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Uniform expansions for nearly locked coupled Josephson weak links

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A uniform perturbation calculation is carried out for two interacting Josephson weak links in the resistively shunted junction model, particular attention being paid to the region of near voltage locking. Good agreement is achieved with numerical simulation. Phenomena observed experimentally are also predicted. A comparison with other theoretical calculations is made.

I. INTRODUCTION

In recent work of Nerenberg *et al.*[1] (hereafter referred to as I) a perturbation-theory calculation of the dc voltage characteristics of two coupled Josephson weak links in the resistively shunted junction (RSJ) model was carried out. Particular attention was focused on the phenomenon of synchronization or dc voltage locking of the junctions. Numerical simulation of the exact equations showed remarkable agreement with the analytical perturbation calculation except in the border region of locking, or what experimentalists refer to as the "voltage pulling"[2] region. This perturbation calculation was used to show that voltage locking could take place for N interacting junctions in a linear array.[3] The locking region predicted by perturbation theory was well substantiated by the numerical simulation.

In this work we turn to the border region of locking which yielded anomalies between perturbation theory and simulation in the original work I. In brief, the perturbation theory predicted the sudden discontinuous onset of locking in parameter space, while the numerical simulation indicated a rapid but continuous onset. The latter calculation indicated that this border region was characterized by a relatively constant short-time average phase difference between junctions followed periodically by rapid slips of 2π. This led to the explanation in I of the observed[4] coherence of microwave radiation from pairs of junction in this shoulder region.[5]

The fact that unusual behavior occurs in this shoulder region is also known in biology where it is referred to as "fringe entrainment"[6] and in coupled chemical oscillators.[7] We therefore turn to the mathematical analysis of this problem in the context of a coupled pair of Josephson weak links in the RSJ model.

The original perturbation calculation I was marked by a "renormalization process" which was necessary in displaying voltage locking and in giving generally very accurate dc voltage characteristics. It was essential in dealing with the linear-array problem.[3] However, this perturbation theory was not uniform in *parameter space* in the transition region to voltage locking. This resulted in the variance there between the perturbation calculation and the numerical simulation of the exact equations. Our approach, using the method of averaging, here eliminates this problem at the expense of some increase in complexity.

The equations of the coupled links expressed in the normalized units of I are

$$\frac{d\phi_j}{dt}=\delta_j I_j-\delta_j I_{cj}\sin\phi_j-\alpha(I_i-I_{ci}\sin\phi_i)\;, \qquad (1)$$

where $i,j=1,2$, $i\neq j$.

Since we seek to describe a succession of superconducting weak links in a filament we have neglected in our model capacitance effects. Equations (1) describe a pair of weak links either coupled by a resistive (nonreactive) shunt in series aiding or opposing configurations,[1] or coupled by the diffusion of quasiparticles generated during the phase slip process.[2] The parameter α measures the strength of the coupling between the pair, and in the former case can be taken to be $(1+R_s/R_2)^{-1}$, where R_s is the shunt resistance and R_2 is the resistance of the second junction, while $\delta_1=1$, and

$$\delta_2=(1+R_s/R_1)/(1+R_s/R_2)\;,$$

R_1 being the resistance of the first junction. Note that dimensionless time is t rather than t^*.

The form of the method of averaging that we use is described by Nayfeh[8] (p. 168). Although the variables of interest in the locking case would be $(\phi_1-\phi_2),(\phi_1+\phi_2)$ the equations (1) in these variables do not have the appropriate form for this method. To correct for this we change the variables ϕ_1,ϕ_2 to ξ_1,ξ_2 based on the nonuniform expansions of I which are valid within and away from the locking zone but not in the near-locking region. The resulting equation for D,ξ where $D=\xi_1-\xi_2$, $\xi=\xi_1+\xi_2$ have the form suitable for the method of averaging [Eqs. (6) and (7) below]. For the next step the near-identity transformation (D,ξ) to $(\bar{D},\bar{\xi})$ is performed [Eqs. (8) and (9)] which has the effect of expressing D,ξ as a sum of a slowly varying term and a term involving fast oscillations with no average value over the period of the oscillations. The voltages $d\phi_1/dt$, $d\phi_2/dt$ have the same decomposition as a sum of fast and slow processes [Eq. (14)]. Thus, this technique provides an excellent uniform description of the physical phenomena throughout the locking zone. Another important advantage of the transformation to the new variables ξ_1,ξ_2 is the natural way in which the form of the asymptotic expansion in α arises [Eq. (4)]. There are three variations of this asymptotic expansion. The more general form is very accurate in comparison with the numerical simulation, while the other two, one of which is used in Ref. 9, are less so.

