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Vortex pinning and the decay of persistent currents in unsaturated superfluid helium films

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We investigate the role played by vortex pinning in modifying the predictions of the Kosterlitz-Thouless theory for thin helium films. We extend the analysis of Huberman, Myerson, and Doniach and of Ambegaokar *et al.* to include vortex pinning. We find that the presence of surface roughness can give rise to pinning sites and that the presence of the sites can modify the predictions of the above authors for the decay of persistent currents. We test these modifications by fitting our formulas to the data of Ekholm and Hallock, and find that the fits are much improved over the fits done without taking into account the effects of pinning.

I. INTRODUCTION

The problem of flow decay in superfluid helium films of a few atomic layers thickness has generated much interest recently. The superfluid component is inviscid so a supercurrent should persist indefinitely. However, there does exist a mechanism for the decay of a superflow through the thermal excitation of vortex pairs of opposite circulation. Thermal fluctuations in the presence of an applied flow can cause the vortices to depair. This will cause the flow to dissipate, because pulling vortices apart to the film edges causes a reduction in the phase of the superfluid wave function and hence in the supercurrent. These ideas, originally set forth by Anderson,[1] constitute the essence of the Iordanski-Langer-Fisher[2] theory of vortex nucleation, which has been applied with success to the decay of flow in three-dimensional channels.[2]

In two dimensions, this theory predicts a power-law decay of velocity with time.[2,8,9] Ekholm and Hallock[3] have found that in experiments on films of 6 — 10 atomic layers that there are substantial deviations from this prediction. Our purpose here is to account for these deviations.

A second impetus to the study of vortices in thin films came from a different direction. Kosterlitz and Thouless[4] (KT) studied the two-dimensional *XY* model, which is a model that also describes the phase of the superfluid wave function in thin films, and found that there was a phase transition of infinite order at a temperature T_{KT}. The interaction between a pair of vortices is modified by the presence of the other vortex pairs, giving a renormalized coupling constant. Below T_{KT}, the coupling constant has a finite value for vortex pairs of infinite separation. Above T_{KT}, this coupling constant is renormalized to zero at large vortex separations, which means the vortex-vortex interaction has been screened out by the other pairs. Below this temperature T_{KT} the vortices are bound in pairs, while above T_{KT} the vortices depair. Above T_{KT}, therefore, the vortices should respond as though they were free when an applied flow is imposed, and thus produce an exponential decay of the flow. Thus above T_{KT} there is no persistent flow and hence no superfluid response. Below T_{KT}, the vortices remain in pairs and are freed only by the flow, resulting in a power-law decay of the flow, and thus the film exhibits a behavior over short times that could be described as superfluid. The vortex-vortex coupling constant is directly proportional to the effective superfluid density in the film. This explained a well-known phenomena[5] in experiments on unsaturated helium films: that the effective superfluid density measured was less than the bulk value and, furthermore, that the superfluid density appeared to jump to zero at a temperature less than the bulk transition temperature. Nelson and Kosterlitz[6] were the first to note that the magnitude of the jump was directly proportional to T_{KT}, with a universal constant of proportionality.[6] The experimental verification[7] of this prediction provided the first confirmation of the Kosterlitz-Thouless theory for helium films.

The dynamic response of the vortex pairs near the transition has been investigated by Ambegaokar, Halperin, Nelson, and Siggia[8] (AHNS) and the nonlinear quasistatic response below T_{KT}

was also worked out by Huberman, Myerson, and Doniach[9] (HMD). The linear response of the vortices at finite frequency was calculated in Ref. 5 and by Ambegaokar and Teitel,[10] and their five parameter fit is in excellent agreement with the oscillating substrate experiments of Bishop and Reppy.[11] The nonlinear static response was treated by both AHNS and HMD, and does not agree with the experiments of Ekholm and Hallock.[3] In particular, the work of HMD and AHNS predicted a decay law $dv/dt=-A(v_s)^{\lambda}$ with $\lambda=(1/2\pi)\langle\rho_s\rangle \times d(h/m)^2$, where v_s is the superfluid velocity, $\langle\rho_s\rangle$ is the effective superfluid density, d is the film thickness, h is Planck's constant, and m is the mass of a ^{4}He atom. The values of λ derived from the persistent-current experiments agree neither with the HMD-AHNS calculations, which assume that the only processes present are the pairing and depairing of vortices, nor with the values calculated from third-sound data.[3] This prompted further work to resolve this difficulty. Donnelly *et al.*[12] attempted to explain the data with a competing-barrier model, but this predicts a decay that is too rapid at long times. In an attempt to improve on the AHNS-HMD nonlinear theory, McCauley[13] has calculated the effect of the finite relaxation rate of the free vortex density on the flow decay, but he finds that the data for thick films is not described at all by the theory. Recently Yu[14] has investigated the effects of vortex nucleation and annihilation at the film boundary on the decay rate, but he also finds that in the thicker films studied, the flow decays more quickly at long times than his calculation predicts.

In an earlier paper[15] we reported a calculation of the effect of vortex pinning, where we found a much improved fit, particularly at long times ($>10^4$ sec), could be obtained by including pinning of the vortices to the substrate. In addition, the value of λ was found to be in much better agreement with those found by third sound and by the HMD-AHNS calculations. This present paper is intended to amplify the brief remarks we made earlier and to show the fits to all of the data of Ref. 3.

This paper is divided into five sections. Section II reviews the mechanism of vortex dissipation and the contributions of both free and bound vortices to the dissipation. Section III describes the modifications of the theory that arise when the irregularities of the substrate are taken into account and how they give rise to the pinning of vortices. In Sec. IV we derive the equations for the flow decay and fit them to the data of Ekholm and Hallock.[3] Section V contains conclusions and indicates areas for future work. There is one Appendix containing the details of the calculation of the depinning and capture rates for the vortices.

II. DISSIPATION THROUGH VORTEX MOTION

In this section we will take up the question of how the vortices in a superfluid film move and how they dissipate the kinetic energy of the superfluid. We will generalize the result of HMD and AHNS to the case of a nonuniform film and also examine the result of giving a finite mass to a vortex. We will assume the usual model of a vortex: a core of radius $\xi\sim 2$ Å, which is filled with normal fluid, and a circulating superfluid outside the core with vorticity $\kappa=h/m$. We will ignore vortices of higher quanta of circulation, since the energy to create a vortex goes like κ^2, and thus vortices of higher circulation are unstable against decaying into vortices with smaller circulation.

In the absence of any interaction between the normal fluid and the superfluid, the vortices will obey the laws of motion for a perfect fluid and each vortex will move with the local superfluid velocity, which is the sum of the velocity fields due to all the other vortices present, plus any flow due to the presence of a boundary in the film (image vortices), and any externally imposed flow. In this case, the kinetic energy of the superfluid is, of course, conserved and there will be no dissipation of the flow.

However, as a real vortex moves, the excitations constituting the normal core can exchange energy and momentum with the normal fluid, with the substrate, and with excitations of the free surface. In addition, a normal fluid excitation moving in the presence of a superflow v_s will find its energy modified by an amount $-\vec{p}\cdot\vec{v}_s$, where $\vec{p}$ is the quasiparticle momentum. Since $\vec{v}_s$ varies with distance from the vortex core, this leads to a scattering of the excitation. This leads to a force on the vortex of the form

$$\vec{F}_D=\rho_s d\frac{h}{m}[\Gamma(\vec{v}_n-\vec{v}_l)+\Gamma' s\hat{z}\times(\vec{v}_n-\vec{v}_l)]\ , \tag{2.1}$$

where ρ_s is the superfluid density, d is the film thickness, $\vec{v}_l$ is the velocity of the vortex line, $\vec{v}_n$ is the velocity of the normal fluid, substrate, and surface excitations. Γ and Γ' are dimensionless coefficients, s is the sign of the vortex circulation, and we have put the vortices int the xy plane with the circulation vector in the z direction. The force is analogous to the mutual friction force derived by Hall and Vinen[16] for the case of rotating bulk helium. We have also assumed that the viscosity of the normal excitations is large enough and the film thin enough that all the excitations move with the same velocity $\vec{v}_n$ as the substrate. The second term in Eq. (2.1) is not a drag force, since it is time-reversal invariant, and therefore not dissipative. It is, however, a mechanism for transmitting momentum to the substrate and so we shall include it as a contribution to the decay of the supercurrent. We shall also take Γ and Γ' as independent of velocity and film thickness.

