satisfy the Langevin equation

$$\dot{f} = -\frac{1}{RC}\Phi'(f) + F_f(t) , \tag{1}$$

where $F_f(t)$ is the delta correlated Gaussian noise with diffusion coefficient $1/2\tau_f$ (related[5] to the parameters of the junction) and $\Phi(f)$ is the generalized potential whose minima, denoted by $\bar{f}$, yield the nonequilibrium steady states:

$$\Phi(f) = \tfrac{1}{2}(f^2-1) + \alpha \tan^{-1}2Q(f-1) - \mu(f-1) . \tag{2}$$

In Eq. (2) μ is the total drive[7]

$$\mu=\left[\frac{N}{N_c}\right]^{1/2}+\frac{2eV_b}{\hbar\omega_c} \tag{3}$$

and

$$\alpha=\frac{8e^2}{\hbar}\left[\frac{T}{\omega_c}\right]^2 R ,$$

$$N_c^{-1/2}=\frac{8e^2}{\hbar}\left|\frac{T}{\omega_c}\right| R , \tag{4}$$

$$T=\frac{1}{2}\frac{I_J}{\sqrt{\hbar C\omega_c}} .$$

The minima of Φ are plotted in Fig. 1 for various values of the parameter $a=(\alpha-\alpha_c)/\alpha_c$, with $\alpha_c=2/3\sqrt{3}Q^2$.

The origin in the figure is the critical point of our system where $\Phi'=\Phi''=\Phi'''=0$, when $\mu=\mu_c=1+\sqrt{3}/2Q$, $\alpha=\alpha_c$, and $f=f_c=1+(2\sqrt{3}Q)^{-1}$. For $a>0$ and $\mu_{c2}<\mu<\mu_{c1}$, Φ has two minima. The locus of points $(f_{c1,2};\mu_{c1,2})$ (defined by $\Phi'=0$, $\Phi''=0$) is the spinodal curve (dashed line) corresponding to the limit of metastability. The coexistence curve denoted by the dotted line in Fig. 1 corresponds to the points where the depth of the two minima of Φ are equal. It can be shown that the width of the coexistence curve is given by ($\alpha>\alpha_c$)

$$f-f_c=\pm\left[\frac{2}{9\sqrt{3}Q^4\alpha}\right]^{1/2}\left[\frac{\alpha}{\alpha_c}-1\right]^{1/2} . \tag{5}$$

It is clear that Fig. 1 represents a nonequilibrium phase diagram with first-order lines ending in a second-order point. The analogy between the above nonequilibrium phase transition and the classic liquid-gas phase transition is clear—pressure $p\leftrightarrow\mu$, volume $v\leftrightarrow f$, temperature $(T-T_c)\leftrightarrow(\alpha_c-\alpha)$; the curve $a=0$ $(\alpha=\alpha_c)$ is similar to the critical isotherm, the spinodal curve is similar to the limit of supersaturation, etc. One would thus expect divergent susceptibilities, exponent behavior, etc., as the critical point (μ_c,f_c,α_c) is approached. Such critical exponents[8] may be calculated (in the mean-field limit) from an analysis of (5) and the minima of (2) which have the property

$$\bar{f}=\begin{cases} f_c\mp\dfrac{1}{\sqrt{3}Q}(\pm\alpha\mp\alpha_c)^{1/3}, & \mu=\mu_c \\ f_c\pm\dfrac{1}{3Q^2}(\pm\mu\mp\mu_c)^{1/3}, & \alpha=\alpha_c \end{cases} . \tag{6}$$

Since the junction is assumed to have transverse dimensions small compared to the coherence length

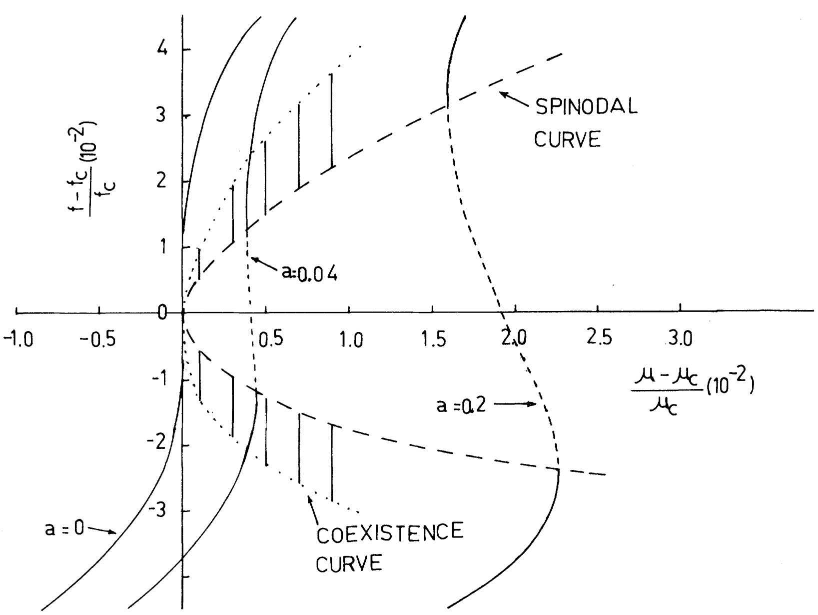

FIG. 1. Nonequilibrium phase diagram. The shaded region between the spinodal curve and the coexistence curve is the metastability region. The minima of Φ are the "isotherms"—the solid (dashed) part denoting stable (unstable) states for various values of $a\equiv(\alpha-\alpha_c)/\alpha_c$, with $Q=5$.