II. ANALYSIS

The first step is to change the dependent variables ϕ_1,ϕ_2 to new variables ξ_1,ξ_2 by the following transformation. We use the identities defined by Eqs. (12), (14), and (15) of I where f_0,g_0 are replaced by ϕ_1,ϕ_2 and $A_1(t),A_2(t)$ replace t^* in f_0,g_0, respectively. Thus, for arbitrary constants ψ_i,

$$\begin{aligned}
&\phi_i=2\tan^{-1}\left[\frac{\delta_i I_{ci}}{\omega_i}+\frac{\Omega_i}{\omega_i}\tan\left[\frac{\Omega_i A_i}{2}+\psi_i\right]\right] ,\\
&\frac{d\phi_i}{dA_i}=\omega_i-\delta_i I_{ci}\sin\phi_i ,\\
&\sin\phi_i=\frac{\delta_i I_{ci}+\omega_i\sin\xi_i}{\omega_i+\delta_i I_{ci}\sin\xi_i} ,\\
&\xi_i=\Omega_i A_i+\theta_i,\quad \theta_i=\tan^{-1}\left[\frac{\delta_i I_{ci}}{\Omega_i}\right]+2\psi_i ,\\
&\omega_i^2=\Omega_i^2+(\delta_i I_{ci})^2 .
\end{aligned} \tag{2}$$

The parameters ω_i,Ω_i, which at this point are not uniquely determined in the above, will depend on α.

From (1) and (2) the equations for ξ_j are

$$\begin{aligned}
&\frac{d\xi_j}{dt}=\Omega_j+\alpha f_i g_j,\quad g_j=\frac{\omega_j+\delta_j I_{cj}\sin\xi_j}{\Omega_j} ,\\
&f_i=\frac{\omega_i}{\delta_i}+(\delta_i I_j-\omega_j)\alpha^{-1}\\
&\qquad -I_i-\frac{\Omega_i^2\delta_i^{-1}}{\omega_i+\delta_i I_{ci}\sin\xi_i} ,
\end{aligned} \tag{3}$$

where $i\neq j$, $i,j=1,2$. The dc voltages V_i are of primary interest where V_i is related to the average value of $d\phi_i/dt$; namely,

$$\begin{aligned}
V_i&=V_0\lim_{t\to\infty}\frac{1}{t}\int_0^t\left[\frac{d\phi_i}{dt}\right]dt\\
&\equiv V_0\left\langle\frac{d\phi_i}{dt}\right\rangle=V_0\lim_{t\to\infty}\left[\frac{\phi_i}{t}\right] ,
\end{aligned}$$

where the constant V_0 is given in I. To show that

$$V_i=V_0\langle d\xi_i/dt\rangle$$

we have, from (2),

$$\frac{d\phi_i}{dt}=g_i^{-1}\frac{d\xi_i}{dt} .$$

The conclusion follows upon exapnding g_i^{-1} in a Fourier series in ξ_i. It suffices, therefore, to deal with the ξ_i in determining the dc voltages.