We can now derive an expression for the rate of decay of the superfluid momentum. The superfluid gives up momentum by the mutual friction on each vortex, so we have

$$\frac{d\vec{J}_s}{dt}=\sum_i \rho_s d(\vec{r}_i)\frac{h}{m}[\Gamma(\vec{v}_n-\vec{v}_{l,i})+\Gamma' s_i\hat{z}\times(\vec{v}_n-\vec{v}_{l,i})]\ , \tag{2.2}$$

where the sum is over all the vortices in the fluid and the subscript i refers to the ith vortex. In anticipation of Sec. III, we have let the film thickness vary; $d(\vec{r}_i)$ is the film thickness at $\vec{r}_i$.

We now need an expression for the velocity of a vortex. In the presence of the mutual friction force, a vortex will no longer move with the local superfluid velocity $\vec{v}_s(\vec{r})$. This results in a Magnus force exerted on the core by the superfluid of the form

$$\vec{F}_M=\rho_s d(\vec{r})\frac{h}{m}s\hat{z}\times[\vec{v}_l-\vec{v}_s(\vec{r})]\ . \tag{2.3}$$

This force is the analog of the Lorentz force on a superconducting vortex.

The equation of motion for a vortex is therefore

$$M_v\frac{d\vec{v}_l}{dt}=\vec{F}_D+\vec{F}_M\ , \tag{2.4}$$

where M_v is the effective mass of a vortex. We can estimate M_v by treating the vortex as a cylinder of radius ξ with circulation κ around it, which yields $M_v \sim \rho_s d\xi^2$. The vortex will reach its drift velocity in a time of order $M_v/\{\rho_s d(h/m)^2 \times[(\Gamma^2+\Gamma'^2)]^{1/2}\}\sim\xi^2 m/h$, which is very short compared to experimental times, and so we can set the right-hand side of Eq. (2.4) to zero. We get the same answer if we assume the vortex has no effective mass, it being merely a flow pattern. We have chosen this description because the idea of forces acting on a vortex assumes that you are treating it as an object with mass. The drift velocity is then

$$\vec{v}_l=\vec{v}_n+A_1[\vec{v}_s(r)-\vec{v}_n]+A_2 s\hat{z}\times[\vec{v}_s(r)-\vec{v}_n]+\vec{\eta}\ , \tag{2.5}$$

where

$$A_1=\frac{1-\Gamma'}{\Gamma^2+(1-\Gamma')^2}\ ,\quad A_2=\frac{\Gamma}{\Gamma^2+(1-\Gamma')^2}$$

and $\vec{\eta}$ is a Gaussian noise term which represents the fluctuations that gave rise to the irreversible part of the drag force. If we require that at long times the vortices be in equilibrium with a distribution $\exp(-U/k_BT)$ with U the kinetic energy of the fluid, we find that $\vec{\eta}$ satisfies[8]

$$\langle \eta_i(t)\eta_j(t')\rangle=(A_2/k_BT)\delta_{ij}\delta(t-t')\ . \tag{2.6}$$

If we substitute back into Eq. (2.2), we find

$$\frac{d\vec{J}_s}{dt}=\sum_i \rho_s d(\vec{r}_i)\frac{h}{m}\{(1-A_1)s_i\hat{z}\times[\vec{v}_s(\vec{r}_i)-\vec{v}_n]-A_2[\vec{v}_s(\vec{r}_i)-\vec{v}_n]\}\ . \tag{2.7}$$

Note that as $\Gamma,\Gamma'\rightarrow 0$, the dissipation vanishes. If we use Eq. (2.5) to solve for $\vec{v}_s(\vec{r})$ in terms of $\vec{v}_n$ and $\vec{v}_l$ and substitute into Eq. (2.2), and if we define an average superfluid velocity $\vec{u}_s=\vec{J}_s/A\bar{d}$, where A is the film area and $\bar{d}$ is the average film thickness, we find

$$\frac{d\vec{u}_s}{dt}=\frac{h}{mA\bar{d}}\sum d(\vec{r}_i)s_i\hat{z}\times[\vec{v}_l-\vec{v}_s(\vec{r}_i)]\ . \tag{2.8}$$

For a neutral system (no net vorticity) of uniform thickness with periodic boundary conditions, the second term can be shown to vanish. This gives the result of AHNS [see their Eq. (2.10)], which was derived from a phase-slip argument. So we see that the decay of the flow can also be derived from purely kinetic arguments, without recourse to the phase-slip argument. In essence, the only quantum feature of the vortices is their quantized vorticity; the rest can be derived from the consideration of the motion of cylinders with circulation about them in an ideal fluid.

Equation (2.10) was studied by AHNS (Ref. 8) and by Ambegaokar and Teitel[10] for the case of a film of uniform thickness. They found that the response of the bound vortex pairs of separation r to a small oscillating $\vec{u}_s - \vec{v}_n$ went like $\omega^2/[(14D/r^2)+\omega^2]$ and therefore vanished as the frequency ω went to zero. So the principal contribution to the quasistatic decay of a persistent current comes from the presence of free vortices. Although the temperature is less than the depairing temperature $T_{\rm KT}$ in the experiments of Ref. 3, there are free vortices present because the flow can depair vortices.

III. VORTEX PINNING

If a vortex becomes pinned to the substrate for some reason, Eq. (2.2) shows that there is no dissipation from such a vortex with $\vec{v}_n = \vec{v}_l$. So the only contribution to the decay of a persistent current will come from free unpinned vortices. In this section we shall investigate how vortex pinning can arise from spatial variations in the film thickness.

The film thickness can vary for two reasons: First, the substrates used experimentally are not smooth on an atomic scale; second, the surface can be inhomogeneous, which would give rise to a spatially varying van der Waals force. It would be energetically favorable for the film to allow the film thickness to vary slowly, so that as a vortex moves over the substrate, it will see a varying fluid depth. Since both the Magnus force and the drag force depend on the film thickness, the forces on the vortex will vary. However, the average drift velocity will not change much, since the coefficients A_1 and A_2 of Eq. (2.6) depend only on Γ and Γ', which are not expected to vary significantly with thickness. We will find, however, that the kinetic energy of the fluid will vary strongly with the location of the vortices and this will provide a pinning potential. To see this effect, we will construct an analogy between the diffusive motion of charges in an inhomogeneous dielectric and the motion of charges in a nonuniform film, similar to that done by AHNS for the case of a uniform film.

