Charged-soliton excitations in statistical mechanics

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(Received 18 February 1981; revised manuscript 30 June 1981)

A method is developed for demonstrating how solitons with some internal periodic motion may emerge as elementary excitations in the statistical mechanics of field systems. The procedure is demonstrated in the context of complex scalar fields which can, for appropriate choices of the Lagrangian, yield charge-carrying solitons with such internal motion. The derivation uses the techniques of the steepest-descent method for functional integrals. It is shown that, despite the constraint of some fixed total charge, a gaslike excitation of such charged solitons does emerge.

I. INTRODUCTION

Recent decades have yielded localized "solitary-wave" and "soliton" solutions to a large number of nonlinear wave equations. This in turn has led, starting from the mid-seventies, to a study of the role of such solutions both in qunatum field theory as well as in statistical mechanics. In quantum field theory soliton solutions have on the one hand led to new types of nonperturbative extended particle states[1] and on the other hand, when used in the Euclidean field-theoretic context, to interesting vacuum-tunnelling phenomena.[2–4] (While conscious of the technical difference between the two terms, we will use the word "solitons" to include solitary waves as well, as is often the practice in the literature.) In statistical mechanics, the contribution of solitonic excitations to the partition function and related thermodynamic quantities of systems which can be approximated by continuum fields, has received increasing attention. In particular, there has been a series of about a dozen articles dealing with soliton excitations in the classical statistical mechanics of one-dimensional field systems.[5–12] This series was initiated by the work of Krumhansl and Schrieffer[5] who studied the one-dimensional ϕ^4 theory. Their suggestion, based on intuitive arguments, was that the solitons (kinks) of this theory form elementary excitations which contribute to the free energy as if they were molecules of a gas. This was in addition to the familiar phonon contribution. Krumhansl and Schrieffer (KS) also computed the partition function independently by transfer integral treating methods, and found that the phenomenological evaluation treating solitons and phonons as elementary excitations agreed with the transfer-integral result to a good approximation. Subsequently, a rapid succession of papers have appeared,[6–12] where the initial work of KS was developed further and improved upon. (A partial list of these papers is given by Refs. 5 to 12. References 11 and 12 review earlier work and contain a fuller list of other references.) These papers applied the KS idea to other one-dimensional field theories such as the sine-Gordon model and the double quadratic system. They also incorporated soliton-phonon interactions in the form of a self-energy addition to the soliton mass.

In this paper we show how a gaslike excitation of "charged" solitons, with some intrinsic time dependence, emerges in the classical statistical mechanics of a *complex* scalar field. The Lagrangian we consider is of the type

$$L=\int[\tfrac{1}{2}(\partial_\mu\phi^+)(\partial^\mu\phi)-U(|\phi|)]d\vec{x} \ , \qquad (1.1)$$

where the form of $U(|\phi|)$ and the space dimensionality can be left arbitrary as long as the system permits charged solitons. Notice that L is invariant under the global $U(1)$ transformation

$$\phi(\vec{x},t)\to e^{i\alpha}\phi(\vec{x},t) \ . \qquad (1.2)$$

Associated with this symmetry is the conserved charge

$$Q=\frac{i}{2}\int\left(\phi^+\frac{\partial}{\partial t}\phi-\phi\frac{\partial}{\partial t}\phi^+\right)d\vec{x} \ . \qquad (1.3)$$

Our work differs from the papers cited above in several respects:

(i) In order that a solution carry nonzero charge, it is evident from (1.3) that it must involve some intrinsic time dependence even in its overall rest frame (zero-momentum frame). By contrast, the KS work and subsequent related papers deal mostly with excitations of *static* solutions. We do not mean that these authors do not include the kinetic (translational) motion of their solitons. What we mean by static solutions is that the solutions themselves (like the ϕ^4-theory "kink") are time independent in their rest frames.

(ii) Static solitons for scalar field systems exist only in one space dimension as per the Derrick-Hobart theorem.[13] [In certain special scale-invariant models, like the nonlinear $O(3)$ model, they can exist in two dimensions as well; but none exist in three dimensions.] But this theorem does not apply to solutions with intrinsic time dependence. Thus, for many choices of $U(|\phi|)$, charged solitons can be found in one, two, or three dimensions.[14] Accordingly, in contrast to the work cited above, our statistical-mechanical derivation will also be tailored to arbitrary dimensions (Sec. III).

(iii) For a real scalar field in one dimension, such as the sine-Gordon or ϕ^4 theory, the partition function behaved,[12] as far as soliton excitations go, as a grand partition function of the soliton particles. In other words, configurations with arbitrary numbers of (well separated) solitons contributed. In the charged case that we will be discussing, the situation is quite different. Since charge is a conserved quantity, the standard procedure of computing the partition function is to either introduce the corresponding chemical potential, or to constrain the charge to have any specific value Q_0. To extract the charged-soliton solutions, we will invoke the latter procedure. But if the total charge Q is fixed at some Q_0, then clearly *arbitrary* numbers of solitons of some *given* charge $Q_{\rm sol}$ cannot contribute, since the total charge must add up to Q_0. The number N must satisfy $NQ_{\rm sol}=Q_0$. Nevertheless, configurations with different soliton numbers will be seen to contribute. However, as N changes, $Q_{\rm sol}=Q_0/N$ will also have to change, and along with it, the shape, the energy and other features of the individual soliton. In short, we will get contributions from a family of different species of solitons, the species varying with the total number N of solitons. This interesting feature is discussed in detail in Sec. IV.

(iv) Last, but not least, we derive the contribution of charged solitons to the partition function Z by a systematic steepest-descent approximation to the functional integral for Z. Of course, given that the field equations of a system yield charged solitons, one would expect on physical grounds that these solitons would contribute as "elementary" excitations in the statistical mechanics of that system. We could introduce them on those intuitive, phenomenological grounds, just as the early papers on the sine-Gordon or ϕ^4 theory did. But it is much more satisfying to derive them systematically from the parent functional integral for the partition function, so that the nature of the approximation involved is clear.

In fact, before describing our calculations in Secs. III and IV, we present a compact rederivation, using the steepest-descent (Gaussian) approximation, of the basic result of Refs. 5 to 12, viz., that a gas of phonons and static solitons emerge as elementary excitations in the partition function of a one-dimensional real scalar field system. All the correction factors obtained in those papers due to phonon-soliton interactions, the kinetic motion of the solitons and zero-mode effects, will be seen to come out automatically. *It should be emphasized however, that there is nothing really new in Sec. II.* The techniques used are borrowed *in toto* from instanton-gas calculations in quantum field theory.[3,4] The final results obtained have as stated, also been given in Refs. 11 and 12, but based on intuitive arguments. All that we do in Sec. II is to obtain the latter results compactly, using the former techniques, for the sake of completeness. It also prepares the ground for the derivation of charged-soliton excitations in Secs. III and IV, which *is* new.

