Specific sine-Gordon soliton dynamics in the presence of external driving forces

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We consider the acceleration of a single sine-Gordon (SG) soliton kink wave by an external time-dependent force $\chi(t)$, first without any dissipation, and then in the presence of a weak damping effect. We use the method of Fogel, Trullinger, Bishop, and Krumhansl [FTBK, Phys. Rev. B **45**, 1578 (1977)] which consists in perturbing the SG equation about its kink solution and solving the resulting linear inhomogeneous equation for the perturbation function by expanding it in the complete set of eigenfunctions of the Schrödinger operator with potential $1-2\,\mathrm{sech}^2 x$. Our results concerning the accelerated soliton dynamics strongly disagree with the FTBK conclusion that the soliton should undergo an acceleration proportional to χ (this is the so-called Newtonian dynamical behavior of SG soliton, which is also predicted by all existing perturbation theories dealing with the perturbed SG equation). On the contrary, we find that this Newtonian acceleration is exactly balanced by a reaction effect of the continuous phonon spectrum excited by the external force χ, upon the moving kink, so that there is no soliton acceleration at all *within the frame of this linear perturbation theory,* i.e., for small time values. Actually, we show by the simple example of a static external force that the acceleration of an initially static kink is a higher-order effect (proportional to χt^2, where t is the time, instead of being constant and proportional to χ). We emphasize that this last result has already been checked by numerical experiments and show, both by theory and by numerical simulations, that it does not qualitatively change when a small damping effect is taken into account.

INTRODUCTION

The sine-Gordon (SG) partial differential wave equation plays a major role in nonlinear physics. Its success in modeling nonlinear physical phenomena has been emphasized in the pioneering papers of Perring and Skyrme[1] and Scott.[2] Its complete integrability and accompanying remarkable soliton properties were established later by Ablowitz, Kaup, Newell, and Segur, using the powerful tools of the inverse scattering technique.[3] We note with historical interest that Ref. 1 already numerically described the basic property of the SG solitary-wave solutions which led to their later name "solitons," namely, their emerging from multi-solitary-wave collisions with the same shapes and velocities with which they entered. In Ref. 2, the emphasis was placed on the exceptional simplicity of the qualitative kinetic study of the SG nonlinear wave solutions. This simplicity broke the widespread belief that there were no basic and elementary problems in the nonlinear wave theory. Reference 2 showed in particular an easily constructed mechanical analog of the Josephson junction transmission line (this line is directly related to the discovery of the physical interest of SG equation in condensed-matter physics[2,4,5]). Here again, the soliton properties of the solitary-wave solutions during pulse interactions were noted, and due to unavoidable energy losses in the mechanical device, the progressive slowing down of SG solitons due to damping and the subsequent Lorentz contraction of their spatial dimensions were emphasized. This mechanical study of weak perturbations of the SG soliton dynamics was continued some ten years later by Nakajima *et al.*[6] They obtained the main characteristic features of these perturbation effects, thus illustrating the fruitful intuitive approach to the SG equation praised by Scott.

The theoretical interest for SG solitons and their perturbations strongly increased after the basic conclusions of Ref. 3 concerning the soliton integrability. The SG kinklike soliton solution was often referred to in order to describe nonlinear excitations, both in elementary-particle physics,[1,7] where it was regarded as an extended relativistic particlelike solution of nonlinear field equations, and in condensed-matter physics; to name a few, we point out its relation with the theories of Bloch walls which separate domains in magnetic materials,[8] with the structural phase transitions,[9,10] with liquid ^{3}He,[11,12] with flux quanta on Josephson transmission lines,[2,5,13–17] and with charge carriers in "one-dimensional" Fröhlich charge-density-wave condensates.[18] People set about studying the interaction of the soliton with spatial inhomogeneities (e.g., due to impurities or defects[19–22]) and with external forces (e.g., external electric or magnetic field[19–26]), which may dramatically influence the soliton dynamics in system of practical interest.