Consider a collection of vortices of circulation $s_i h/m\hat{z}$, length d_i, at postions $\vec{r}_i = (x_i, y_i)$. The equations governing the superfluid are

$$\begin{aligned} &\vec{J}_s(\vec{r}) = \rho_s d(\vec{r})\vec{v}_s(\vec{r}) , \\ &\vec{\nabla}\cdot\vec{J}_s = 0 , \\ &\vec{\nabla}\times\vec{v}_s = \frac{h}{m}\hat{z}\sum_i s_i \delta(\vec{r}-\vec{r}_i) , \end{aligned} \tag{3.1}$$

where J_s and v_s are defined as averages over the thickness of the film:

$$\vec{J}_s(\vec{r}) = \int_0^{d(\vec{r})} \vec{J}_{3s}(\vec{r},z)dz ,$$

$$\vec{v}_s(\vec{r}) = \frac{1}{d(r)}\int_0^{d(\vec{r})} \vec{v}_{3s}(\vec{r},z)dz ,$$

where j_{3s} and v_{3s} are the true three-dimensional current and velocity. In doing the average over the second two equations in (3.1), which are exact for the unaveraged quantities j_{3s} and v_{3s}, we have dropped terms in both expressions that are proportional to $(\nabla d)/d \sim \xi/\bar{d}$, which is small for the films studied here. We have also ignored any z dependence, for example, due to bending of the vortices. We will ignore the compressibility of the liquid, allowing the areal superfluid density $\sigma_s(\vec{r})$ $[=\rho_s d(\vec{r})]$ to vary only because the thickness is varying. Since $d(r)$ is not constant, we do not have the usual relation $\nabla\cdot\vec{v}_s = 0$, so the solution is not the same as in AHNS. Define fields $\vec{D}(x,y), \vec{E}(x,y)$ with $D_z, E_z = 0$ by

$$\vec{v}_s(\vec{r}) = -\hat{z}\times\vec{D}, \quad \vec{J}_s(\vec{r}) = -\rho_s\bar{d}\,\hat{z}\times\vec{E} . \tag{3.2}$$

If we define a quantity $\epsilon(r) = \bar{d}/d(r)$, then Eq. (3.1) becomes

$$\begin{aligned} &\vec{\nabla}\times\vec{E} = 0 , \\ &\vec{\nabla}\cdot\vec{D} = \frac{h}{m}\sum_i s_i\delta(\vec{r}-\vec{r}_i) , \\ &\vec{D} = \epsilon\vec{E} . \end{aligned} \tag{3.3}$$

So we have rewritten the problem in terms of charges of magnitude $h/2\pi m$ in an inhomogeneous dielectric $\epsilon(\vec{r})$. A region of large dielectric constant corresponds to a region where the film is thin.

We can understand the effect of the inhomogeneity by a simple example. Suppose there is a single thin region of radius a surrounded by a uniform region with thickness $\bar{d}$ with a single vortex present. The equivalent Coulomb problem is shown in Fig. 1. The solution of Eq. (3.3) is immediate by the method of images and gives for the electrostatic potential Φ for a vortex at r_1

$$\Phi=\begin{cases} -s_i \dfrac{h}{2\pi m}\ln(\,|\vec{r}-\vec{r}_1|\,), & r<a \\[2ex] -s_i \dfrac{\epsilon-1}{\epsilon+1}\dfrac{h}{2\pi m}\ln\left[\left|\vec{r}-\dfrac{a^2}{r_1^2}\vec{r}_1\right|\right], & r>a \end{cases} \tag{3.4a}$$

for a vortex outside the thin region, while for a vortex inside the thin region

$$\Phi=\begin{cases} -\dfrac{h}{2\pi m}\left[\ln(\,|\vec{r}-\vec{r}_1|\,)-\dfrac{\epsilon-1}{\epsilon+1}[\ln(\,|\vec{r}-\vec{r}_1|\,)-\ln(r)]\right], & r>a \\[2ex] -\dfrac{h}{2\pi m\epsilon}\left[\ln(\,|\vec{r}-\vec{r}_1|\,)+\dfrac{\epsilon-1}{\epsilon+1}\ln\left[\left|\dfrac{r_1^2}{a^2}\vec{r}_1-\vec{r}\right|\right]\right], & r<a\ . \end{cases} \tag{3.4b}$$

The velocity field this induces at the vortex is therefore

$$\vec{v}_s(\vec{r}_1)=\begin{cases} -s\dfrac{h}{2\pi m}\dfrac{\epsilon-1}{\epsilon+1}\left[\dfrac{a^2}{r_1^2}-1\right]\hat{z}\times\hat{r}_1, & r_1<a \\[2ex] -s\dfrac{h}{2\pi m}\dfrac{\epsilon+1}{\epsilon-1}\left[\dfrac{a^2}{r_1^2-a^2}\right]\hat{z}\times\hat{r}_1, & r_1>a\ . \end{cases} \tag{3.5}$$

We see that in the drag-free case the vortex will circle the pinning center. With an applied supercurrent $\vec{u}_s$, the vortex will be deflected by the pinning center. If we include drag forces, the vortex can be captured if it passes close enough to the pinning site.

Once on the pinning site, the vortex can still move if the site is of finite extent, but its dissipation is much less since it is now much shorter and the drag forces, which produce the dissipation and which are proportional to the film thickness, are much smaller. Furthermore, the vortex will stay on the pinning site until a random fluctuation can free it.

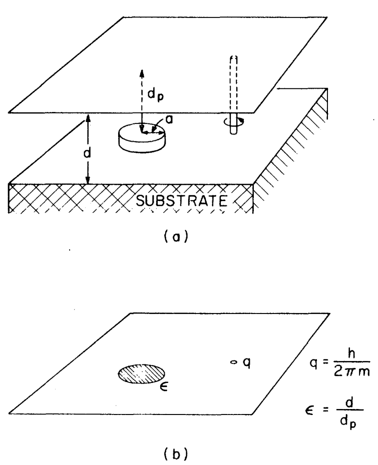

FIG. 1. (a) A single pinning site in an otherwise uniform film. (b) Equivalent Coulomb problem.

This can be seen by considering the energy of a vortex. The kinetic energy of the fluid is, using Eq. (3.2)

$$U=\tfrac{1}{2}\int \vec{v}_s\cdot\vec{J}_s\,d^2r=\tfrac{1}{2}\rho_s\bar{d}\int \vec{E}\cdot\vec{D}\,d^2r$$
$$=\tfrac{1}{2}\rho_s\bar{d}\frac{h}{m}\sum_i s_i\Phi(\vec{r}_i)+\mathscr{S}\ , \tag{3.6}$$

where $\mathscr{S}$ represents a surface term. If we now break up the various contributions to U from vortices, pinning sites, and an applied flow u_s, we get

$$U=U_{\text{self}}+U_{v\text{-}v}+U_{v\text{-pc}}+U_{\text{pc-pc}}+U_{\text{ext}}\ , \tag{3.7}$$

where the subscript v signifies vortex and the subscript pc pinning center. The core energy U_{self} of the vortex, or its self-interaction with its own velocity field, is given by

$$U_{\text{self}}=\frac{1}{2\pi}\rho_s\left[\frac{h}{m}\right]^2 d(\vec{r}_i)\ln\left[\frac{R}{\xi}\right]\ , \tag{3.8}$$

where R is the linear size of the system and where we have smeared out the vorticity uniformly over a region of radius ξ. We see from this that the regions where d is small, that is, on a pinning site, are local minima of the core energy, or chemical potential, of a vortex.

The vortex-pinning center interaction can be derived from (3.4) and we obtain

$$U_{v\text{-pc}}=\begin{cases}\dfrac{1}{2\pi}\rho_s\bar{d}\left[\dfrac{h}{m}\right]^2\dfrac{\epsilon-1}{\epsilon+1}\ln\left[1-\dfrac{a^2}{r^2+\xi^2}\right], & r>a\\[2ex] \dfrac{1}{2\pi}\rho_s\bar{d}\left[\dfrac{h}{m}\right]^2\dfrac{1}{\epsilon}\dfrac{\epsilon-1}{\epsilon+1}\left[\ln\left[\dfrac{a^2}{a^2+\xi^2-r^2}\right]+\epsilon\ln\left[\dfrac{\xi^2}{a^2+\xi^2}\right]\right], & r<a\ ,\end{cases} \tag{3.9}$$

where we have again smeared the vortex over a size ξ. A picture of the potential $U_{v\text{-pc}}$ is shown in Fig. 2. The depth of the well is approximately

$$U_{v\text{-pc}}(0)-U_{v\text{-pc}}(\infty)=\frac{1}{2\pi}\rho_s\bar{d}\left[\frac{h}{m}\right]^2\times\left[1-\frac{1}{\epsilon}\right]\ln\left[\frac{\xi^2}{a^2+\xi^2}\right]. \tag{3.10}$$

Since this interaction falls off rapidly with distance ($\sim r^{-2}$), we will ignore the interaction of a vortex with more than one pinning site. For the same reason, we shall ignore the pinning-site—pinning-site interaction which arises from the induced field from one site polarizing another site. It is easy to see that the interaction energy between sites would fall off as least as fast as the fourth power of their separation.