Since $\omega_i^2=\Omega_i^2+(\delta_i I_{ci})^2$ it is natural to define ω_i,Ω_i as an asymptotic expansion in terms of α in the form

$$\begin{aligned}
&\omega_i=\delta_i I_{ci}\cosh\lambda_i ,\\
&\lambda_i=\sum_{k=0}^{n}\lambda_i^k(\alpha)\alpha^k+O(\alpha^{n+1}) ,\\
&\Omega_i=\delta_i I_{ci}\sinh\lambda_i,\quad \lambda_i^k(\alpha)=O(1) ,
\end{aligned} \tag{4}$$

where $\cosh\lambda_i,\sinh\lambda_i$ are the hyperbolic cosine and sine functions, respectively. We will show later that the coefficients $\lambda_i^k(\alpha)$ are determined by requiring that the average value of certain expressions are zero to a certain order. In particular, λ_i^0 is defined by $I_i=I_{ci}\cosh\lambda_i^0$ which ensures that $f_i=O(1)$ as $\alpha\to 0$ in (2). ω_i,Ω_i are now defined to first order in α.

The equations suitable for asymptotic expansions are obtained from (3) upon expanding f_i in a Fourier series in ξ_i. Then (3) can be expressed as

$$\frac{d\xi_j}{dt}=\Omega_j^i-\alpha\gamma_{j1}^i\cos(\xi_i-\xi_j)$$
$$-\alpha\sum e^{-\lambda_i}J_{jrs}^i\cos[(2r+2)\xi_i+s(\xi_i-\xi_j)]+L_{jrs}^i\sin[(2r+1)\xi_i+s(\xi_i-\xi_j)]\ , \quad (5)$$

where the summation is over $r=0,1\ldots$; $s=-1,0,1$. The coefficients in (5) are given by

$$\gamma_{js}^i=I_{ci}\frac{\sinh\lambda_i}{\sinh\lambda_j}e^{-s\lambda_i}\times(-1)^s(2\cosh\lambda_j)^{\delta_{s,0}}\ ,$$

$$K_{js}^i=(-\gamma_{js}^i+e^{\lambda_i}\sigma_j^i)\delta_{s,-1}\ ,$$

$$\Omega_j^i=\Omega_j+\alpha\sigma_j^i\cosh\lambda_j\ ,$$

$$\delta_{i,j}=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases}$$

$$\sigma_j^i=[I_{ci}e^{-\lambda_i}-I_i+(I_j-I_{cj}\cosh\lambda_j)\delta_j\alpha^{-1}](\sinh\lambda_j)^{-1}\ ,$$

$$J_{jrs}^i=(-1)^{r+1}e^{-(2r+1)\lambda_i}\gamma_{js}^i\ ,$$

$$L_{jrs}^i=J_{jrs}^i-e^{-\lambda_i}K_{js}^i\delta_{r,0}\ .$$

The dc voltage locking occurs in (5) whenever the time average of

$$d(\xi_j-N\xi_i)/dt,\quad N=1,2,\ldots$$

is zero. In the sequel only the case $N=1$ is considered, that is, equal-voltage locking; the other cases can be handled in a similar fashion.

Since we want to examine the behavior of the system in the neighborhood of voltage locking we define new variables

$$\xi_j=(\xi-\epsilon_{ij}D)/2,\quad \epsilon_{12}=1,\quad \epsilon_{21}=-1\ .$$

Then the method of averaging is applied to Eq. (3) for ξ,D. In the new variables (3) becomes

$$\frac{d\xi}{dt}=\Omega-\alpha(\gamma_{21}^1+\gamma_{11}^2)\cos D-\alpha\sum Z_{jrs}^i\ , \quad (6)$$

$$\frac{dD}{dt}=\alpha(\gamma_{21}^1-\gamma_{11}^2)(\beta+\cos D)-\alpha\sum\epsilon_{ji}Z_{jrs}^i\ , \quad (7)$$

where

$$Z_{jrs}^i=e^{-\lambda_i}J_{jrs}^i\cos[(r+1)\xi+(r+1+s)\epsilon_{ij}D]+L_{jrs}^i\sin[(r+\tfrac{1}{2})\xi+(r+\tfrac{1}{2}+s)\epsilon_{ij}D]\ ,$$

$$\Omega=\Omega_1^2+\Omega_2^1,\quad \beta=(\Omega_1^2-\Omega_2^1)\alpha^{-1}(\gamma_{21}^1-\gamma_{11}^2)^{-1}\ ,$$

and the summation is over

$$s=-1,0,1,\quad r=0,1,\ldots,$$
$$i,j=1,2,\quad i\neq j\ .$$

These equations are in a form suitable for the method of averaging where ξ is the "fast" variable and D is the "slow" variable.