II. STATIC SOLITON EXCITATIONS DERIVED USING FUNCTIONAL INTEGRALS

We give a short summary of how the contribution of static soliton excitations in classical statistical mechanics may be obtained compactly, using functional integrals. Consider, for illustration, the one-dimensional sine-Gordon field $\phi(x)$ with canonical momentum $\pi(x)$, and a Hamiltonian given by

$$H(\pi,\phi)=\int dx\left[\frac{1}{2}[\pi(x)]^2+\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+U(\phi)\right] , \tag{2.1}$$

with

$$U(\phi)=\frac{m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right] .$$

The classical partition function is

$$Z=\int \mathfrak{D}[\phi(x)]\mathfrak{D}[\pi(x)]\exp[-\beta H(\pi,\phi)] .$$

The $\pi(x)$ integration can be done exactly since the integral is just a Gaussian, to yield a factor of $\sqrt{2\pi/\beta}$ at each point x. Thus,

$$Z=Z_\pi Z_\phi , \tag{2.2}$$

where

$$Z_\pi\equiv\prod_x\left(\frac{2\pi}{\beta}\right)^{1/2} ,$$

$$Z_\phi\equiv\int \mathfrak{D}[\phi(x)]\exp(-\beta E[\phi]) ,$$

and

$$E[\phi]\equiv\int\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+U(\phi)\right]dx . \tag{2.3}$$

The ϕ integration cannot in general be done exactly and is treated in the Gaussian approximation by expanding $E[\phi]$ about its minima. The minimization condition is just the classical static field equation

$$\frac{-d^2\phi}{dx^2}+\frac{dU}{d\phi}=0 \ . \tag{2.4}$$

Let $\phi_{\rm cl}(x)$ be a solution of (2.4), and let $\eta_n^2(x)$ and ω_n^2 be the complete set of eigenfunctions and eigenvalues of the operator

$$\left[\frac{-d^2}{dx^2}+\left(\frac{d^2U}{d\phi^2}\right)_{\phi_{\rm cl}}\right] \ .$$

Then one can expand

$$\phi(x)=\phi_{\rm cl}(x)+\sum_0^\infty c_n\eta_n(x) \tag{2.5a}$$

in which case

$$E[\phi]=E[\phi_{\rm cl}]+\tfrac{1}{2}\sum_0^\infty c_n^2\omega_n^2+O(c_n^3) \ . \tag{2.5b}$$

Defining the integration measure $\mathfrak{D}[\phi(x)]$ as $\prod_n dc_n$, and neglecting the $O(c_n^3)$ terms in (2.5), we have

$$Z=Z_\pi Z_\phi=Z_\pi e^{-\beta E[\phi_{\rm cl}]}\int\prod_n\left[dc_n\exp\left(-\frac{\beta}{2}c_n^2\omega_n^2\right)\right]=Z_\pi e^{-\beta E[\phi_{\rm cl}]}\prod_{n=0}^\infty\left(\frac{2\pi}{\beta\omega_n^2}\right)^{1/2} \ . \tag{2.6}$$

In general, (2.4) may have more than one finite-energy solution, in which case it is assumed that each will make an additive contribution to Z, as given by (2.6). In the sine-Gordon case, $\phi(x)=2N\pi(m/\sqrt{\lambda})$ is a zero-energy solution for any integer N. In addition, $\phi_s(x-X)=(4m/\sqrt{\lambda})\tan^{-1}(e^{m(x-X)})$ is a static soliton solution of finite energy $M\equiv 8m^3/\lambda$. The antisoliton is $\phi_A(x)=-\phi_S(x)$. These are the only static solutions. The contribution of the $\phi(x)=0$ solution to the partition function, called the "phonon" contribution, is obtained by noting that the eigenvalues of

$$\left[\frac{-d^2}{dx^2}+\left(\frac{d^2U}{d\phi^2}\right)_{\phi=0}\right]$$

are

$$\omega_n^2=k_n^2+m^2, \text{ with } k_nL=2n\pi, \quad L\to\infty \ . \tag{2.7}$$

Hence, using (2.6)

$$\begin{aligned}Z_{\rm ph}&=Z_\pi\prod_{n=0}^\infty\left(\frac{2\pi}{\beta\omega_n^2}\right)^{1/2}\\ &=Z_\pi\exp\left[-\frac{L}{2\pi}\int_{-\infty}^\infty dk\frac{1}{2}\ln\left(\frac{\beta(k^2+m^2)}{2\pi}\right)\right] \ .\end{aligned} \tag{2.8}$$

To obtain the soliton contribution, let $\tilde{\eta}_n(x)$ and $\tilde{\omega}_n^2$ be the eigenfunctions and eigenvalues of

$$\left[-\frac{d^2}{dx^2}+\left(\frac{d^2U}{d\phi^2}\right)_{\phi_s(x)}\right] \ .$$

This is a Schrödinger operator, and $\tilde{\omega}_n^2$ and $\tilde{\eta}_n(x)$ can all be exactly obtained for the sine-Gordon case. There is a discrete zero-mode $\tilde{\omega}_0=0$, followed by a continuum, whose density is related to the phase shift $\Delta(k)$ of the Schrödinger problem

$$\tilde{\omega}_n^2=k_n^2+m^2 \text{ for } n>0, \quad k_nL+\Delta(k)=2n\pi, \quad L\to\infty \ . \tag{2.9}$$

The zero-mode $\tilde{\omega}_0$, related to translation symmetry, clearly renders the naive formula (2.6) divergent. This problem is handled by using collective coordinates.[4] Instead of (2.5), one expands

$$\phi(x)=\phi_s(x-X)+\sum_{n=1}^\infty b_n\tilde{\eta}_n(x-X) \tag{2.10a}$$

with

$$E[\phi]=M+\frac{1}{2}\sum_1^\infty b_n^2\tilde{\omega}_n^2+O(b_n^3) \ . \tag{2.10b}$$

In the place of the set $\{c_n; n=0,1,\ldots\infty\}$, one uses the set $\{X, b_n; n=1,\ldots\infty\}$ as the variables. The Jacobian associated with this change is given by[4]

$$\prod_0^\infty dc_n=\sqrt{M}dX\prod_1^\infty db_n \ . \tag{2.11}$$

Then, instead of the form (2.6), the contribution $Z_{1\phi_s}$ of the single soliton to the partition function is given by

$$Z_{1\phi_s}=Z_\pi e^{-\beta M}\sqrt{M}\left(\prod_1^\infty\int db_n e^{-(\beta/2)b_n^2\tilde{\omega}_n^2}\right)\int_{-L/2}^{L/2}dX=Z_\pi e^{-\beta M}\sqrt{M}L\prod_{n=1}^\infty\left(\frac{2\pi}{\beta\tilde{\omega}_n^2}\right)^{1/2} \ . \tag{2.12}$$

Using (2.8), this becomes

$$Z_{1\phi_s}=Z_{\rm ph}Le^{-\beta M}\left(\frac{\beta M}{2\pi}\right)^{1/2}\exp\left(\sum_0^\infty\ln\omega_n-\sum_1^\infty\ln\tilde{\omega}_n\right) \ . \tag{2.13}$$

Recall that the set $\omega_n(n=0, \ldots \infty)$ and the set $\tilde{\omega}_n(n=1, \ldots \infty)$ form continua in the $L \rightarrow \infty$ limit, with densities given by (2.7) and (2.9), respectively. Using these, (2.13) becomes

$$Z_{1\phi_s} = Z_{\text{ph}} L \left(\frac{\beta}{2\pi}\right)^{1/2} \exp[-(\beta M + \sigma)] \ , \quad (2.14)$$

where

$$\sigma \equiv -\ln\sqrt{M} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{d}{dk}[\Delta(k)] \ln(k^2+m^2)^{1/2} \ . \quad (2.15)$$

Notice that σ is independent of the temperature β^{-1}. In the low-temperature limit, σ will be much smaller than βM. Also, the integrals over k can diverge in a truly continuum field theory, but they can be rendered finite by adding the usual counter-terms.[15] In the condensed-matter context there is a natural ultraviolet cutoff.