The interaction between two vortices will also be modified by the presence of a pinning site. From Eqs. (3.4) and (3.6) we obtain, for two vortices outside a pinning site

$$U_{v\text{-}v}(\vec{r}_1,\vec{r}_2)=\frac{1}{2\pi}\rho_s d\frac{h}{m}s_1s_2\left\{\ln(|\vec{r}_1-\vec{r}_2|)+\frac{\epsilon-1}{\epsilon+1}\left\{\ln\left[\left|\vec{r}_2-\left[\frac{a}{r_1}\right]^2\vec{r}_1\right|\right]-\ln(r_2)\right\}+\frac{\epsilon-1}{\epsilon+1}\left\{\ln\left[\left|\vec{r}_1-\left[\frac{a}{r_2}\right]^2\vec{r}_2\right|\right]-\ln(r_1)\right\}\right\}. \tag{3.11}$$

The first term is the usual vortex-vortex interaction, the second and third terms represent the change due to the presence of the pinning site. The correction decreases rapidly with distance and can be ignored because the depairing of bound vortices occurs on large length scales where the correction is unimportant.

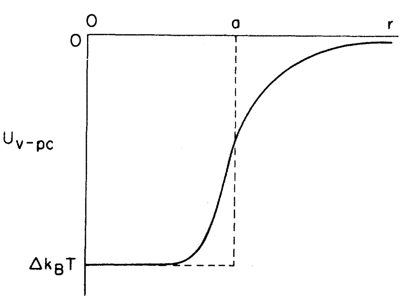

FIG. 2. Approximate form of the pinning potential versus distance for the problem of Fig. 1. The dashed line shows the sqaure well with which we model the pinning potential to calculate the pinning and depinning rates.

The external superflow yields an interaction of the form

$$U_{\text{ext}}=\rho_s\bar{d}\sum_i s_i\hat{z}\cdot(\vec{r}_i\times\vec{u}_s)\ . \tag{3.12}$$

So our expression (3.7) looks like the energy of two-dimensional Coulomb particles interacting with an approximate $\ln(r)$ potential with an external field and a random polarizable medium provided by the thickness variations.

Although we have shown that the energetics are identical to charges moving in an inhomogeneous media, the analogy is not complete unless the dynamics are the same. The equations of diffusive motion for charges in an inhomogeneous dielectric are

$$\frac{d\vec{r}_i}{dt}=q_i\mu\vec{E}(\vec{r}_i)+\vec{\eta}(t), \quad \langle\eta_i\eta_j\rangle=2\mu k_B T\delta_{ij} ,$$

$$\vec{\nabla}\times\vec{E}=0, \quad \vec{D}=\epsilon(\vec{r})\vec{E} , \tag{3.13}$$

$$\vec{\nabla}\cdot\vec{D}=2\pi\sum_i q_i\delta(\vec{r}-\vec{r}_i) .$$

There are two difficulties with establishing contact for the dynamics. The first was already noted in AHNS, and is the fact that the drift of the vortices along with the flow, represented by the second term in Eq. (2.6), has no analog in the Coulomb problem. For bound pairs, McCauley[17] has shown that this term drops out of the dissipation equation entirely. For free vortices, this is not possible. If we use the bulk values for A_1 and A_2, the experimental situation[18] is such that $A_2<1$ and $A_1\approx 1$ (drag is small), so we cannot ignore A_1 compared to A_2 in the vortex equation of motion (2.6). The possibility does exist that the vortex interactions with the substrate and the free surface in a thin film are sufficient to increase A_2 and decrease A_1 so as to reach the large-drag regime. Since we have no microscopic theory of these interactions we will simply assume that we are in the large-drag limit. The second difficulty is that the quantity that the vortex feels as a force is the local value of $\vec{D}$, not the local electric field as in Eq. (3.13). This is the same as saying that the coefficients A_1 and A_2 in Eq. (2.6) are independent of d. So we are forced to ignore the screening of a vortex on a pinning site by the dielectric representing that pinning site when examining the dynamics. In fact, we shall replace the entire inhomogeneous dielectric by a set of square wells distributed at random, all of the same depth $\Delta k_B T$ and radius a for simplicity. This is our major simplifying assumption. A square well was chosen to represent the sum of the core energy and the induced potential since those two energies vary rapidly only near the regions where the dielectric constant, that is, the film thickness, varies significantly.

So we have shown that the thickness variations give rise to a random potential. By comparing the equations of motion for the vortices [Eq. (2.6)] and those for the Coulomb problem [Eq. (3.13)], we can identify the analog of the diffusion constant and mobility. By comparing the interaction energy of the vortices, we find corresponding quantities for the electric charge and the external electric field E_{ext} [see Eq. (3.12)]. The results of this comparison yield

$$D\leftrightarrow\frac{A_2}{k_B T}, \quad q\leftrightarrow\frac{h}{2\pi m} ,$$
$$\vec{E}_{\text{ext}}\leftrightarrow\rho_s\bar{d}\frac{h}{m}\hat{z}\times\vec{u}_s . \tag{3.14}$$

IV. KINETIC EQUATIONS

In this section we will bring together the ideas of the preceding sections to construct a set of equation to describe the dissipation of a persistent current. We already have an equation for the flow dissipation, Eq. (2.9). We have noted in Sec. III that the pinned vortices cannot drift indefinitely and are thus removed as a source of dissipation, or sink of momentum. Like the bound vortices, the collection of pinned vortices can only act as a thermally activated source of free vortices.

From (2.9) we have the following equation for the decay of the superfluid velocity u_s:

$$\frac{du_s}{dt}=-\frac{h}{m}n_f v_{d\perp} , \tag{4.1}$$

where $v_{d\perp}$ is the mean-drift velocity of the vortices in a direction perpendicular to the applied flow and n_f is the density of free vortices of both signs. We can estimate $v_{d\perp}$ from (2.6) as roughly $A_2 u_s$.

We now write down a simple kinetic equation describing the time dependence of the free vortex density n_f.

$$\frac{dn_f}{dt}=R_b+R_p n_p-\Gamma_b n_f^2-\Gamma_p n_f , \tag{4.2}$$

where R_b is the generation rate of free vortices from bound pairs, R_p is the rate of escape of vortices from pinning sites, n_p is the number of pinned vortices, Γ_b is the rate of recombination of free vortices into bound pairs, and Γ_p is the capture rate of free vortices by pinning site. We ignore here diffusion of vortices due to a nonuniform vortex distribution, which restricts the validity of Eq. (4.2) to slowly varying processes.

Expressions for R_b and Γ_b have been calculated by several workers[8,9,19] by looking at the Brownian motion of a pair of vortices in their mutual $\ln(r)$ potential with an external potential $\sim u_s x$ due to the applied flow u_s. The results are

$$\Gamma_b = 2\pi D\lambda,\ R_b = \nu D \left[\frac{\lambda}{\xi}\right]^2 \left[\frac{2\pi m u_s \xi}{h}\right]^\lambda ,$$
$$\lambda = \frac{h^2 d\rho_s}{2\pi k T} m^2 , \tag{4.3}$$

where ν is the density of bound vortex pairs, D is the vortex diffusion constant, and u_s is the applied flow velocity. We shall also calculate the rates of pinning and depinning of vortices by a Brownian motion method, the details of which are relegated to the Appendix. We get a constant pinning rate and a depinning rate that grows exponentially with u_s.