For the method of averaging we substitute the following expression into (6) and (7) and equate terms in α. It is very useful at this stage to regard β and $\lambda_l, l=1,2$ as parameters and not as functions of α. This does not alter the sorting process into fast and slow terms. The consequences of doing this will be seen below. Following the usual procedures[8]:

$$D=\bar{D}+\alpha F_1(\bar{D},\bar{\xi},\lambda_l)+\alpha^2F_2(\bar{D},\bar{\xi},\lambda_l)+O(\alpha^3)\ ,$$
$$\xi=\bar{\xi}+\alpha H_1(\bar{D},\bar{\xi},\lambda_l)+\alpha^2H_2(\bar{D},\bar{\xi},\lambda_l)+O(\alpha^3)\ , \quad (8)$$

$$\frac{d\bar{D}}{dt}=\alpha Q_1(\bar{D},\lambda_l)+\alpha^2Q_2(\bar{D},\lambda_l)+O(\alpha^3)\ ,$$
$$\frac{d\bar{\xi}}{dt}=\Omega+\alpha P_1(\bar{D},\lambda_l)+\alpha^2P_2(\bar{D},\lambda_l)+O(\alpha^3)\ . \quad (9)$$

Q_i,P_i are used to ensure that F_i,H_i are periodic functions in $\bar{\xi}$ with no average value over the period. Upon substituting (8) and (9) into (6) and (7) and expanding and then equating terms in α we have

$$Q_1=(\gamma_{21}^1-\gamma_{11}^2)(\beta+\cos\bar{D})\ ,$$

$$P_1=-(\gamma_{21}^1+\gamma_{11}^2)\cos\bar{D}\ ,$$

$$H_1=-\frac{1}{\Omega}\sum Y_{jrs}^i\ ,$$

$$F_1=-\frac{1}{\Omega}\sum\epsilon_{ji}Y_{jrs}^i\ ,$$

where

$$Y^i_{jrs} = \frac{J^i_{jrs} e^{-\lambda_i}}{(r+1)} \sin[(r+1)\bar{\xi} + (r+1+s)\epsilon_{ij}\bar{D}] - \frac{L^i_{jrs}}{(r+1/2)} \cos[(r+\tfrac{1}{2})\bar{\xi} + (r+\tfrac{1}{2}+s)\epsilon_{ij}\bar{D}] ,$$

and the summation is the same as that in (6) and (7). In a similar way the expressions for F_2, H_2, and P_2, Q_2 can be determined. In (8) only $\bar{D}, \bar{\xi}$ contribute to the average value of D, ξ. From (9) the equations for $\bar{D}, \bar{\xi}$ are

$$\frac{d\bar{\xi}}{dt} = (\Omega_1+\Omega_2) - \alpha(\gamma^1_{21}+\gamma^2_{11})\cos\bar{D} + \alpha(\sigma^2_1 \cosh\lambda_1 + \sigma^1_2 \cosh\lambda_2) + \alpha^2 P_2 + O(\alpha^3) ,$$
$$\frac{d\bar{D}}{dt} = \alpha(\gamma^1_{21}-\gamma^2_{11})(B+\cos\bar{D}) + \alpha(\sigma^2_1\cosh\lambda_1 - \sigma^1_2\cosh\lambda_2) + \alpha^2 Q_2 + O(\alpha^3) , \quad (10)$$