A set of N widely separted solitons will also solve the sine-Gordon equation to arbitrary accuracy as the separation tends to infinity. Repeating the arguments made above, the N-soliton contribution can be seen to be

$$Z_{n\phi_s} = Z_{\text{ph}} \frac{1}{N!} \left[L \left(\frac{\beta}{2\pi}\right)^{1/2} \exp(-\beta M - \sigma) \right]^N \ , \quad (2.16)$$

where the $N!$ appears because permuting the location of the N solitons will yield the same configuration. An identical contribution will arise from antisolitons. If we use open boundary conditions (i.e., zero chemical potential), then the system can permit an arbitrary number of solitons and antisolitons. The sum of all such contributions in a "dilute soliton gas" is clearly

$$Z = Z_{\text{ph}} \sum_{N_1 N_2} \frac{1}{N_1! N_2!} \left[L \left(\frac{\beta}{2\pi}\right)^{1/2} e^{-\beta M - \sigma} \right]^{N_1+N_2}$$

$$= Z_{\text{ph}} \exp\left[2L \left(\frac{\beta}{2\pi}\right)^{1/2} e^{-\beta M - \sigma} \right] \ . \quad (2.17)$$

An important assumption has been that the solitons are widely separated, i.e., that the gas is dilute. The mean density, obtained by maximizing (2.16) with respect to N is

$$\frac{N_{\max}}{L} = \left(\frac{\beta}{2\pi}\right)^{1/2} e^{-\beta M - \sigma} \ .$$

At low temperatures ($\beta \rightarrow \infty$), the gas will satisfy the requisite diluteness. We have thus compactly derived the basic result on *static* soliton excitations [as for instance in Eqs. (3.21) to (3.26) of Ref. 12].

III. CHARGED-SOLITON EXCITATIONS

In the preceding section, we derived in the Gaussian approximation to the partition function in classical statistical mechanics, the contribution from static soliton solutions. These solutions are static in their rest frame, i.e., the zero-momentum frame. The static nature of the solitons that emerged in that derivation was clearly because the time derivative of the field $\dot{\phi}(x) \equiv \pi(x)$ was separated and completely integrated out, leaving behind Z_ϕ which depended only upon $\phi(x)$. Naively, it may appear that in such a derivation, contributions to the partition function of solitons carrying some intrinsic time dependence (in the rest frame) may not emerge, even if such soliton solutions are permitted by the field equations; whereas, on physical grounds, we would expect that such solitons, if they exist, would contribute to the partition function.

This section is devoted to generalizing the derivation given in Sec. II so as to yield the contributions of solitons with some periodic internal motion. We illustrate the procedure using the simplest example of this kind, namely, charge-carrying solitons of a complex scalar field.

Consider the complex scalar field

$$\phi(\vec{x},t) = \phi_1(\vec{x},t) + i\phi_2(\vec{x},t) \quad (3.1)$$

in an arbitrary number of space dimensions. Let the Lagrangian have the form

$$L = \int [\tfrac{1}{2}(\partial_\mu \phi^+)(\partial^\mu \phi) - U(|\phi|)] d\vec{x} \ . \quad (3.2)$$

We note that this Lagrangian is invariant under global $U(1)$ transformations,

$$\phi(\vec{x},t) = e^{i\alpha}\phi(\vec{x},t) \ . \quad (3.3)$$

Associated with this symmetry, there will be a conserved charge

$$Q = \frac{i}{2}\int (\phi^+\dot{\phi} - \dot{\phi}^+\phi) d\vec{x} \ . \quad (3.4)$$

We require $U(|\phi|)$ to have its absolute minimum at $|\phi| = 0$. In that case, it is clear that finite energy soliton solutions must approach $|\phi| = 0$ at the boundary of space, and consequently will carry a finite value of the charge Q. The only other requirement on $U(|\phi|)$ is that it must permit charge carrying soliton solutions. That is to say, consider the equation of motion

$$\Box \phi + \frac{\partial U(|\phi|)}{\partial(|\phi|)} \frac{\phi}{|\phi|} = 0 \quad (3.5)$$

and a solution to this equation of the form

$$\phi(\vec{x},t) = \rho_\nu(\vec{x}) e^{-i\nu t} \quad (3.6)$$

Then $\rho_\nu(\vec{x})$ must obey

$$\vec{\nabla}^2\rho_\nu - \left.\frac{\partial U(\rho)}{\partial\rho}\right|_{\rho=\rho_\nu} + \nu^2\rho_\nu = 0 \ . \tag{3.7}$$

The condition on $U(\rho)$ is therefore that this equation should yield a localized solution for $\rho_\nu(\vec{x})$ for some range of ν. An explicit example of a potential that supports such solutions will be considered in the next section. If this condition is satisfied, then soliton solutions of the form (3.6) with periodic intrinsic time dependence will exist. They will carry a charge

$$Q = \nu \int \rho_\nu^2(\vec{x})\, d\vec{x} \ . \tag{3.8}$$

There will also be other charged-soliton solutions to Eq. (3.5) with a more complicated space-time dependence (even in the rest frame) than the family of solutions given in (3.6). However, it can be shown[14] that the latter carry the least energy for a given charge. Consequently, on physical grounds, we would expect these to be the most significant contributors to the partition function for any given total charge of the system. Let us proceed to show how their contribution to the partition function may be extracted in the functional integral formalism.

The Hamiltonian for the system is given in terms of the real and imaginary parts of $\phi(\vec{x})$ by

$$H = \int \{ \tfrac{1}{2}\pi_1^2 + \tfrac{1}{2}\pi_2^2 + \tfrac{1}{2}(\vec{\nabla}\phi_1)^2 + \tfrac{1}{2}(\vec{\nabla}\phi_2)^2 + U[(\phi_1^2+\phi_2^2)^{1/2}] \}\, d\vec{x} \ , \tag{3.9}$$

where the momenta, $\pi_1(\vec{x})$ and $\pi_2(\vec{x})$, conjugate, respectively, to $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$ are given by

$$\pi_1(\vec{x}) = \frac{\partial\phi_1}{\partial t} \equiv \dot{\phi}_1(\vec{x}), \quad \pi_2(\vec{x}) = \frac{\partial\phi_2}{\partial t} \equiv \dot{\phi}_2(\vec{x}) \ . \tag{3.10}$$

The appropriate procedure to compute the partition function in the presence of a conserved charge is to either constrain the charge or to introduce a suitable chemical potential. We will adopt the former method. The partition function for this system is then given by the following path integral

$$Z = \int \mathfrak{D}[\phi_1(\vec{x})]\mathfrak{D}[\phi_2(\vec{x})]\mathfrak{D}[\pi_1(\vec{x})] \times \mathfrak{D}[\pi_2(\vec{x})]\delta(Q-Q_0)e^{-\beta H} \ , \tag{3.11}$$

where Q, the charge given in (3.4) reduces to

$$Q = \int [\phi_2(\vec{x})\pi_1(\vec{x}) - \phi_1(\vec{x})\pi_2(\vec{x})]\, d\vec{x} \ . \tag{3.12}$$

In contrast to the uncharged case in (2.4), we notice that now the canonical momenta $\pi_1(\vec{x})$ and $\pi_2(\vec{x})$ are coupled to the fields $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$ through the delta function and thus cannot be integrated separately. Therefore, let us change variables to polar fields $\rho(\vec{x})$ and $\theta(\vec{x})$ defined by

$$\phi(\vec{x},t) = \rho(\vec{x},t)e^{-i\theta(\vec{x},t)} \ . \tag{3.13}$$

The Lagrangian given by (3.2) in terms of ρ and θ is

$$L = \int [\tfrac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) + \tfrac{1}{2}\rho^2(\partial_\mu\theta)(\partial^\mu\theta) - U(\rho)]\, d\vec{x} \ . \tag{3.14}$$