We must include a kinetic equation for the time dependence of the number of pinned vortices n_p,

$$\frac{dn_p}{dt} = -R_p n_p + \Gamma_p n_f . \tag{4.4}$$

We now scale Eqs. (4.1), (4.2), and (4.4) by

$$V = \frac{u_s(t)}{u_s(0)},\ N = \frac{n_f}{N_0},\ M = \frac{n_p}{M_0},\ T = \frac{t}{T_0} ,$$

where $N_0/T_0 = \nu D(\lambda/\xi)^2[m\xi u_s(0)/h]^\lambda$, $N_0 T_0 = 1/(2\pi D\lambda)$, and $M_0/N_0 = e^\Delta$. The kinetic equations become

$$\frac{dN}{dT} = V^\lambda - N^2 + G(e^{\beta V}M - N) ,$$
$$\frac{dM}{dT} = G'(N - e^{\beta V}M) , \tag{4.5}$$
$$\frac{dV}{dT} = -NV ,$$

where $G = \Gamma_p N_0/T_0$, $G' = \exp(-\Delta)G$, and $\beta = 2\pi\lambda m a u_s(0)/h$. We now ignore the equation for M, since $G' << G$, and take M to be constant in the first equation. This approximation assumes that n_p is large enough (since Δ is large) that it will not change significantly during the decay.

If $G = 0$ we have the equations originally discussed by AHNS and HMD and that were examined in close detail by McCauley,[13] where he finds that the thinnest films are reasonably fit by the old expressions, but the thicker films ($d > 7.2$ atomic layers) do not follow the predicted curves. If $G >> 1$, the pinning-depinning process is much faster than the pairing-depairing process and is also much faster than the decay of the velocity. So we can set dN/dT to zero and solve for N, which results in the expression for dv/dt given in our previous paper.[15] For large values of G we find $N \sim Me^{\beta V}$, which implies that the decay is dominated by pinning rather than vortex depairing processes.

In the fits to the data, there were five parameters: the time scale T_0, $N(T=0)$, G, M, and β. The data in Ref. 3 is normalized so that $V = 1$ at the first piece of data, which is taken 15 sec after the removal of the driving current. We will use this as our origin of time. λ was taken from the theoretical expression[20] [see Eq. (4.3)]. As was discussed in the Introduction, the vortex-vortex coupling constant λ is renormalized by the presence of the other vortices. The transition occurs at a value for the renormalized λ of 4. In the experiments of Ref. 3, λ is between 14 and 26; in the language of Kosterlitz and Thouless, this means that the experiments are performed at a temperature much less than the transition temperature T_{KT}. For very low temperature, their analysis[4] shows that there are very few vortex pairs and λ is not renormalized significantly. So we can use the bare value and not be troubled by trying to assign a properly renormalized value for a free vortex.

The results of the nonlinear least-squares fits are shown in Figs. 3 and 4. In doing the fits, it was found that including the equation for M did not improve the fits, nor did they improve by adjusting λ. The values of the parameters found in the fits are tabulated in Table I. As can be seen, the fits are quite close. In fact, the error in the fits is of the same order of magnitude as the scatter in the velocity. The most important result is that G is large for all the decays, so that we are seeing a process dominated by pinning. We also see that

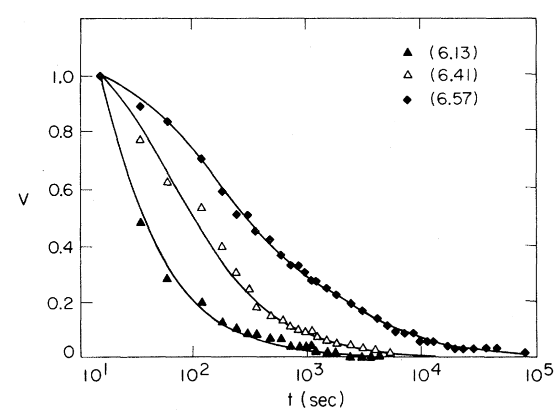

FIG. 3. Data of Ref. 3 for the thinnest films together with our fits. The numbers in parentheses denote film thicknesses in atomic layers.

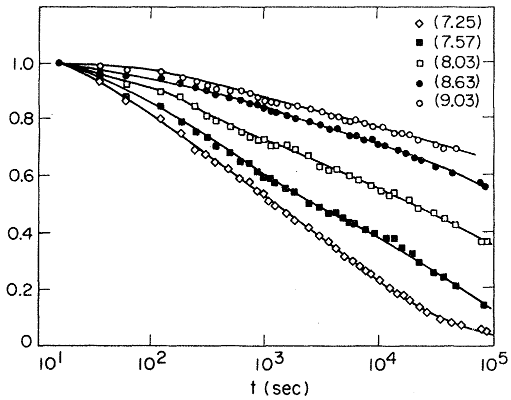

FIG. 4. Data of Ref. 3 for the thicker films together with our fits.

the density of pinned vortices, M is much smaller than the number of free vortices. If we assume that n_f and n_p are given by their steady-state values at $t=0$ [$n_f \approx \exp(\beta v - \Delta) n_p$], we find that Δ is about 10 to 12, so the wells are quite deep. In Fig. 5 we have plotted the velocity and the free vortex density for the parameters found for the data of 7.25 atomic layers. The dashed line is the expression $N = M\exp(\beta V)$, so we see that this relation holds well except at short times. The additional free vortex density at short times comes from the depairing of vortices, since that term only contributes near $V=1$.

Since $G >> 1$ we can also understand the initial logarithmic decay seen by Ekholm and Hallock[3]. As G becomes large, we have

$$N = Me^{\beta v}, \quad \frac{dV}{dT} = -Me^{\beta v} v \;, \tag{4.6}$$

which, if $\beta >> 1$ and $v - 1 << 1$, gives a decay law

$$v = -\beta^{-1}\ln(t) + \text{const} \;. \tag{4.7}$$

If we use this expression as an interpolation formula for the initial decays, starting at $t = t_0$, we have

$$v(t) = 1 - \beta^{-1}\ln(t/t_0) \;, \tag{4.8}$$

and we can identify the Ekholm and Hallock ξ as $\beta^{-1} = h/[2\pi m a \lambda u_s(0)]$, which explains qualitatively the dependence of ξ on film thickness and temperature. The difficulty is that our expression predicts that ξ should vary inversely with the initial velocity, which is not seen.[3] This is probably an artifact of our simplified picture of identical pinning sites distributed at random. If we included a distribution of sizes of pinning sites, we would find a different expression for the escape and capture rates, which could give rise to a velocity-dependent initial decay because the number of sites that could effectively participate in the depinning at short times would depend on the velocity.

We can also see that, if the temperature is lowered during the decay, the decay rate will be lowered dramatically. This is because the escape rate, which is proportional to $\exp(-\Delta)$ [cf. Eq. (A21)], will become quite small, while the capture rate, which is relatively independent of the well depth [cf. Eq. (A22)], will not change much. So the density of unpinned vortices will become quite small, which results in a small dissipation. Raising the temperature again will inject a large number of free vortices as the vortices depin and will result in a rapid initial decay until the steady-state value $N \sim M\exp(\beta v)$ is re-established. This rapid decay will last a relatively short time because the pinning processes occur at a much faster rate than the velocity decay. This provides an explanation for the "memory effect" seen by Ekholm and Hallock[3] (see especially Fig. 20 of Ref. 3).

In Figs. 6 and 7 we show the results of a computer simulation of this memory effect, using the

TABLE I. Table of parameters from the least-squares fits shown in Figs. 3 and 4. χ^2 is the sum of the squared error between the data and the fitted curve.