where

$$B = (\Omega_1-\Omega_2)\alpha^{-1}(\gamma^1_{21}-\gamma^2_{11})^{-1} .$$

The asymptotic expansion for λ_j is determined in the sequel. As indicated previously λ_j and hence Ω_j are determined to first order by requiring $\sigma^i_j = O(1)$. That is,

$$I_j - I_{cj}\cosh\lambda_j = 0, \quad \lambda_j = \lambda^0_j .$$

For the next order λ_j, Ω_j are defined by requiring

$$\langle \sigma^2_1 \cosh\lambda_1 + \sigma^1_2 \cosh\lambda_2 \rangle = O(\alpha) ,$$
$$\langle \sigma^2_1 \cosh\lambda_1 - \sigma^1_2 \cosh\lambda_2 \rangle = O(\alpha) ,$$

or equivalently $\sigma^i_j = O(\alpha)$. This is achieved by truncating the expansion for λ_j and solving

$$\sigma^i_j = 0, \quad \lambda_j = \lambda^0_j + \alpha\lambda^1_j(\alpha) . \quad (11)$$

This condition is equivalent to the "renormalization" process which leads to the equation (19), (20) in I. Using λ_j in (11) the equations for $\bar{\xi}, \bar{D}$, neglecting terms of order $O(\alpha^2)$, are

$$\frac{d\bar{\xi}}{dt} = (\Omega_1+\Omega_2) - \alpha(\gamma^1_{21}+\gamma^2_{11})\cos\bar{D} ,$$
$$\frac{d\bar{D}}{dt} = \alpha(\gamma^1_{21}-\gamma^2_{11})(B+\cos\bar{D}) . \quad (12)$$

It is the parameter

$$B = (\Omega_1-\Omega_2)\alpha^{-1}(\gamma^1_{21}-\gamma^2_{11})^{-1}$$

that determines if the dc voltages are locked ($|B| \le +1$) or unlocked ($|B| > 1$). This procedure can be carried out to higher orders. λ_j is defined to the next order by imposing the conditions on Eq. (10)

$$\langle \alpha\sigma^2_1 \cosh\lambda_1 + \alpha\sigma^1_2 \cosh\lambda_2 + \alpha^2 P_2 \rangle = 0 ,$$
$$\langle \alpha\sigma^2_1 \cosh\lambda_1 - \alpha\sigma^1_2 \cosh\lambda_2 + \alpha^2 Q_2 \rangle = 0 ,$$

where $\lambda_j = \sum_{k=0}^{2} \lambda^k_j(\alpha)\alpha^k$. Then Eqs. (12) determine the dc voltages where the terms of order $O(\alpha^3)$ are neglected. The algebraic details are somewhat involved for this case. The requirement that λ^k_j in Eq. (4) depends on α is important for accurate results in comparison with the numerical simulation. If λ^k_j is independent of α (i.e., λ_j is a power series in α) then it is necessary to exapnd σ^i_j in (10) as a power series in α. The consequence of this approach is illustrated in Fig. 1.

The solution of (12) for the slow processes $\bar{D}$ when $|B| > 1$ in the unlocked region is readily shown to be

$$\bar{D} - \pi/2 = 2\tan^{-1}\{ B^{-1} + (1-B^{-2})^{1/2} \tan[(\Omega_1-\Omega_2)(1-B^{-2})^{1/2} t + \psi] \} , \quad (13)$$

where ψ is determined by the initial conditions for $\bar{D}$. In the locked region $|B| < 1$ the solution for $\bar{D}$ approaches the solution of $B + \cos\bar{D} = 0$ for large t. The connection between ϕ_1, ϕ_2, and $\bar{\xi}, \bar{D}$ follows from

$$\frac{d\phi_i}{dt} = \frac{g_i^{-1} d\xi_i}{dt} .$$

Upon expanding g_i^{-1} in a Fourier series and then using the Eqs. (8) and (9) after some algebra we have