for the transition, the system is "zero dimensional," and spatially varying thermal fluctuations are relatively suppressed. True divergences thus would not occur, and deviations from mean-field exponent behavior are expected within a critical region. For $\alpha=\alpha_c$, a naive Ginzburg criterion yields an estimate $|\Delta\mu| \lesssim (RC/\tau_f)^{1/2}Q^2$, which is $\sim 10^{-3}$ for a typical choice of parameters.

From Eq. (2), a steady-state point, $\mu=\mu_{ci}+\Delta\mu$, $f=f_{ci}+\Delta f$, $\alpha=\alpha_{ci}+\Delta\alpha$ close to a critical ($\mu_{ci}\to\mu_c$, etc.) or spinodal ($\mu_{ci}\to\mu_{ci,2}$, etc.) point, obeys the equation

$$\Delta\mu=\tfrac{1}{6}\alpha_{ci}L^{(3)}(f_{ci})(\Delta f)^3 + L^{(1)}(f_{ci})\Delta f\Delta\alpha+\tfrac{1}{2}L^{(2)}(f_{ci})(\Delta f)^2 + [1+\alpha_{ci}L^{(1)}(f_{ci})]\Delta f+L(f_{ci})\Delta\alpha \; , \tag{7}$$

where $L(f)\equiv 2Q/[1+4Q^2(f-1)^2]$ and $L^{(i)}$ denotes the ith derivative of $L(f)$ with respect to f. The third and fourth terms vanish at the critical point, and the exponents of Table I follow at once for constant μ, α, or f, or along the coexistence curve. Equation (7) is in complete correspondence with the equation of state of a van der Waals gas near its critical point[9]

$$\Delta P=-A\Delta T\Delta V-B(\Delta V)^3+C(\Delta T) \; . \tag{8}$$

Divergent response functions at second-order critical points are associated with flattening of Ginzburg-Landau potential minima and an increase in fluctuations. However, a weakening of restoring forces also occurs at the spinodal curve, or limit of metastability, in the *first-order* transition region, where one of the bistable minima flattens and disappears. It is thus also possible to define another class of what might be termed "spinodal" exponents near the points $(f_{ci,2},\mu_{ci,2})$. Here we consider exponents for systems externally driven far from equilibrium, rather than metastable states of equilibrium systems.[9,10] In Eq. (7), with $f_{ci}=f_{ci,2}$ the third term vanishes, and the spinodal exponents of Table I follow.

An increase in relaxation times ("critical slowing down") occurs at both critical and spinodal points. From Eq. (1) the relaxation rate $T_1^{-1}=RC/\Phi''\sim\chi$, so this dynamic exponent[11] is the same as that of the static susceptibility as in the classical (equilibrium) von Hove case. Another characteristic time scale is T_2, the lifetime of the metastable state in the bistable region. This vanishes[5] similar to $(\mu-\mu_{ci})^{1/2}$, with the exponent $\frac{1}{2}$ coming from the spinodal exponent δ^{-1}. However, the stochastic switching region $(\mu-\mu_{ci})\lesssim 10^{-4}$ within which this behavior occurs is probably too small to be probed accurately.

For the critical exponent ν we must consider nonuniform fluctuations, $f\to f(r)\equiv\bar{f}+\tilde{f}(r)$. Expanding Φ of Eq. (2) about $\bar{f}=\bar{f}_{ci}$, with $\Delta\mu$, $\Delta\alpha$ small, one gets only $\Delta\alpha\tilde{f}^2(r)$, $\tilde{f}^4(r)$ terms in the critical case ($\bar{f}_{ci}=f_c$), while cubic terms $\tilde{f}^3(r)$ occur in addition, in the spinodal case. For a simple

TABLE I. Exponents exhibited by physical quantities as the critical or spinodal point is approached along the specified paths on the nonequilibrium phase diagram.