Thickness	6.13	6.41	6.57	7.25	7.57	8.03	8.63	9.03
λ	14.3	15.3	15.9	18.5	19.7	21.4	23.7	25.2
$u_s(0)$ (cm/sec)	15.2	23.0	26.6	36.7	29.0	25.9	29.3	25.4
T_0 (sec)	2×10^5	10^4-10^6	10^5-10^6	3.5×10^5	1.5×10^5	10^6	7×10^6	4×10^6
G	1.2×10^3	17–20	2×10^3	1.6×10^4	2.3×10^4	3×10^4	10^5	2×10^4
β	4.8	6–13	3.90	6.19	6.3	10.7	13.8	12.9
$n(0)$	0.98	0.2–0.7	7×10^{-2}	7×10^{-2}	3×10^{-2}	3.4×10^{-5}	1.2×10^{-2}	1.0×10^{-2}
M	9×10^{-2}	3.4×10^{-2}	2×10^{-2}	1.5×10^{-4}	4.7×10^{-5}	5.1×10^{-7}	6.2×10^{-9}	7.0×10^{-9}
Γ_p (sec^{-1})	4.7×10^{-3}	$10^{-2}-10^{-1}$	4.6×10^{-3}	4.6×10^{-2}	4.9×10^{-3}	2.4×10^{-2}	4×10^{-2}	5×10^{-2}
χ^2 ($\times10^{-3}$)	2.8	10.8	5.5	7.0	9.5	7.2	1.7	2.6

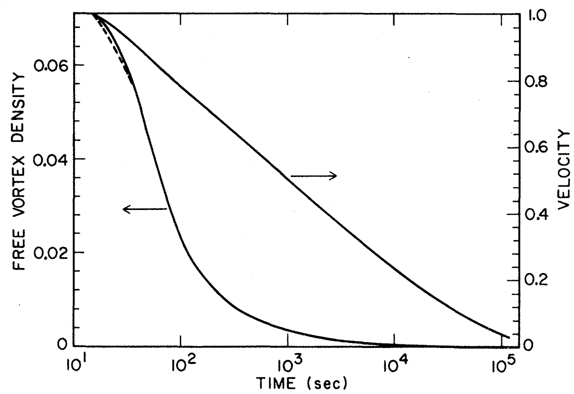

FIG. 5. Variation of the free vortex density and velocity as a function of time for the film of 7.25 layers. The dashed curve is the relation $N = M\exp(\beta V)$.

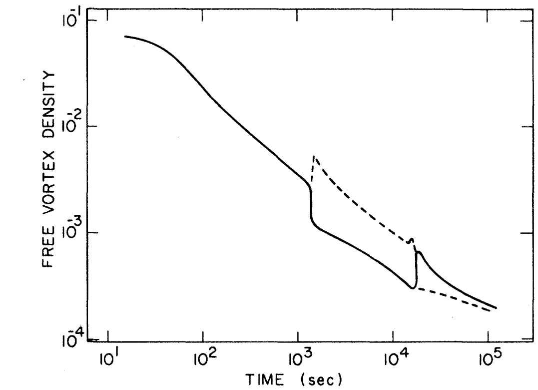

FIG. 7. Free vortex density versus time for the decays shown in Fig. 6.

parameters found for 7.25 atomic layers. The temperature was changed by 10% from the initial temperature at 6000 sec, and the original temperature was re-established at 1.65×10^4 sec. The solid curves are for a 10% reduction of the initial temperature and the dashed lines are for the case of a 10% increase.

We can use this memory effect to directly test for pinning processes in the film. If there were no depairing, then the total number of vortices would be conserved, and all we are doing by varying the temperature is modifying for a limited time the fraction of vortices that are free. Since n_p and n_f are related by $n_f=(R_p/\Gamma_p)n_p$ in steady state, and their sum is conserved, this implies that

$$n_f = n_{\text{tot}}/(1+e^{(\Delta-\beta v)}) \ . \tag{4.9}$$

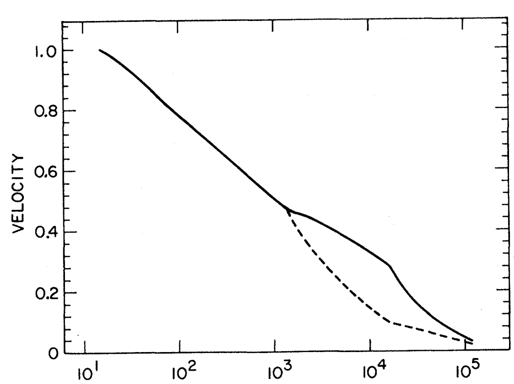

FIG. 6. Superflow velocity versus time for the using the parameters for the film of 7.25 layers. The solid curve is the decay if the temperature is lowered 10% for $6\times10^3 < t < 1.65\times10^4$ sec. The dashed curve is the case of the temperature being raised 10% over the same time interval.

Since $G >> 1$, this implies that long after the temperature has been restored, the free vortex density is given by Eq. (4.9), and so $(1/v)dv/dt$ is the same regardless of the past history. If, however, we include the depairing and repairing processes, the total number of vortices is no longer constant. If the temperature were raised for a while, the rate of production of free vortices from bound pairs would increase, so the decay would be faster than the decay without vortex depairing. So after the temperature was restored, the velocity would be lower than without the depairing taken into account. In addition, the number of free vortices would no longer be given by Eq. (4.9), because the past history (the heating) will have changed the total number of vortices. Similar conclusions can be found for the case of cooling the film for a while, then reheating it. In each case, $(1/v)dv/dt$ is independent of past history for the pinning and depinning processes, but the addition of depairing and repairing will make the heating-cooling curves show "hysteresis."

In light of the fact that we have five parameters available in the fits, it would increase our confidence in the theory presented here if the fitted values agreed with our theoretical expressions. This at least would reduce the possibility that we have included too many parameters. To partially check this, we have plotted in Fig. 8 β vs $\lambda u_s(0)$, which should be a straight-line of slope $2\pi ma/h$. We derive from this a value for a, the pinning-center size, of 300 ± 50 Å. This seems quite large, although the analysis of Sec. III would indicate that a fairly large site would have a considerable energy barrier, although it also allows some motion of the vortex. It might be possible that the pinning sites were clustered by having the roughness

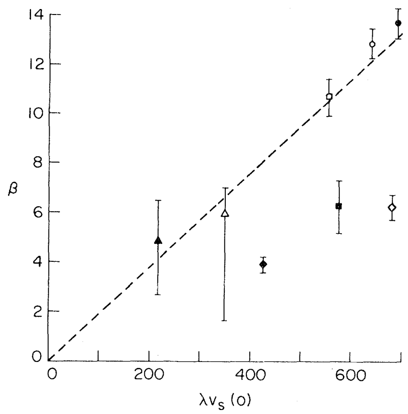

FIG. 8. Graph of β vs $\lambda u_s(0)$ (cm/sec) for the fits done. The slope yields a value of the pinning site size of 300±50 Å. The points plotted use their corresponding symbol from Figs. 3 and 4.

produce a number of sites near each other and that this figure represents an average cluster size. A vortex may be able to hop from one site to the next, but it is confined by the cluster of pinning sites. It is also likely that this is an artifact of our simplified model for the pinning-site distribution.

We can also calculate Γ_p from G and T_0. This quantity should not show any significant variation with thickness and we find that it is of order 10^{-2} sec for all the curves. Given that G is so large in these fits so that the dominant process is pinning and that depairing is a small correction, we do not find that the theoretical expressions for T_0 will be accurately followed by our parameters.

Donnelly *et al.*[12] considered a "permanent barrier" model which gave fair agreement with the data of Ref. 3. Their permanent barrier results from the image potential of a vortex near a wall, and the decay of the flow comes from vortices moving to the wall and annihilating. We have shown that their results are qualitatively correct, for we have just such a permanent barrier from the pinning sites. While their model is best applied to thin channels where the image forces on the vortex ring give rise to the barrier, the same idea of a permanent barrier is present here.