The momenta, $p(\vec{x})$ and $q(\vec{x})$, canonically conjugate to $\rho(\vec{x})$ and $\theta(\vec{x})$ are thus

$$p(\vec{x}) = \dot{\rho}(\vec{x}), \quad q(\vec{x}) = \rho^2(\vec{x})\dot{\theta}(\vec{x}) \tag{3.15}$$

and the charge in (3.4) is now given by

$$Q = \int q(\vec{x})\, d\vec{x} \ . \tag{3.16}$$

The Hamiltonian of the system given in (3.9) reduces to

$$\begin{aligned} H &= \int [\tfrac{1}{2}\dot{\rho}^2 + \tfrac{1}{2}\rho^2\dot{\theta}^2 + \tfrac{1}{2}(\vec{\nabla}\rho)^2 + \tfrac{1}{2}\rho^2(\vec{\nabla}\theta)^2 + U(\rho)]\, d\vec{x} \\ &= \int [\tfrac{1}{2}p^2 + \tfrac{1}{2}\frac{q^2}{\rho^2} + \tfrac{1}{2}(\vec{\nabla}\rho)^2 + \tfrac{1}{2}\rho^2(\vec{\nabla}\theta)^2 + U(\rho)]\, d\vec{x} \ . \end{aligned} \tag{3.17}$$

Note that the Jacobian of the transformation from the variables $\phi_1, \phi_2, \pi_1, \pi_2$ to ρ, θ, p, q is unity. Thus the partition function in these new variables is

$$Z = \int \mathfrak{D}[\rho(\vec{x})]\mathfrak{D}[\theta(\vec{x})]\mathfrak{D}[p(\vec{x})]\mathfrak{D}[q(\vec{x})]\delta(Q-Q_0)e^{-\beta H} \ . \tag{3.18}$$

Using the integral representation for the delta function

$$\delta(Q-Q_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[i\alpha(Q-Q_0)]\, d\alpha \ . \tag{3.19}$$

We can write the partition function as

$$Z = \frac{1}{2\pi}\int d\alpha\, \mathfrak{D}[p(\vec{x})]\mathfrak{D}[q(\vec{x})]\mathfrak{D}(\rho(\vec{x}))\mathfrak{D}[\theta(\vec{x})]e^{-\beta H}e^{i\alpha(Q-Q_0)} \ . \tag{3.20}$$

The path integral over $p(\vec{x})$ is just a product of Gaussian integrals, and can be easily evaluated

$$\int \mathfrak{D}[p(\vec{x})]\exp\left[-\frac{\beta}{2}\int d\vec{x}\,p^2(\vec{x})\right]=\prod_{\vec{x}}\left[\frac{2\pi}{\beta}\right]^{1/2} . \tag{3.21}$$

The path integral over $q(\vec{x})$ is easily converted to a product of Gaussians and then evaluated as follows:

$$\int \mathfrak{D}[q(\vec{x})]\exp\left[i\alpha\int d\vec{x}\,q(\vec{x})-\frac{\beta}{2}\int d\vec{x}\frac{q^2(\vec{x})}{\rho^2(\vec{x})}\right]$$

$$=\int \mathfrak{D}[q(\vec{x})]\exp\left[-\frac{\beta}{2}\int d\vec{x}\left[q-\frac{i\alpha\rho^2}{\beta}\right]^2\right]\exp\left[-\frac{\beta}{2}\frac{\alpha^2}{\beta^2}\int\rho^2(\vec{x})\,d\vec{x}\right]$$

$$=\exp\left[-\frac{\alpha^2}{2\beta}\int\rho^2(\vec{x})\,d\vec{x}\right]\prod_{\vec{x}}\left[\left[\frac{2\pi}{\beta}\right]^{1/2}\rho(\vec{x})\right] . \tag{3.22}$$

Substituting (3.21) and (3.22) in (3.20)

$$Z=\left[\prod_{\vec{x}}\left[\frac{2\pi}{\beta}\right]\right]\int\rho(\vec{x})\mathfrak{D}[\rho(\vec{x})]\mathfrak{D}[\theta(\vec{x})]\,d\alpha\exp\left[-i\alpha Q_0-\frac{\alpha^2}{2\beta}\int\rho^2(\vec{x})\,d\vec{x}\right]$$

$$\times\exp\left[-\frac{\beta}{2}\int[\tfrac{1}{2}(\vec{\nabla}\rho)^2+\tfrac{1}{2}\rho^2(\vec{\nabla}\theta)^2+U(\rho)]\,d\vec{x}\right] . \tag{3.23}$$

The integral over α is also converted to a Gaussian and then evaluated

$$\int d\alpha\exp\left[-i\alpha Q_0-\frac{\alpha^2}{2\beta}\int\rho^2(\vec{x})\,d\vec{x}\right]$$

$$=\exp\left[-\frac{\beta}{2}\frac{Q_0}{\int\rho^2(\vec{x})\,d\vec{x}}\right]\int d\alpha\exp\left[-\left[\int\rho^2(\vec{x})\,d\vec{x}\right]\left[\alpha+i\beta Q_0\Big/\int\rho^2(\vec{x})\,d\vec{x}\right]^2(2\beta)^{-1}\right]$$

$$=\sqrt{2\pi\beta}\frac{1}{\left[\int\rho^2(\vec{x})\,d\vec{x}\right]^{1/2}}\exp\left[-\frac{\beta}{2}Q_0^2\Big/\int d\vec{x}\,\rho^2(\vec{x})\right] . \tag{3.24}$$

The partition function is thus reduced to a path integral over only $\rho(\vec{x})$ and $\theta(\vec{x})$

$$Z=\left[\frac{\beta}{2\pi}\right]^{1/2}\left[\prod_{\vec{x}}\frac{2\pi}{\beta}\right]\int\frac{\rho(\vec{x})\mathfrak{D}[\rho(\vec{x})]\mathfrak{D}[\theta(\vec{x})]}{\left[\int\rho^2(\vec{x})\,d\vec{x}\right]^{1/2}}e^{-\beta E} , \tag{3.25}$$

where

$$E=\int[\tfrac{1}{2}(\vec{\nabla}\rho)^2+\tfrac{1}{2}\rho^2(\vec{\nabla}\theta)^2+U(\rho)]\,d\vec{x}+Q_0^2\Big/\left[2\int d\vec{x}\,\rho^2(\vec{x})\right] . \tag{3.26}$$

Rewritten in terms of $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$,

$$Z=\left[\frac{\beta}{2\pi}\right]^{1/2}\left[\prod_{\vec{x}}\frac{2\pi}{\beta}\right]\int\frac{\mathfrak{D}[\phi_1(\vec{x})]\mathfrak{D}[\phi_2(\vec{x})]}{\left[\int(\phi_1^2+\phi_2^2)\,d\vec{x}\right]^{1/2}}e^{-\beta E} \tag{3.27}$$

with

$$E=\int\{\tfrac{1}{2}(\vec{\nabla}\phi_1)^2+\tfrac{1}{2}(\vec{\nabla}\phi_2)^2+U[(\phi_1^2+\phi_2^2)^{1/2}]\}\,d\vec{x}+Q_0^2\Big/\left[2\int(\phi_1^2+\phi_2^2)\,d\vec{x}\right] . \tag{3.28}$$