There are now a great number of extensive review papers[21,27,28] or books[29,30] which summarize the progress and main results obtained in the past few years in the analytical treatment of weakly perturbed SG soliton or multisoliton solutions. Unfortunately, the number of experimental—or numerical—papers dealing with the same subject is by far smaller than the number of the former. To take a simple example which is the subject of this article, we note that there are to our knowledge only two papers[17,23] giving numerical results concerning the simplest case of acceleration of a simple SG (static) kink, namely, its perturbation by a static driving force in the absence of any damping. Only one of them shows the soliton velocity plotted versus time.[17] Moreover, even in this last reference, the authors chose too small a time scale in their figure 2, so that the basic result of the present paper concerning the anomalous kinetic behavior of SG solitons (at nonrelativistic velocities) with regard to existing theoretical predictions could not be checked.

Actually we are faced with the following situation. In a recent paper (Ref. 31, hereafter referred to as I), we showed by performing several numerical experiments on the "forced" SG equation (written in reduced units),

$$\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + \sin u = \text{const} = \chi\,, \tag{1}$$

that a single nonrelativistic kink moving with velocity $V(t)$, defined as the velocity of the kink's point of steepest slope and assumed small compared to its limit value 1, does not undergo the so-called Newton acceleration which is predicted by all existing theories[19-30]:

$$\frac{d}{dt}V(t) \simeq \frac{\pi}{4}\chi(1-V^2)^{1/2} \simeq \frac{\pi}{4}\chi\,. \tag{2}$$

To the contrary, it moves according to a time-dependent acceleration (proportional to t^2). We also showed that a weak damping, $\Gamma(\partial/\partial t)u$ ($\Gamma > 0$), added to the left-hand side (lhs) of (1) does not qualitatively change this result. We gave some theoretical arguments which helped in the understanding of this unexpected soliton dynamics and recovering its quantitative formulation. These arguments lay on the crucial assumption that the kink profile distortion function could be considered, in first approximation, as uniform function of time, i.e., independent of the spatial coordinate in the soliton rest frame. This assumption, obvious for small time values, is by no means evident when taking into account recent results dealing with the soliton shape change [tail or "continuum spectrum" growth due to the presence of the force χ in Eq. (1), Refs. 19–21 and 27]. So the aim of the present paper is to discuss its validity and recover the accuracy of the statement that SG solitons do not behave like Newtonian (classical) particles when driven by external forces[31] (see *Note added in proof*).

It is, of course, impossible to enter into a critical study of the whole theoretical literature devoted to the acceleration of SG solitons by external forces. Some authors (actually, those who first announced the classical Newton response of SG kinks to perturbations[19,20]) used a linear perturbation scheme having an analogy with inverse scattering techniques restricted to a single soliton situation, while others treated the problem of perturbation of multisoliton solutions by extending the inverse scattering transform to these situations.[21,22] The same general situation was considered in Refs. 27 and 29, where the authors developed a perturbation formalism based on a Green's-function technique.

It is clear that such multisoliton perturbation techniques have the great advantage of proposing a wide "spectrum" of new results concerning the single perturbed soliton case as well as the effect of perturbations on kink-antikink collisions[21,29] and their subsequent possible decay into a breather mode,[21] on breathers themselves, leading to breather decay,[29] and so on. It is out of our scope to consider these multisoliton effects and, for the sake of clear physical interpretation, we shall concentrate on the simplest possible example directly related to our previous numerical results, namely, *the acceleration of a single SG kinklike soliton by a time-dependent external force*, first in the absence of damping, and then in the presence of a weak damping.

We shall use the linear perturbation formalism of Ref. 19. Indeed it is a quite clear and adequate technique for a single soliton situation. The complete set $\mathcal{F}$ of eigenfunctions of the Schrödinger operator with potential $1-2\,\text{sech}^2 x$, which spans the linearized perturbation function ψ about the SG kink solution, is entirely appropriate to our problem since it leads to a system of simple second-order differential equations in time, the space derivatives being removed due to the particular Schrödinger potential. This system may then be exactly integrated in the absence of damping, thus leading to an *exact* solution of the problem.