Yu[14] has investigated the results of adding the possibility of escape and annihilation of vortices at the film edges. His equations for N and V are similar to ours, except for the replacement of $Me^{\beta V}$ by $V^{\lambda/2}$ and G by GV. He finds much improved fits over those of McCauley[13] at the cost of introducing the additional parameter G, which he finds grows rapidly with thickness. Since the experiments of Ekholm and Hallock are done on the surface of a torus, it is difficult to see where these boundaries might arise. It can be asserted that they may arise through puddling of the film, but there is no strong evidence for this happening in thin helium films. The principle difference between his results and ours is that as V goes to zero, the rates for free vortex production in his equations vanish; this results in a slow (power-law) decay of the velocity with time. In our model, the number of free vortices becomes small but never vanishes because it is possible for a pinned vortex to be freed even at zero velocity. So we find that N goes to a very small but finite value as V vanishes, which results in an exponential decay of the flow with a very long time constant. The time constant can be found from Eq. (4.5) where we find $N \approx M$, so the time constant is T_0/M and is the order of 10^7 sec. We find that our theory gives a better fit at long times ($> 10^4$ sec) than the fits of Yu, where his predicted decay is too slow due to the reason discussed above.

It has been previously assumed, and demonstrated by Kosterlitz and Thouless,[4] that the vortices are all paired below T_{KT}. We are claiming that a small fraction of them may be pinned and not paired, an effect which arises solely from the effect of the nonuniform substrate. The number of pinned vortices is always too small to affect the thermodynamics, so that the predictions of Kosterlitz and Thouless, which treat the unbinding of the pairs on large length scales, are essentially unchanged. The dynamics can be affected, however, because the pinned vortices need not respond like the bound pairs. The presence of these pinned vortices, which could respond as free at finite frequency, may be the source of the additional dissipation below the transition seen by Bishop and Reppy,[11] even at low driving amplitude.

V. CONCLUSIONS

We have seen how the inclusion of substrate irregularity can lead to the pinning of vortices in superfluid helium films of a few atomic layers thickness. We have also found that these pinning effects can quite drastically change the dynamic response of a thin superfluid film. This is reminiscent of the situation in type-II superconductors, where the pinning of vortices leads to large critical currents.

Due to the predominance of pinning effects over pairing-depairing effects, the experiments of Ekholm and Hallock are not a strict test of the predictions of the Kosterlitz-Thouless theory. In order to render persistent current experiments more sensitive to the vortex pairing-depairing process, the pinning effect must be reduced. This could be done by overplating the substrate, which would undoubtably reduce the inhomogeneities, or by performing experiments not sensitive to the presence of pinned vortices.

One possibility for such an experiment is to perform the persistent-current experiment close to T_{KT}, where any change in the dynamic response would come primarily from depairing, since the pinning process has no unusual behavior near T_{KT}. This would cause experimental difficulties with the present experimental setup for measuring the decay of persistent currents because the decay becomes quite rapid as T approaches T_{KT}, and so would require a faster technique to measure the flow velocity than the one currently employed. An experiment done at finite frequency, such as that of Bishop and Reppy,[11] is a more sensitive test of the depairing transition, since at frequencies above the typical rates for pinning and depinning, the pinned vortices would respond as free at all temperatures and could be subtracted out as a background term below T_{KT}. A good extension of the Bishop and Reppy experiment is to investigate the dissipation of third sound as the amplitude is increased. The calculation of the nonlinear third-sound response will be the subject of a subsequent paper.

It may also be possible to use the memory effect described in Sec. IV to look at the depairing effects since $d(\ln v)/dt$ is a known function of v [Eq. (4.9)] if there are only pinning processes present. The function will only change if the total number of vortices changes. Since this change of the total number is due only to depairing, by varying the temperature we vary the rate of generation of additional free vortices from bound pairs, and thus comparing results with and without temporary heating and cooling of the film can lead to the detection of pinning effects.

ACKNOWLEDGMENTS

We would like to thank R. Hallock for stimulating our interest in this problem, for supplying us with copies of his data, and for several useful conversations. We would also like to thank S. Trugman for many useful insights into the dynamics of vortices and for suggesting the inhomogeneous dielectric model discussed in Sec. III, R. Donnelly for several stimulating and useful discussions, C. Caroli for her hospitality at the Universite de Paris VII, where some of this work was completed, and Lu Yu for his discussions and for providing his results on the boundary effects. This work was partially supported by the National Science Foundation under Grant No. DMR-80 07934.

APPENDIX

In this appendix we shall give a detailed calculation of the depinning rate and the capture for pinned vortices. The usual saddle-point approximation is not vaid here, since we shall need expressions for the rates in both strong and weak external fields. Also, since the potential is discontinuous, it is difficult to decide how to describe the particle density near the well lip since it is sensitive to the density inside, which is not the usual case when making a Kramers-style approximation. For this calculation, we shall follow the method of McCauley,[19] who used it to calculate the depairing and repairing rates.

We write down a kinetic equation for the number of pinned vortices N_p and density of free, unpinned vortices n_f,

$$\frac{dN_p}{dt} = -R_p N_p + \Gamma_p n_f \; . \tag{A1}$$

The coefficients R_p and Γ_p can be calculated through the following boundary value problem: we solve the Fokker-Planck equation for the density of vortices $f(r)$ in the (x,y) plane

$$\frac{\partial f}{\partial t} = D\vec{\nabla}\cdot(\vec{\nabla} f + f\vec{\nabla} U) \; , \tag{A2}$$

where U is the potential energy divided by $k_B T$, and is composed of a square well of depth Δ and radius a and a constant external force of magnitude $2F$ (the factor of 2 is for later convenience)

$$U = -\Delta\Theta(a-r) - 2Fx \; . \tag{A3}$$

For the helium situation, we have

$$2F = \rho_s d \frac{h}{mk_B T} u_s = 2\pi\lambda \frac{m}{h} u_s \; .$$

We will solve (A2) in the steady-state limit $(\partial f/\partial t = 0)$ with a current source $J_0 \delta^2(\vec{r})$ at the origin. The number of vortices trapped in the well is given by

$$N_p = \int_{r<a} f(\vec{r}) d^2 r \tag{A4}$$

and

$$\frac{dN_p}{dt}=\int_{r<a}\frac{\partial f}{\partial t}d^2r=-a\int_0^{2\pi}\vec{J}\cdot\hat{r}\,d\theta=J_0\ , \tag{A5}$$

where

$$\vec{J}(\vec{r})=-D(\vec{\nabla}f+f\vec{\nabla}U)\ . \tag{A6}$$

If we solve (A2) in the form

$$\vec{\nabla}\cdot(\vec{\nabla}f+f\vec{\nabla}U)=-\frac{J_0}{D}\delta^2(\vec{r})\ , \tag{A7}$$

with f vanishing far from the origin so that f represents a density of trapped vortices, then

$$J_0=\frac{dN_p}{dt}=-R_pN_p\ , \tag{A8}$$

and by finding f to calculate N_p, we can calculate R_p. Similarly, if we let f tend to n_f far from the well and put $J_0=0$ (steady state, no sources), then

$$\frac{dN_p}{dt}=0=-R_pN_p+\Gamma_p n_f\ , \tag{A9}$$

so that we have

$$\Gamma_p=\frac{N_p}{n_f}R_p\ , \tag{A10}$$

and so by calculating N_p in this case we can use our previous result for R_p to calculate Γ_p.

For our problem, there is an important point to be made. The equations we have described here are for a single pinning site. Let R denote the mean distance between pinning centers. We really want to describe a vortex as belonging to a particular pinning site only if it is much closer than R to the pinning site, since vortices farther away are dominated by the presence of a different pinning site. So our boundary conditions on f should not be applied at $r=\infty$, but at $r=R$. The reason we need to make this point is that as the external field vanishes with only a single pinning site in an infinite region, both Γ_p and R_p vanish. This is because a particle can never diffuse far enough away without the aid of the external field so as to not return to the pinning site. Physically, a particle need only diffuse beyond R, for then it will be dominated by the presence of a different pinning site and not return to the original one. This is the reason for introducing R. If the external field is strong enough, R is unnecessary since the applied current will drag the particle away from the site.