	Critical exponents				Spinodal exponents		
Physical quantity	$\mu=\mu_c$	$\alpha=\alpha_c$	$f=f_c$	Coexistence curve[a]	$\mu=\mu_{c1,2}$	α fixed, $>\alpha_c$	$f=f_{c1,2}$
Order parameter f	$\frac{1}{3}(\beta)$	$\frac{1}{3}(\delta^{-1})$		$\frac{1}{2}(\beta)$	$\frac{1}{2}(\beta)$	$\frac{1}{2}(\delta^{-1})$	
Susceptibility[b] $\chi=\frac{\partial f}{\partial\mu}$	$\frac{2}{3}(\gamma)$	$\frac{2}{3}$	1	1	$\frac{1}{2}(\gamma)$	$\frac{1}{2}$	1
Specific heat $C_v=\frac{\partial^2\Phi}{\partial\alpha^2}$	$\frac{2}{3}(\alpha)$	$\frac{2}{3}$	0	0	$\frac{1}{2}(\alpha)$	$\frac{1}{2}$	0

[a]Dynamical exponent for T_1 same as that of χ; the linewidth of Josephson radiation $\sim T_1^{1/2}$, whereas the linewidth of voltage fluctuations $\sim 1/T_1$. We find $\xi\sim T_1^{1/2}$ yielding for example $\nu=\frac{1}{3}(\frac{1}{4})$ for critical (spinodal) exponent, for constant μ.

[b]See Eq. (5) of the text.

static gradient term $\sim\xi^2(0)\,|\,\vec{\nabla}\tilde{f}(r)\,|^2$ the critical dimensions are[12] $d_c=4$ and 6, respectively. However, the gradient term enters here less directly.[13]

For the Josephson junction, with a small varying drive component $\mu+\delta\mu(r,t)$, and $\phi=\omega_c f$,

$$\dot{f}=-\frac{1}{RC}[f-\mu+\alpha L(f)]$$
$$+\frac{\bar{c}^2}{\omega_c}(\nabla^2\phi-\lambda_J^{-2}\sin\phi) \tag{9}$$

Here the Josephson length $\lambda_J\equiv 2\pi\bar{c}/\omega_J$ is large, and treating λ_J^{-2} as small, a linear response analysis in $\delta\mu(k,\,\omega)$ yields a "most divergent" susceptibility $\chi(k,\,\omega=\omega_J)\sim(\omega_J T_1)^{-1}\,[(\bar{c}k/\omega_J)^4+(\omega_J T_1)^{-2}]^{-1}$. Thus the correlation length is $\xi\equiv\lambda_J[\omega_J T_1(\alpha,\mu)]^{1/2}$, which diverges at transition, with $\nu=\frac{1}{3}$ and $\frac{1}{4}$ in the critical and spinodal cases. It is clear that the critical and spinodal exponents independently satisfy the relations $\alpha+2\beta+\gamma=2$, $\delta=(2-\alpha+\gamma)/(2-\alpha-\gamma)$, $\beta=(\nu/2)\times(d_c-2+\eta)$, and $\nu d_c=2-\alpha$, with the critical dimensionality d_c set equal to 4 and 6, respectively.

The measurement of some exponents could be done directly from the dc voltage versus drive nonequilibrium phase diagram, with estimates indicating that the region where mean-field exponents can be seen may be of the order of the hysteresis region itself. Dynamic exponents might also be seen through other means. In the Gaussian approximation Eq. (1) leads to the voltage fluctuations

$$\langle[f(t)-\bar{f}][f(t+\tau)-\bar{f}]\rangle=(T_1/2\tau_f)e^{-\tau/T_1}\,. \tag{10}$$

The Lorentzian voltage fluctuation spectrum will therefore narrow (independent of τ_f) at transition. The line shape of Josephson radiation is related to $\langle\exp[i\Phi(t+\tau)-\Phi(t)]\rangle$ and in the relevant parameter regime[5] the spectrum is Gaussian $\sim$ exp $[-(\omega-\omega_c\bar{f})^2/(T_1/\tau_f)]$ with width proportional[14] to $(T_1/\tau_f)^{1/2}$. The linewidth behavior can be understood from the Josephson relation $\omega=2ev/\hbar$ and (10), since $\langle(\delta\omega)^2\rangle\sim\langle(\delta f)^2\rangle\sim(T_1/\tau_f)$.

Finally, it would be interesting if the micsroscopic results on the bistable behavior of Josephson junctions can be obtained using phenomenological equations so that the present work can be placed in perspective with other dc effects in Josephson junctions.[15] Work in this direction is in progress.

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[7]The system could, for example, be biased near the spinodal point with a battery, and switched by a weak pulse of radiation.

[8]A brief discussion of the exponents along the coexistence curve in the context of optical bistability was given by G. S. Agarwal, L. M. Narducci, D. H. Feng, and R. Gilmore, in *Coherence and Quantum Optics IV,* edited by L. Mandel and E. Wolf (Plenum, New York, 1979), p. 281.

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[13]Compare M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975), Chap. 6.

[14]In the nonbistable regime and far away from the $(f_{c1,2},\,\mu_{c1,2})$ points, the line shape turns out to be Lorentzian and the lines are much narrower; compare also P. A. Lee and M. O. Scully, Phys. Rev. B 3, 769 (1971); M. J. Stephen, Phys. Rev. 182, 531 (1969).

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