In Sec. I, we briefly recall the construction of the set $\mathcal{F}$, which is physically equivalent to the set of small oscillations in the presence of a static soliton (kink) free of external influences. We emphasize, as is usually done,[19-30] the role of the zero-frequency bound state (Goldstone) mode $f_b(x)$ in the soliton dynamics. We compare the nature of this "translation" mode,[32] corresponding to the single discrete eigenvalue $\omega_b = 0$, to the "continuum" states $f_k(x)$ (extended modes) labeled by a wave vector k and describing phase-shifted phonon waves having the usual dispersion relation $\omega_k^2 = 1 + k^2$ [in our reduced units, see Eq. (1)].

Then, in Sec. II, we consider the (small) function ψ describing the change of the soliton state due to the presence of the external force χ in Eq. (1) and, following Ref. 19, we expand it in the complete set $\mathfrak{F}=\{f_b, f_k\}$. The time evolution of ψ is therefore determined by the time dependance of the amplitudes $\psi_b=(\psi, f_b)$ and $\psi_k=(\psi, f_k)$, where the parentheses here mean the scalar product defined in the set $\mathfrak{F}$. The kinetic equations for ψ_b and ψ_k are obtained by substitution of the solution $u_{(0)}+\psi$ into Eq. (1), linearization in ψ in order to obtain a partial differential equation of the form

$$\mathfrak{L}(\psi)=\chi\,, \tag{3}$$

(where $\mathfrak{L}$ is a definite linear operator), and subsequent *ad hoc* integration over the space in order to use the orthonormality relations between the elements of $\mathfrak{F}$.[19]

Considering an initial static ($V=0$) kink, these equations easily lead to the expected perturbation function

$$\psi(x,t)=\psi(t)=\int_0^t dt' \int_0^{t'} dt''\chi(t'')$$

for small time values ($t \gtrsim 0$). The principal new result of our paper is the *physical* interpretation of the spectral expansion of ψ in the complete set $\mathfrak{F}$. In order to illustrate the difficulty, consider the expansion in $\mathfrak{F}$ of a constant (for example, unity, since the problem of the development of a function independant of the space variable x in $\mathfrak{F}$ may always be reduced to this case by obvious linearity of the scalar product). We obtain by use of the completeness relation of $\mathfrak{F}$:

$$1=\frac{\pi}{\sqrt{2}}f_b+\int_{-\infty}^{\infty}dk\,\psi_k f_k=\frac{\pi}{\sqrt{2}}f_b+\left(1-\frac{\pi}{\sqrt{2}}f_b\right), \tag{4}$$

and verify that $1-(\pi/\sqrt{2})f_b$ is orthogonal to f_b. Expansion (4) looks obvious, but its mathematical and physical meaning are not obvious at all. Indeed, expression (4) *mathematically* means that the *projection* of the function 1 on the basis vector f_b is $\pi/\sqrt{2}$, while the *dynamical* contribution of 1 to the soliton motion is zero because so is the coefficient of f_b in expression (4). We emphasize that this coefficient is the quantity which has a *physical* meaning since it measures the (small) translation of a soliton, the perturbation function ψ of which would be 1.

Therefore the above result applies to the start of the soliton motion, and we find that the soliton remains static even when the perturbation χ is "switched on," while an erroneous interpretation[19] of relation (4) was to predict the Newtonian behavior $V(t)\propto \pi\chi t/\sqrt{2}$, by retaining only the mathematical *projection* of the force χ onto f_b. As a consequence, *the soliton does not undergo any acceleration,* at least in the leading order of the present linear nonrelativistic perturbation scheme [actually, it does, but this acceleration appears as a second-order t^2 term in this theory, where t is always assumed small compared to unity in order to have the kink velocity $V(t)\ll V_{\text{limit}}=1$].