If we now write

$$f=e^{Fr\cos\theta}g(r,\theta)\ , \tag{A11}$$

then (A7) becomes

$$\nabla^2 g-F^2 g=-\frac{J_0}{D}\delta^2(\vec{r})\ . \tag{A12}$$

The solution to this equation is a sum of modified Bessel functions. For $r<a$ we find

$$g=\frac{J_0}{2\pi D}K_0(Fr)+\sum_{n=-\infty}^{+\infty}A_nI_n(Fr)e^{in\theta} \tag{A13a}$$

and for $r>a$ we find

$$g=\sum_{n=-\infty}^{+\infty}[C_nI_n(Fr)+B_nK_n(Fr)]e^{in\theta}\ . \tag{A13b}$$

At $r=a$ the matching conditions are

$$\begin{aligned}&f(r\to a-)=e^{\Delta}f(r\to a+)\ ,\\ &\vec{J}(r\to a-)\cdot\hat{r}=\vec{J}(r\to a+)\cdot\hat{r}\ ,\end{aligned} \tag{A14}$$

while at $r=R$ we have

$$g(R,\theta)=N_fe^{-FR\cos\theta}=N_f\sum_{n=-\infty}^{+\infty}(-)^nI_n(FR)e^{in\theta}\ . \tag{A15}$$

Using these three conditions [(A14) and (A15)] we can eliminate B_n and C_n to produce the following set of linear equations for the quantities A_n:

$$\begin{aligned}&A_n[(1-e^{-\Delta})I_n'+\alpha_ne^{-\Delta}]-\tfrac{1}{2}(1-e^{-\Delta})(A_{n+1}I_{n+1}+A_{n-1}I_{n-1})\\ &\quad=\frac{J_0}{2\pi D}\left[\left[(1-e^{-\Delta})K_1-\frac{K_0(FR)}{I_0(FR)}e^{-\Delta}\alpha_0\right]\delta_{n,0}+\tfrac{1}{2}\delta_{|n|,1}(1-e^{-\Delta})K_0\right]+N_f(-1)^n\alpha_n\ ,\quad |n|<\infty\ ,\end{aligned} \tag{A16}$$

where

$$\alpha_n^{-1}=Fa\left[K_n(Fa)-\frac{K_0(FR)}{I_0(FR)}I_n(Fa)\right], \tag{A17}$$

and where the Bessel functions have argument Fa unless explicitly indicated and a prime denotes differentiation with respect to the argument.

The number of bound particles can be found from (A4), (A11), and (A13) and is

$$N_p=J_0\frac{a^2}{D}[K_0(Fa)I_0(Fa)+K_1(Fa)I_1(Fa)]+A_0\,{}_1F_2(\tfrac{1}{2};1,2;F^2a^2)$$
$$+\sum_{n=1}^{\infty}(A_n+A_{-n})\frac{(Fa/2)^2}{n!(n+1)!}\,{}_1F_2(n+\tfrac{1}{2};2n+1,n+2;F^2a^2)\,, \tag{A18}$$

where ${}_1F_2$ is a generalized hypergeometric function.

The rates in zero applied field are found in a straightforward manner from Eq. (A7) with $F=0$. The solution of this that satisfies the conditions (A14) and (A15) is easily found to be

$$f=\begin{cases}\dfrac{1}{2\pi D}J_0[\ln(a/r)+e^{\Delta}\ln(R/a)]+n_f e^{\Delta}, & r<a\\[2mm] \dfrac{1}{2\pi D}J_0\ln(R/r)+n_f\,, & r>a\,,\end{cases} \tag{A19}$$

which yields for the number of bound particles

$$N_p(F=0)=\frac{a^2J_0}{2D}[\tfrac{1}{2}+e^{\Delta}\ln(r/a)]+n_f\pi a^2e^{\Delta}\,. \tag{A20}$$

For the escape problem ($n_f=0$) this yields

$$R_p^0=\frac{4D}{a^2}\frac{1}{1+2e^{\Delta}\ln(R/a)}\,, \tag{A21}$$

and for the steady-state problem ($J_0=0$) we find

$$\Gamma_p^0=4\pi D\frac{e^{\Delta}}{1+2e^{\Delta}\ln(R/a)}\,. \tag{A22}$$

While the above equations (A16) and (A18) are analytically quite complicated, they are easily solved by computer. We show the results of such a calculation in Fig. 9. As is evident from Fig. 9, Γ_p is roughly constant for $2Fa<\Delta$ and R_p grows exponentially with the applied field, for large field. The rise at small velocities is due to the increased importance of the external flow in sweeping the vortices along, as opposed to the presence of the boundary at $r=R$. As soon as $FR>1$, the effect of the boundary is negligible. The source of the factor of 8 rise can be traced to the $2\ln(R/a)$ term in Eqs. (A21) and (A22), since we chose R/a to be 20 in making the fits. So the magnitude of the rise is dependent on our model and will be ignored. If we use the large-field results for the weak-field regime, we overestimate both Γ_p and R_p but their ratio is still correct. We show this result in Fig. 10. So while we overestimate the rate of relaxation to

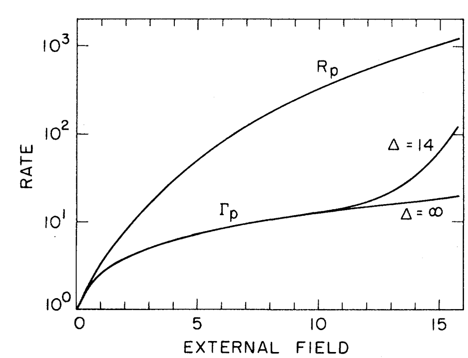

FIG. 9. Graph of escape rate (R_p) and capture rate (Γ_p) for a square well of depth Δk_BT. The rates are normalized to their values in zero external field. The applied flow is measured in units of $2Fa/k_BT$.

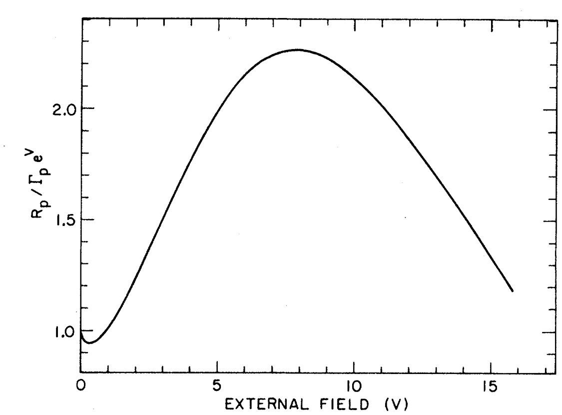

FIG. 10. Plot showing the relative accuracy of the relation $R_p/\Gamma_p=\mathrm{const}\exp(V)$. The rates are normalized to their values in zero field [see Eqs. (A21) and (A22)].

the steady state for a given flow when the flow is small, the corrent steady-state solution is still found. We shall use the strong-field results over the entire range of applied flows. At small slows, the pinning is important as the sole source of free vortices, since the depairing rate is extremely small. In this regime we are using an incorrect rate, but the errors in the data are becoming of the order of the velocities themselves, so the fits are not sensitive to our error.

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[20]$\langle \rho_s \rangle$ is calculated by an empirical formula that agrees well with the values of λ at onset (see Ref. 7) and which is due to H. Scholtz, E. D. McLean, and I. Rudnick [Phys. Rev. Lett. 32, 1471 (1974)]. The formula is $\langle \rho_s \rangle = \rho_{sb}(1-D/d)$, where ρ_{sb} is the bulk superfluid density and D and d are in atomic layers (3.6 Å). d is the film thickness and D is a "healing length" and is given by $D = 0.5 + 1.13T(\rho/\rho_{sb})$, where T is the temperature in degrees Kelvin and ρ is the density of helium. A factor of T_λ should be removed from their expression (R. B. Hallock, private communication).