The functional integral in (3.27) is evaluated using the method of steepest descent. In this, the dominant contributions to the partition function are from the neighborhood of those configurations of $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$ that minimize E in (3.28), that is, from the solutions to the equations

$$-\vec{\nabla}^2\phi_1+\frac{\partial U(|\phi|)}{\partial(|\phi|)}\frac{\phi_1}{|\phi|}-\frac{Q_0^2}{\left[\int(\phi_1^2+\phi_2^2)\,d\vec{x}\right]^2}\phi_1=0\ , \tag{3.29}$$

$$-\vec{\nabla}^2\phi_2+\frac{\partial U(|\phi|)}{\partial(|\phi|)}\frac{\phi_2}{|\phi|}-\frac{Q_0^2}{\left[\int(\phi_1^2+\phi_2^2)\,d\vec{x}\right]^2}\phi_2=0\ . \tag{3.30}$$

Equations (3.29) and (3.30) are integro-differential equations for $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$, and as such, it may appear to be a difficult task to obtain solutions to them. However, a family of simple but nontrivial solutions may be obtained by the following trick.[16] Writing $\phi(\vec{x})$ in polar form, with a space-independent phase

$$\phi(\vec{x})\equiv\phi_1(\vec{x})+i\phi_2(\vec{x})=\rho(\vec{x})e^{i\theta} \tag{3.31}$$

we note that both (3.29) and (3.30) reduce to

$$-\vec{\nabla}^2\rho+\frac{\partial U(\rho)}{\partial\rho}-\frac{Q_0^2}{\left[\int\rho^2(\vec{x})\,d\vec{x}\right]^2}\rho=0\ . \tag{3.32}$$

Configurations with a space-dependent phase θ have higher free energy because they give an extra positive contribution $\frac{1}{2}\int\rho^2(\vec{\nabla}\theta)^2d\vec{x}$ to E in (3.26), and will therefore not be considered here.

However, note that Eq. (3.32) is the same as Eq. (3.7) [which has solutions of the form (3.6)] provided we make the following identification

$$Q_0=\nu\int\rho_\nu^2(\vec{x})\,d\vec{x}\ . \tag{3.33}$$

That is, a solution $\rho_\nu(\vec{x})$ to Eq. (3.7) that has a charge equal to Q_0 will also be a solution to (3.32).

For a fixed Q_0, Eq. (3.32) can have several solutions with different frequencies ν. Consider any one such solution $\rho_\nu(\vec{x})$ with a frequency ν. Let the energy [given in (3.26)] associated with this solution be denoted by E_ν. Then the contribution to the partition function from this solution and its Gaussian neighborhood (in function space) will be

$$Z_\nu=\left[\prod_{\vec{x}}\frac{2\pi}{\beta}\right]\left[\frac{\beta}{2\pi}\right]^{1/2}\left[\frac{\beta}{2\pi}\right]\times\exp\left[-\beta\left[E_\nu+\frac{\sigma}{\beta}\right]\right]\Big/\left[\int\rho_\nu^2(\vec{x})\,d\vec{x}\right]^{1/2}\ , \tag{3.34}$$

where $(\beta/2\pi)e^{-\sigma}$ is the contribution of the Gaussian fluctuations about $\rho_\nu(\vec{x})$ and arises as follows. As explained in Sec. II, there is a factor of $\sqrt{\beta/2\pi}$ associated with the zero-frequency translation mode. Unlike the uncharged case dealt with there, where there was only one zero-frequency eigenmode, now the spectrum of the fluctuations consists of two zero-frequency eigenvalues—one corresponding to uniform translation in ordinary space and the other corresponding to uniform rotation in internal space. These give rise to a factor $[(\beta/2\pi)^{1/2}]^2$ in the partition function. The β-independent contributions from the zero-frequency modes are absorbed in $e^{-\sigma}$, which also includes the contributions from the remaining nonzero frequency eigenmodes of the fluctuations. σ is independent of β. Though there is a specific prescription, analogous to that given in Sec. II, for evaluating σ, it is difficult to evaluate it in practice, partly because of the added complications due to the presence of two zero-frequency eigenmodes. Note, however, that σ is one order of β smaller than βE; we will therefore neglect it in comparison to βE in further computations.

In the following section, we will identify all the solutions to Eq. (3.32) and evaluate their contribution to the partition function only to leading order in β. If the solution that we picked here had corresponded to a configuration of N identical, well-separated solitons, then an extra multiplying factor of $V^N/N!$ will have to be introduced in the expression (3.34) for the corresponding partition function. This factor arises from the integration over the locations of the N solitons.

If other scalar fields are also present in the theory, then they will be unaffected by the $U(1)$ symmetry transformation of $\phi(\vec{x})$. Therefore the charge associated with this symmetry remains the same. The procedure given in this section for the computation of the partition function in the presence of a conserved charge still holds. These other fields will be unaffected in the early stages of the procedure and will appear in the final result (3.25) along with the radial field $\rho(\vec{x})$ with an energy functional which depends on these fields and $\rho(\vec{x})$.

IV. CHARGED SOLITONS AS CONTINUOUS GAS SPECIES

When it comes to the question of the solitons forming a gas, there are two differences between the uncharged solitons of Sec. II and the charged solitons of Sec. III. Firstly, in the uncharged case there is a unique static soliton function (as for instance, in the sine-Gordon case) whereas, for the charged soliton [Eq. (3.32)], one can have a continuous family of solitons by continuously varying ν. Secondly, in the uncharged case, one could add the contributions from an arbitrary number of solitons and antisolitons under the boundary conditions specified in Sec. II.

In the charged case, one cannot add the contributions from an arbitrary number of the same single soliton since their charge will not add up to Q_0. Remember that the demand of charge conservation persists in our result through Eqs. (3.7), (3.32), and (3.33); unless (3.33) is satisfied, a solution of (3.7) will not satisfy (3.32).

Given these differences, it is still possible to have contributions from configurations consisting of different numbers of solitons in the following sense. The simplest possibility is to have a one-soliton configuration with some frequency ν_1 such that $Q_0=\nu_1\int\rho_{\nu_1}^2(\vec{x})\,d\vec{x}$. The next possibility is to have two identical widely separated solitons each rotating with some frequency ν_2 and having a charge $Q_0/2$, i.e., $Q_0=2\nu_2\int\rho_{\nu_2}^2(\vec{x})\,d\vec{x}$. Remember that $\rho_{\nu_2}(\vec{x})$ is a solution to Eq. (3.7) with $\nu=\nu_2$ and is a different function from $\rho_{\nu_1}(\vec{x})$. In this double-soliton configuration, the two solitons are identical to one another but different from the single soliton $\rho_{\nu_1}(\vec{x})$ mentioned earlier. Though the charge of each ν_2 soliton is half that of the ν_1 soliton, ν_2 is not half of ν_1 nor is the energy of each ν_2 soliton half that of the ν_1 soliton. These depend on the exact form of the potential. Similarly, one can have three solitons all rotating with a frequency ν_3 and carrying a charge $Q_0/3$ each, and so on. Of all these configurations that can contribute to the partition function for a given Q_0, the one that has the minimum free energy will dominate.