As an application of the present study, we consider in Sec. III the case of an oscillating field $\chi(t)=\chi\cos\omega_0 t$ and give the expression of the corresponding function ψ. The case $\omega_0=0$ allows us to recover the results of I. Finally (Sec. IV), by using arguments similar to those in Refs. 19 and 20, we remark that the presence of a weak damping Γ in the nonrelativistic case ($V\ll 1$) does not change the conclusions of the dissipationless study, since it introduces additional coupling terms proportional either to ΓV or to $\Gamma\psi V$ in the kinetic equation for ψ_b and ψ_k. These terms are at least of second order in the small quantities Γ, V, and ψ. We give the approximate equation of motion of the weakly damped soliton in the case of a static field $\omega_0=0$,

$$V(t)=\tanh\left[\pm\frac{\pi\chi t^3}{24}\left(1-\frac{\Gamma}{2}t\right)\right] \tag{5}$$

(+ and − signs for antikinks and kinks, respectively), and check it by numerical simulations.

I. REVIEW OF THE FREE SOLITON LINEAR PERTURBATION THEORY

We briefly recall the nature of small oscillations in the presence of a soliton (kink) free of external influences.[19,20] The soliton has the following profile, where the plus and minus sign is used, respectively, for a kink and an antikink:

$$S_V(z)=4\tan^{-1}\exp(\pm z)\,. \tag{6}$$

z is the Lorentz-transformed spatial variable:

$$z=(1-V^2)^{-1/2}(x-Vt)\,. \tag{7}$$

The soliton (6) is solution of the reduced SG equation

$$\frac{\partial^2}{\partial t^2}u-\frac{\partial^2}{\partial x^2}u+\sin u=0\,. \tag{8}$$

Since this equation is relativistically covariant, we write it in the soliton rest frame,

$$\frac{\partial^2}{\partial \tau^2}u-\frac{\partial^2}{\partial z^2}u+\sin u=0\,, \tag{9}$$

where the "local" time τ is deduced from the laboratory time t by a Lorentz transformation:

$$\tau=(1-V^2)^{-1/2}(t-Vx)\,. \tag{10}$$

Now we consider, in the soliton rest frame, a small perturbation function $\psi(z,\tau)$ that we write as[19,20,28]

$$\psi(z,\tau)=f(z)e^{-i\omega\tau}, \tag{11}$$

and we assume that the function $u=S_V+\psi$, describing the free soliton (6) surrounded by a cloud of small oscillations (11), is the solution of Eq. (9). After linearization in ψ, the amplitude $f(z)$ is found to obey the Schrödinger-type eigenvalue equation:

$$-\frac{d^2}{dz^2}f+(1-2\,\mathrm{sech}^2 z)f=\omega^2 f. \tag{12}$$

This equation allows the existence of a single "bound" state,

$$\omega_b=0, \quad f_b(z)=\frac{1}{\sqrt{2}}\,\mathrm{sech}\,z, \tag{13}$$

and the presence of a set of "scattering" or "continuum" states (expanded modes) labeled by a wave vector k:

$$\omega_k^2=1+k^2, \quad f_k(z)=\frac{1}{\sqrt{2\pi}}\frac{e^{ikz}}{\omega_k}(k+i\tanh z). \tag{14}$$

The bound state (13) is usually called the "translation" mode,[19-30] since the addition to S_V of a perturbation proportional to this mode, $\psi(z,\tau)=\alpha f_b(z)$, corresponds to the translation of the soliton S_V by an amount proportional to α. Therefore, when we deal in the next section with solitons perturbed by external forces, we shall concentrate on the value of the *coefficient* of f_b, which appears in the development of the perturbation function ψ [and which may differ from the value of the *projection* of ψ on f_b, see Eq. (4)], since this coefficient will give indications about the kinetic response of the perturbed soliton to the external force.