$$\frac{d(\phi_1-\phi_2)}{dt} = \frac{d\bar{D}}{dt} + O(\alpha^2) + \mathscr{F}\mathbf{P}' , \quad (14a)$$

$$\frac{d(\phi_1+\phi_2)}{dt}=\frac{d\bar{\xi}}{dt}+O(\alpha^2)+\mathscr{F}\mathrm{P}\ , \tag{14b}$$

where $\mathscr{F}\mathrm{P}=O(1)$ in α and $\mathscr{F}\mathrm{P}$ is periodic in $\bar{\xi}$ with no average value over the period. This term $\mathscr{F}\mathrm{P}$ represents the fast oscillations superimposed on a slowly varying term. The dc voltages V_i are given by

$$\left\langle\frac{d\bar{D}}{dt}\right\rangle=(\Omega_1-\Omega_2)\begin{cases}(1-B^{-2})^{1/2}, & |B|>1\\ 0, & |B|\le 1\end{cases}$$
$$\left\langle\frac{d\bar{\xi}}{dt}\right\rangle=\Omega_1+\Omega_2+\alpha(\gamma_{21}^1+\gamma_{11}^2)B\begin{cases}1-(1-B^{-2})^{1/2}, & |B|>1\\ 1, & |B|\le 1\end{cases} \tag{15}$$
$$V_1=\frac{V_0}{2}\left[\left\langle\frac{d\bar{D}}{dt}\right\rangle+\left\langle\frac{d\bar{\xi}}{dt}\right\rangle\right]\ ,\quad V_2=\frac{V_0}{2}\left[\left\langle\frac{d\bar{\xi}}{dt}\right\rangle-\left\langle\frac{d\bar{D}}{dt}\right\rangle\right]\ .$$

For $|B|>>1$ we have, as expected, the approximations

$$V_1=V_0\Omega_1+O(\alpha^2),\quad V_2=V_0\Omega_2+O(\alpha^2)$$

of I. Before discussing these results we would like to comment on an alternative approach.

Recent work[9] has used the following method in dealing with interacting junctions [cf. Refs. (10) and (11) for further examples]. We define $\phi_j=\phi_j^0+\phi_j^1+\cdots$, where $\phi_j^1=O(\alpha)$. The equation for ϕ_j^0 is

$$\frac{d\phi_j^0}{dt}+\delta_j I_{cj}\sin\phi_j^0=\delta_j I_j-\hat{i}_{1j},\quad \hat{i}_{1j}=O(\alpha) \tag{16}$$

where $\hat{i}_{1j}$ ensures that ϕ_j^1 involves only fast processes which is the same requirement for F_i,H_i in Eq. (8). Upon changing to the new variables θ_i [Eq. (7) in Ref. 9] the resulting reduced equation is the first two terms of (5) where $\xi_i=\theta_i-\pi/2$ and $\lambda_j=\lambda_j^0$. Upon adding and subtracting these equations we have (12) where $\lambda_j=\lambda_j^0$ and Ω_j is replaced by Ω_j^i. Instead of following this last step the transformation (2) can be applied to (16) and this leads to Eqs. (11) and (12). The consequences of these different reduced equations are shown in Fig. 1 and will be discussed below.