The most interesting situation is where in the thermodynamic ($V\to\infty$) limit, Q_0 is proportional to V, i.e., the charge density $q_0\equiv Q_0/V$ is finite. One can still ask the question whether an infinite number of solitons are necessary for this or not. Typically, one finds that in the allowed range of frequencies, the charge of a single soliton varies from a finite to an infinite value, i.e., a charge of the order of the volume of the system (see for instance the example below). In that case, one can achieve a total charge of Q_0 proportional to V with either some finite numbers of solitons each having a charge of order V, or an infinite number of solitons each with finite charge. Intuitively, one would expect the latter possibility, because of the enormous contribution to entropy from the infinite number of solitons. This indeed turns out to be true. Even then, it still remains to find the optimal number density of solitons which minimizes the free energy. Once we find this optimal n_0 and an associated ν_0, we essentially have a gas of solitons each of charge q_0/n_0 and density n_0 that dominates the partition function.

As mentioned towards the end of Sec. III, the contribution of a configuration of N solitons (each rotating with a frequency ν_N, and having a charge Q_N and energy E_N) to the partition function will be proportional to

$$Z_N=\frac{V^N}{N!}\frac{e^{-\beta NE_N}\sqrt{\nu_N}}{\sqrt{Q_0}}\ . \tag{4.1}$$

The factors of $2\pi/\beta$ which are common to all configurations, have been left out. For ready reference, we rewrite the expressions for Q_N and E_N here

$$Q_N=N\int\rho_{\nu_N}^2(\vec{x})\,d\vec{x}\ , \tag{4.2}$$

$$E_N=\int[\tfrac{1}{2}(\vec{\nabla}\rho_{\nu_N})^2+U(\rho_{\nu_N})]\,d\vec{x} + Q_N^2\Big/\left(2\int\rho_{\nu_N}^2(\vec{x})\,d\vec{x}\right)\ , \tag{4.3}$$

where $\rho_{\nu_N}(\vec{x})$ is the one-soliton function rotating with a frequency ν_N. Explicit expressions will be given for Q_N and E_N in the specific model discussed later in this section.

The partition function is given by a sum of the contributions of all such configurations with arbitrary N's

$$Z=\left(\prod_{\vec{x}}\frac{2\pi}{\beta}\right)\left(\frac{\beta}{2\pi}\right)^{3/2}\sum_N Z_N\ , \tag{4.4}$$

where each of the N-soliton configurations being summed over must have a total charge of Q_0. It is, in practice, difficult to evaluate the sum to give a closed-form expression for Z. We will therefore approximate it by its peak value multiplied by a typical width which will soon be defined precisely.

We now proceed to find the optimal number density of solitons. For this, we have to maximize Z_N with the constraint

$$NQ_N=Q_0 \tag{4.5}$$

on N, or equivalently on ν_N. Notice that N and ν_N are related through (4.5). The optimal value N_0 is the solution to the equation

$$\frac{dZ_N}{dN}=0 \tag{4.6}$$

or equivalently, to the equation

$$\frac{d(\ln Z_N)}{dN}=0\ . \tag{4.7}$$

We have taken N to vary continuously, which we are allowed to do for large N. From the expression (4.1) for Z_N, we have

$$\ln Z_N=N\ln V-\beta NE_N+\tfrac{1}{2}\ln\nu_N-N(\ln N-1) -\tfrac{1}{2}\ln(2\pi)-\tfrac{1}{2}\ln N-\tfrac{1}{2}\ln Q_0\ , \tag{4.8}$$

where the Stirling's approximation for $N!$ has been used. (4.7) then becomes

$$\ln V-\beta E_N-\beta N\frac{dE_N}{dN}+\frac{1}{2\nu_N}\frac{d\nu_N}{dN}-\ln N-\frac{1}{2N}=0\ . \tag{4.9}$$

We express all the quantities in this equation in terms of ν_N; as we shall soon see, it will be simpler to solve this equation for ν_N than for N. As N varies, the frequency ν_N also varies according to (4.5), and we have

$$Q_N dN + NQ_N' d\nu_N = 0 \ ,$$

i.e.,

$$\frac{d\nu_N}{dN} = -\frac{Q_N}{NQ_N'} = -\frac{Q_N^2}{Q_0 Q_N'} \ , \tag{4.10}$$

where the prime on Q_N denotes the frequency derivative, evaluated at $\nu = \nu_N$, of the single soliton charge function. Further, dE_N/dN may also be expressed completely in terms of ν_N and Q_N through Eqs. (4.3) and (4.10) as follows. We can write

$$\frac{dE_N}{dN} = \left.\frac{\partial E_N}{\partial \nu}\right|_{\nu_N} \frac{d\nu_N}{dN} \ . \tag{4.11}$$

The energy E_N of the single soliton depends on the frequency implicitly via its dependence on Q_N. From (4.3) we have

$$\frac{dE_N}{dQ_N} = \nu_N \ , \tag{4.12}$$

$$\frac{dE_N}{dN} = \frac{dE_N}{dQ_N} Q_N' \frac{d\nu_N}{dN} = -\nu_N Q_N' \frac{Q_N^2}{Q_0 Q_N'} = -\frac{\nu_N Q_N^2}{Q_0} \ . \tag{4.13}$$

Equation (4.9), which determines the optimal (peak) value then takes the form

$$-\beta E_N + \beta \nu_N Q_N - \frac{Q_N^2}{2\nu_N Q_0 Q_n'} - \ln q_0 + \ln Q_N - \frac{Q_N}{2Q_0} = 0 \ . \tag{4.14}$$

The third and last terms are of order $1/Q_0$, i.e., of order $1/V$ and may be dropped in the thermodynamic limit. This equation then reduces to

$$\ln Q_N - \ln q_0 - \beta E_N + \beta \nu_N Q_N = 0 \ . \tag{4.15}$$

For a given β and q_0, this can be solved for the frequency and a solution ν_0 obtained. Then, the optimal number density, n_0, of solitons can be obtained from (4.5) and (4.15) to be

$$n_0 \equiv \left.\frac{q_0}{Q_N}\right|_{\nu_N = \nu_0} = \exp[-\beta(E_N - \nu_N Q_N)_{\nu_N = \nu_0}] \ . \tag{4.16}$$

As mentioned earlier, the soliton number $N_0 (= n_0 V)$ that minimizes the free energy for a finite nonzero charge density *does* in fact correspond to a finite number density n_0 of solitons. At this number density, Z_N in (4.1) has a maximum value given by

$$Z_{N_0} = \frac{V^{N_0}}{N_0!} \frac{\sqrt{\nu_0} e^{-\beta N_0 E_{N_0}}}{\sqrt{Q_0}} = \left(\frac{\nu_0}{2\pi Q_0 N_0}\right)^{1/2} \exp\{ V(1 - \beta \nu_0 Q_{N_0}) \exp[-\beta(E_{N_0} - \nu_0 Q_{N_0})]\} \ . \tag{4.17}$$

The partition function given in (4.4) is approximated by the area under the Gaussian function centered at N_0 and having a peak value Z_{N_0} at that point. To get a rough estimate, we approximate this area by the product of the peak value and the half-peak width. We already have an expression for the peak value.