We note that the existence of the translation mode (13) is related to the property that an arbitrary rigid translation necessarily yields a solution of Eq. (9). Since expressions (13) and (14) are eigenfunctions of the self-adjoint spatial operator $-(d^2/dz^2)+(1-2\,\mathrm{sech}^2 z)$, they form a complete set which spans the space of functions of z. The orthonormality and completeness relations are

$$\int_{-\infty}^{\infty}dz\, f_b^2(z)=1,$$

$$\int_{-\infty}^{\infty}dz\, f_k^*(z)f_{k'}(z)=\delta(k-k'), \tag{15}$$

$$\int_{-\infty}^{\infty}dz\, f_b(z)f_k(z)=0,$$

$$f_b(z_1)f_b(z_2)+\int_{-\infty}^{\infty}dk\, f_k^*(z_1)f_k(z_2)=\delta(z_1-z_2), \tag{16}$$

and, according to (15) and (16), any function $\psi(z,\tau)$ may now be expanded in the complete set $\mathfrak{F}=\{f_b(z),f_k(z)\}$:

$$\psi(z,\tau)=\psi_b(\tau)f_b(z)+\int_{-\infty}^{\infty}dk\,\psi(k,\tau)f_k(z). \tag{17}$$

II. NONRELATIVISTIC RESPONSE TO AN EXTERNAL DRIVING FORCE: THE DEVIATION FROM NEWTON'S LAW

We allow the presence of a (small) time-dependant torque $\chi(\tau)$ at the right-hand side (rhs) of Eq. (9) and consider the corresponding perturbation function $\psi_\chi(z,\tau)$ developed as shown by Eq. (17). *We suppose a nonrelativistic dynamics:* We assume, as in Ref. 19, the soliton velocity is always small compared to its limiting value equal to unity:

$$V\ll 1. \tag{18}$$

Therefore we may approximate the soliton rest time (10) with the laboratory time t.

The small perturbation function $\psi_\chi(z,t)$ satisfies the following linearized kinetic equation:

$$\frac{\partial^2}{\partial t^2}\psi_\chi-\frac{\partial^2}{\partial z^2}\psi_\chi+(1-2\,\mathrm{sech}^2 z)\psi_\chi=\chi(t). \tag{19}$$

The substitution of expression (17) of ψ_χ into Eq. (19) and the subsequent multiplication by either $f_b(z)$ or $f_k^*(z)$ and integration over z leads, with the aid of the orthonormality relations (15), to the kinetic equations for the amplitudes $\psi_b(z)$ and $\psi_k(t)$, respectively:

$$\frac{\partial^2}{\partial t}\psi_b(t)=\frac{\pi}{\sqrt{2}}\chi(t), \tag{20}$$

$$\frac{\partial^2}{\partial t}\psi(k,t)+\omega_k^2\psi(k,t)=\chi(t)\int_{-\infty}^{\infty}dz' f_k^*(z'). \tag{21}$$

These equations are immediately integrated and yield, once we assume an exact soliton waveform (6) (moving at velocity V) as the initial condition $u(z,0)$ in Eq. (9),

$$\psi_b(t)=\frac{\pi}{\sqrt{2}}F(t), \tag{22}$$

$$\psi_k(t)=F_k(t)\int_{-\infty}^{\infty}dz' f_k^*(z'), \tag{23}$$

where

$$F(t)=\int_0^t dt'\int_0^{t'}dt''\chi(t''), \tag{24}$$

$$F_k(t)=g(t)-g(0)\cos\omega_k t-\omega_k^{-1}\dot{g}(0)\sin\omega_k t, \tag{25}$$

and the function $g(t)$ is a particular solution of

$$\ddot{g}+\omega_k^2 g=\chi(t)\,. \tag{26}$$

It is worth noting that the integration constants in (22) and (23) are, respectively, due to the following.