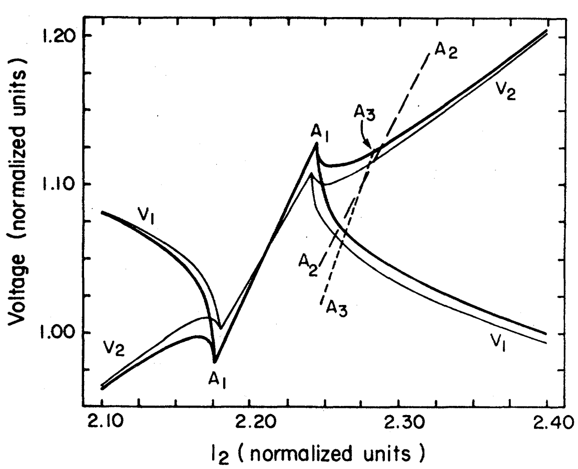

FIG. 1. Various solutions of the dc voltages V_1,V_2 vs I_2 for the region of locking as in Fig. 2 of I where $I_1=2$, $I_{c1}=1.2$, $I_{c2}=0.8$, $\delta_1=1$, $\delta_2=\delta=\frac{2}{3}$, $\alpha=0.2$ in Eq. (1). Voltages are in units of V_0. Numerical simulation (thin curves). A_1 (thick curves): asymptotic solution as given by Eqs. (11) and (12). A_2 (dashed line): asymptotic result for locking region where λ_i in Eq. (4) is a power series in α. A_3 (dashed line): asymptotic result for locking region using the approach of Ref. 9.

III. RESULTS AND COMPARISON WITH NUMERICAL SIMULATION

The basic disagreement between the numerical simulation (repeated again here as in I) and perturbation theory in the voltage pulling region is seen to be eliminated. Perturbation theory now also predicts a rapid but continuous transition to voltage locking. Figure 1 illustrates the voltage characteristics of a coupled pair in the region of voltage locking. The case illustrated in Fig. 2 of I is the one used. We see good agreement has been attained between numerical simulation and perturbation theory. Not such good agreement in the form of a shift of the voltage locked interval is shown both for our method of averaging employing a more conventional asymptotic expansion for the λ_j as mentioned above and the method of Likharev *et al.*[9]

Figure 2 illustrates the solution for D as a function of time in the border region of locking. By

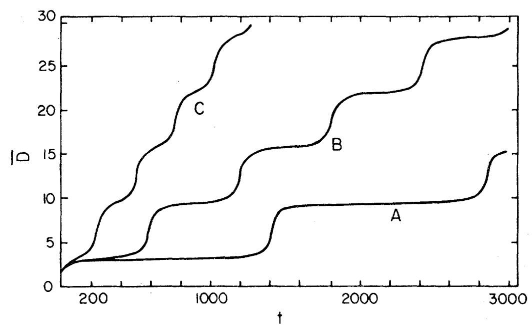

FIG. 2. Graph of the slow component of the phase difference $\bar{D}$ versus time t indicating slippage just outside the region of locking. The parameters are those of Fig. 1. A: $I_2=2.17600$. B: $I_2=2.17500$. C: $I_2=2.17000$. Locking occurs at $I_2=2.17623$ for A_1 in Fig. 1.

Eq. (15) $\bar{D}$ is closely related to $\phi_1-\phi_2$, but with the latter averaged over the fast period. $\phi_1-\phi_2$ behaves like $\bar{D}$ in time except for superposed fast terms with frequencies $(V_1+V_2)/4\pi V_0$ (and its harmonics) in this region. We see that $\bar{D}$ has the peculiar behavior of $\phi_1-\phi_2$ (averaged over the fast period) as first pointed out in I, of being constant for relatively long times and then rapidly slipping by 2π. This explains the observed phenomenon of coherence[4] just outside the locking zone. We see extremely long "constant" $\bar{D}$ intervals close to locking with a dc relative voltage difference of $\sim\frac{1}{2}\%$ (case A). Case B with $\sim 1\%$ voltage difference still shows this same effect of long periods of coherence. Case C with $\sim 2\%$ voltage difference shows a marked diminution of this effect. We would like to comment here that Fig. 3 of I is somewhat in error in indicating that the numerical simulation of $\phi_1-\phi_2$ gave rapid oscillations of the order of thirteen times the "jump" frequency. The perturbation theory tells us that this cannot be the case. This error was an artifice of the intervals at which $\phi_1-\phi_2$ was printed. Printing the results at sufficiently short intervals indicated that the high-frequency was, as expected by perturbation theory, of the order of $(V_1+V_2)/4\pi V_0$ which gives a period in these dimensionless units of 2π approximately, 200 times smaller than the jump period $2\pi V_0/(V_1-V_2)$.

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