To obtain the half-peak width, note that the Gaussian approximation amounts to writing Z_N for any N as

$$Z_N = Z_{N_0} + \frac{1}{2} \left.\frac{d^2 Z_N}{dN^2}\right|_{N_0} (N - N_0)^2 \ . \tag{4.18}$$

The half-peak width Δ is then given by

$$\Delta = 2(\tilde{N} - N_0) \ , \tag{4.19}$$

where, as the name implies, $\tilde{N}$ is that value of N for which

$$Z_{\tilde{N}} = \tfrac{1}{2} Z_{N_0} \ . \tag{4.20}$$

We thus have, from (4.18), for the half-width

$$\Delta = -2\left(Z_{N_0} \Big/ \left.\frac{d^2 Z_N}{dN^2}\right|_{N_0} \right)^{1/2} \ . \tag{4.21}$$

Now, $(d^2 Z_N/dN^2)_{N_0}$ is given by

$$\left.\frac{d^2 Z_N}{dN^2}\right|_{N_0} = Z_{N_0}\left[\left(\frac{d \ln Z_N}{dN}\right)^2_{N_0} + \left(\frac{d^2 \ln Z_N}{dN^2}\right)_{N_0}\right]$$

$$= Z_{N_0} \left.\frac{d^2 \ln Z_N}{dN^2}\right|_{N_0} \ , \tag{4.22}$$

since N_0 is a solution to the equation $d \ln Z_N / dN = 0$. Using (4.9), (4.15), (4.10), and (4.13) this is

$$\left.\frac{d^2Z_N}{dN^2}\right|_{N_0} = Z_{N_0}\left(-\beta\frac{dE_N}{dN} - \frac{1}{N} + \beta Q_N\frac{d\nu_N}{dN} + \beta\nu_N Q_N'\frac{d\nu_N}{dN}\right)_{N_0}$$

$$= Z_{N_0}\left(\beta\nu_N\frac{Q_N^2}{Q_0} - \frac{Q_N}{Q_0} - \beta\frac{Q_N^3}{Q_0Q_N'} - \beta\nu_N\frac{Q_N^2}{Q_0}\right)_{N_0} = -Z_{N_0}\frac{Q_{N_0}}{Q_0}\left(1+\beta\frac{Q_{N_0}^2}{Q_{N_0}'}\right) \tag{4.23}$$

leading to

$$\Delta = 2\sqrt{N_0}\bigg/\left(1+\beta\frac{Q_{N_0}^2}{Q_{N_0}'}\right)^{1/2} . \tag{4.24}$$

The product of (4.17) and (4.24) gives an estimate of the partition function as

$$Z = \left(\prod_{x}\frac{2\pi}{\beta}\right)\left(\frac{\beta}{2\pi}\right)^{3/2}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\nu_0}{Q_0}\right)^{1/2}\left(1+\frac{\beta Q_{N_0}^2}{Q_{N_0}'}\right)^{-1/2}\exp\{V(1-\beta\nu_0 Q_{N_0})\exp[-\beta(E_{N_0}-\nu_0 Q_{N_0})]\} . \tag{4.25}$$

Once Eq. (4.15) is solved and the optimal frequency ν_0 found, then the charge and the energy at that frequency can be evaluated from (4.2) and (4.3). The contribution of the soliton solutions to the partition function in the steepest-descent approximation may then be obtained from (4.25).

Besides the soliton solutions, which are space-dependent functions, Eq. (3.32) will also, in general, have space-independent (constant ρ) solutions. $\rho(x)=0$ is a solution to the equation, but only with $Q_0=0$, which is of no interest. The other extrema of $U(\rho)-\nu^2\rho^2/2$, being solutions of (3.7) will also be space-independent solutions to (3.32) for suitable values of Q_0. From Eq. (3.33) it is clear that the charge associated with any such solution corresponds to a finite charge density. Further, from (3.26) note that the energy of any constant solution is also proportional to the volume, i.e., a finite energy density is associated with such solutions.

Thus, for a value of Q_0 proportional to the volume, (3.32) is likely to have constant ρ solutions. However, their contributions to the partition function will be of the order of $e^{-\beta V \mathcal{E}}$, $\mathcal{E}$ being the energy density associated with any such solution, and will therefore be negligible in the thermodynamic limit. The partition function will therefore receive contributions only from the space-dependent soliton solutions.

We will now consider a specific model[14,17] in one space dimension to illustrate with an example the main points of this section. We choose this particular model because explicit expressions are available in this model for the charged soliton solution and the charge and energy associated with it. The potential $U(|\phi|)$ in this model is

$$U(|\phi|) = \frac{m^2}{2}|\phi|^2 - b|\phi|^4 + c|\phi|^6 . \tag{4.26}$$

If the values of the parameters m, b, and c are restricted by the condition

$$b^2 < 2m^2c \tag{4.27}$$

the potential U satisfies all the conditions stated in Sec. III to be necessary for charged soliton solutions to exist. Equation (3.7) for $\rho(x)$ takes the form

$$\frac{d^2\rho}{dx^2} = (m^2-\nu^2)\rho - 4b\rho^3 + 6c\rho^5 . \tag{4.28}$$

This has two space independent solutions, the extrema of $U(\rho)-\nu^2\rho^2/2$, given by

$$\rho_\pm^2(x) = \frac{b}{3c} \pm \frac{(b^2-3ac)^{1/2}}{3c} \tag{4.29}$$

where $a \equiv (m^2-\nu^2)/2$. These solutions are, of course, functions of the frequency ν and exist in certain ranges of ν.

$$m > \nu > \tilde{\nu} \text{ for } \rho_- \tag{4.30}$$

and

$$\nu > \tilde{\nu} \text{ for } \rho_+ , \tag{4.31}$$

where $\tilde{\nu}$ is defined by

$$\tfrac{3}{2}(m^2-\tilde{\nu}^2)c = b^2 . \tag{4.32}$$

Equation (4.28) has also nontopological soliton solutions with $\rho(x)\to 0$ as $x\to\pm\infty$, for a range of ν given by

$$m > \nu > \nu_{\min} , \tag{4.33}$$

where $\nu_{\min}$ satisfies

$$2(m^2-\nu_{\min}^2)c = b^2 . \tag{4.34}$$

The solution is

$$\rho_\nu^2(x) = \frac{m^2-\nu^2}{b+[b^2-2(m^2-\nu^2)c]^{1/2}\cosh[2(m^2-\nu^2)^{1/2}x]} . \tag{4.35}$$

This has a half-width given by

$$\xi = \frac{1}{\sqrt{2a}} \ln \left[\frac{2b + 4(b^2 - 4ac)^{1/2}}{(b^2-4ac)^{1/2}} \right] . \tag{4.36}$$

This solution carries a charge

$$Q_\nu = \frac{\nu}{2\sqrt{2}\sqrt{c}} \ln \left[\frac{b + \sqrt{4ac}}{b - \sqrt{4ac}} \right] \tag{4.37}$$

and has an energy

$$E = \frac{2m^2c + 2\nu^2 c - b^2}{8\sqrt{2}c\sqrt{c}} \ln \left[\frac{b + \sqrt{4ac}}{b - \sqrt{4ac}} \right] + \frac{\sqrt{a}\, b}{2\sqrt{2}c} . \tag{4.38}$$

It can easily be shown that

$$\begin{aligned} &E_\nu, Q_\nu \to 0 \quad \text{as} \quad \nu^2 \to m^2 , \\ &E_\nu, Q_\nu \to \infty \quad \text{as} \quad \nu^2 \to \nu^2_{\min} , \\ &E_\nu - \nu Q_\nu \to C \quad \text{as} \quad \nu^2 \to \nu^2_{\min} , \end{aligned} \tag{4.39}$$

where C has a positive finite value.