Equation (22). (i) The vanishing velocity of the soliton at $t=0$ measured with respect to the Galilean moving frame of velocity V:

$$v(t)\bigg|_{t=0}=\mp\frac{1}{2\sqrt{2}}\,\frac{\partial}{\partial t}\psi_b(t)\bigg|_{t=0}=0 \tag{27}$$

(kink: −sign; antikink: +sign, see Ref. 19).

(ii) A regular adjusting of the solitary wave profile to the kink waveform (6) as $t\to 0$.

Equation (23). (i) The stability of the soliton waveform, which implies that, at each point of abscissa z in the soliton rest frame, there is a local minimum of energy, i.e., that

$$\frac{\partial}{\partial t}\psi(k,t)\bigg|_{t=0}=0\,. \tag{28}$$

(ii) The choice of a kink as initial condition $u(z,0)$.

According to expansion (17), the response of the soliton to the external driving field $\chi(t)$ is

$$\psi_\chi(z,t)=\frac{\pi}{\sqrt{2}}f_b(z)F(t)+\int_{-\infty}^{+\infty}dz'\int_{-\infty}^{+\infty}dk\,F_k(t)f_k(z)f_k^*(z')\,. \tag{29}$$

It is now of interest to consider the start of the dynamics, since it helps to understand the origin of the non-Newtonian aspect of the soliton behavior by recovering Eq. (4). Therefore, let us assume as a first stage of our kinetic study

$$\epsilon=\omega_k t\ll 1\,. \tag{30}$$

Then Eq. (29), with the aid of Eqs. (24)–(26), reduces to lowest order in ϵ to

$$\psi_\chi(z,t)=F(t)\left(\frac{\pi}{\sqrt{2}}f_b(z)+\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dk\,f_k(z)f_k^*(z')\right)\,. \tag{31}$$

Using the completeness relation (16) we obtain

$$\psi_\chi(z,t)=\frac{\pi}{\sqrt{2}}F(t)f_b(z)\left(1-\frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty}dz'f_b(z')\right)+F(t)\,. \tag{32}$$

The first term in in the rhs of Eq. (32) describes the solitary-wave position in the Galilean frame of velocity V [this position is proportional to the translation mode amplitude: see definition (27)]. Since $\int_{-\infty}^{\infty}dz'f_b(z')=\pi/\sqrt{2}$, we see that *it vanishes identically*. This result may be understood as follows: When neglecting the effect of the "continuous" perturbation spectrum described by Eq. (21), Eq. (20) leads to the so-called Newtonian law of motion $\psi_b(t)=(\pi/\sqrt{2})F(t)$ predicted by all existing theories.[19-30] The fact that this motion is identically cancelled by the kinetic (i.e., proportional to f_b) contribution of the phonon spectrum $-F(t)f_b(z)\int_{-\infty}^{\infty}dz'f_b(z')$ means that there is actually no classical (i.e., Newtonian) particlelike behavior of SG solitons for small time values t in the presence of external driving forces, a result which agrees with our numerical experiments in I.

There is an important consequence of Eq. (32). In the nonrelativistic description of the soliton response to applied fields, the perturbation function ψ_χ may be considered in first approximation in ϵ [see definition (30)] as z independent: It is only a function of time. This was precisely the basic assumption of the analytical study performed in I.

In order to free ourselves from the rather drastic assumption (30) and add some more comments to the above results, we need to consider definite expressions of the force $\chi(t)$. We take as a basic example a periodic force[21,24]

$$\chi(t)=\chi\cos\omega_0 t\,. \tag{33}$$

III. EXPLICIT CALCULATION OF THE SOLITON SHAPE DISTORTION IN THE PRESENCE OF AN OSCILLATING FIELD

In this section, we explicitly calculate the integrals of the singular functions involved in Eq. (29) by use of the method of contours when $\chi(t)$ is given by Eq. (33). We recover all the above-mentioned results which were obtained for $t\to 0$ [see inequality (30)].