The contribution of these charged soliton solutions to the partition function for a given charge density Q_0/V, can be obtained from Eq. (4.25). For this, we need to know the optimal number density n_0 of the solitons and the corresponding optimal frequency ν_0. These are obtained by solving for ν_0 from (4.15) which is

$$\ln Q_N - \ln q_0 - \beta E_N + \beta \nu_N Q_N = 0$$

and then using (4.5) to evaluate n_0. Since we know explicitly how the charge Q_N and the energy E_N of the soliton in this model vary with ν_N, we can solve this equation for a given β and q_0. We must, of course, make sure that Eq. (4.15) does indeed have solutions. We rewrite (4.15) as

$$Q_N / q_0 = e^{\beta(E_N - \nu_N Q_N)} \tag{4.40}$$

and examine this graphically for specific values $b = c = m = 1$ of the parameters in the potential U. In Fig. 1 we have plotted the right-hand side as a function of ν for a particular value of β. The left-hand side has also been plotted in the same figure for three different values of q_0. It is clear that there is at least one point of intersection, that is, at least one solution to Eq. (4.15). This can also be seen directly from Eq. (4.40) as follows. The left-hand side, for any q_0, is zero at $\nu = \nu_{\max} (= m)$ and increases monotonically as ν decreases, tending towards infinity as $\nu \to \nu_{\min}$. The quantity $(E_N - \nu_N Q_N)$ is zero at $\nu = \nu_{\max}$ and it also increases monotonically as ν decreases, but it has a finite positive value at $\nu_{\min}$. Thus, for any β, the right-hand side will be unity at $\nu_{\max}$ and will continuously increase with decreasing ν. At $\nu_{\min}$ it has some finite value. Thus Eq. (4.40) will have at least one solution ν_0 for any q_0 and β. From

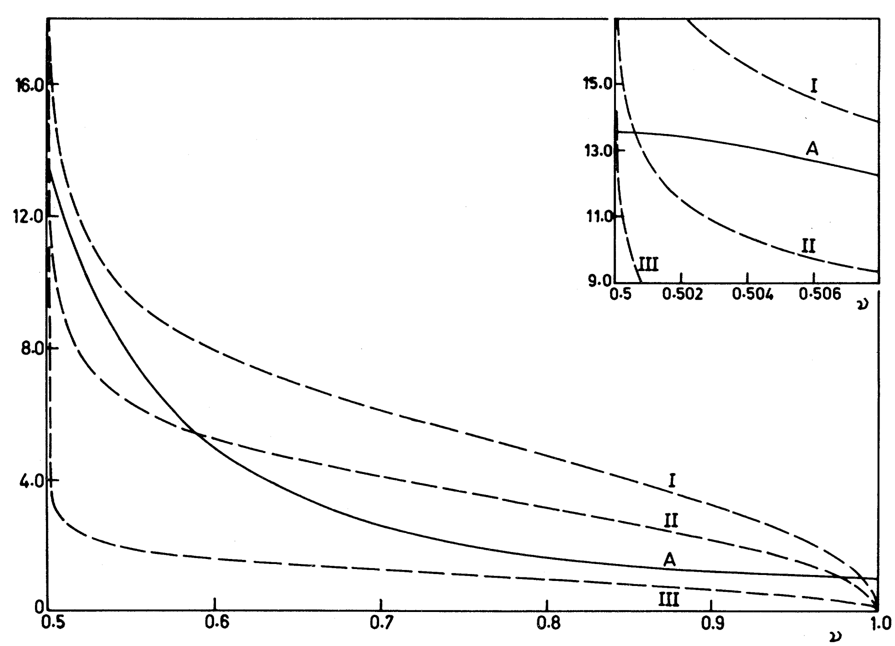

FIG. 1. A numerical solution to Eq. (4.40) for $\beta = 15$ and three values of q_0. Curve A is the plot of $e^{\beta(E_N - \nu_N Q_N)}$ vs the frequency ν for $\beta = 15$. The dashed curves I, II, III are the plots of Q_N/q_0 for $q_0 = 0.1$, 0.15, 0.5, respectively. All the four curves are for parameter values $b = 1$, $c = 1$, $m = 1$.

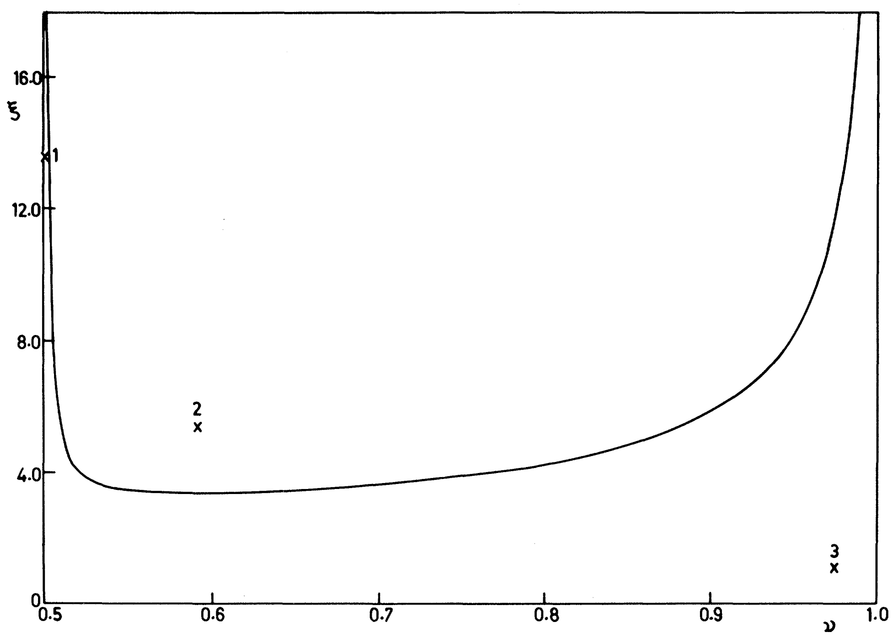

FIG. 2. Comparison of the intersoliton separation Q_N/q_0 and the soliton width ξ. The solid curve shows the soliton width ξ as a function of the frequency ν. The three crosses denote the three points of intersection of the curves A and II in Fig. 1, i.e., for $\beta=15$ and $q_0=0.15$. The ordinates of the crosses are the intersoliton separations. Of these, only the cross marked "2" represents a valid N-soliton solution, i.e, where the intersoliton separation is much larger than the soliton width.

Fig. 1, we can see that for a certain range of q_0 (this range depends on β) there are as many as three intersections denoted by crosses in Fig. 2. However, as we shall now show, only one of these corresponds to a true solution. Note that the average separation between any two solitons in a configuration of N solitons is given by Q_N/q_0. For this configuration to be a solution of (4.15), this average separation must be much larger than the soliton width. This must be true for the separation and the width evaluated at the optimal frequency (i.e., at the values of ν corresponding to the intersections). In Fig. 2 we have plotted the soliton width as a function of ν. The width becomes arbitrarily large at both limits, $\nu_{\min}$ and m, of the frequency range. As is clear from this figure, only one of the points of intersection, the central one, satisfies the criterion for representing a solution to Eq. (4.15).

Having found the optimal frequency ν_0, one can easily evaluate (4.25) to give the solitonic contribution to the partition function.

Let us now consider the case of a finite charge instead of a finite charge density discussed so far. Following the same procedure, we have to look for solutions to Eq. (4.40) for a finite charge Q_0. Note that though the right-hand side remains unaffected, the q_0 in the denominator of the left-hand side which was finite earlier is now vanishingly small. The solution to (4.40) will then tend to be the trivial solution; so the dominant contribution to the partition function will be from small fluctuations about the trivial solution, i.e., from the phonons with a total given finite charge. The system will then behave like a gas of phonons rather than a gas of solitons.

ACKNOWLEDGMENTS

We thank Dr. H. R. Krishnamurthy for helpful discussions.

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