The function $F_k(t)$ defined by Eq. (25) reads, when $\chi(t)$ is given by Eq. (33);

$$F_k(t)=\frac{\chi}{\omega_k^2-\omega_0^2}(\cos\omega_0 t-\cos\omega_k t)\,. \tag{34}$$

In order to calculate the "continuum" contribution ψ_χ^c to $\psi_\chi(z,t)$, given by the second term in the rhs of Eq. (29), we first evaluate $\int_{-\infty}^{\infty}dz'f_k^*(z')$ using the method of contours. We obtain

$$\int_{-\infty}^{\infty}dz'f_k^*(z')=\frac{1}{\sqrt{2\pi}\,\omega_k}\left(2\pi k\delta(k)-\frac{\pi}{\sinh\frac{1}{2}\pi k}\right)\,. \tag{35}$$

Then the integration over k reads

$$\psi_\chi^c(z,t)=\int_{-\infty}^{\infty}dk\,F_k(t)f_k(z)\int_{-\infty}^{\infty}dz'f_k^*(z')$$
$$=-\tfrac{1}{2}\chi\int_{-\infty}^{\infty}dk\frac{(\cos\omega_0 t-\cos\omega_k t)(k+i\tanh z)e^{ikz}}{(\omega_k^2-\omega_0^2)\omega_k^2\sinh\frac{1}{2}\pi k}\,. \tag{36}$$

This integral is also evaluated by the method of contours and yields

$$\psi_\chi^c(z,t) = -\tfrac{1}{2}\chi\pi \operatorname{sech}|z|\,\frac{1-\cos\omega_0 t}{\omega_0^2} + 2\chi\left(\sum_{n=1}^{\infty}(-1)^n e^{-2n|z|}(2n+\tanh|z|)\,\frac{\cos\omega_0 t - \cosh(4n^2-1)^{1/2}t}{(1-4n^2)(1-4n^2-\omega_0^2)}\right)$$
$$+\chi\tanh|z|\,\frac{\cos\omega_0 t - \cos t}{1-\omega_0^2}. \qquad (37)$$

Since the function $F(t)$ is [cf. (24)]

$$\chi\frac{1-\cos\omega_0 t}{\omega_0^2}, \qquad (38)$$

the first term in (37) yields

$$-\frac{\pi}{\sqrt{2}}F(t)f_b(z) \qquad (39)$$

and therefore cancels, as expected from the preceding section, the Newtonian kinetic term $(\pi/\sqrt{2})F(t)f_b(z)$ related to the amplitude of the translation mode [cf. Eqs. (20) and (29)]. It is worth noting that this first term in (37) is obtained by the method of contours from the root $k=+i$ located in the upper half plane, and therefore corresponds to a zero-frequency mode [see the dispersion relation (14)], which is precisely the characterization of the translation (Goldstone) mode. The presence of explosive terms proportional to $\cosh(4n^2-1)^{1/2}t$ implies that the present theory is only relevant for not too large time values,[19,20] a condition which has already been emphasized earlier [cf. (18)]. Let us describe both following particular cases.

(i) $t\to 0$. We have

$$\psi_\chi^c(z,t) = \tfrac{1}{2}\chi t^2\left(-\frac{\pi}{\sqrt{2}}f_b(z) + 2\sum_{n=1}^{\infty}(-1)^n\frac{e^{-2n|z|}(2n+\tanh|z|)}{1-4n^2} + \tanh|z|\right), \qquad (40)$$

and therefore [since $\psi_\chi(z,t)$ is equal to both remaining terms in the rhs of Eq. (40); see formulas (29) and (39)]

$$\psi_\chi(z,t) = \tfrac{1}{2}\chi t^2\left(2\sum_{n=1}^{\infty}(-1)^n\frac{e^{-2n|z|}(2n+\tanh|z|)}{1-4n^2} + \tanh|z|\right) = \tfrac{1}{2}\chi t^2. \qquad (41)$$

Note that the large parentheses are identically equal to 1. We obtain result (32) as expected because we consider small time values.

(ii) $z\to\infty$. We obtain in a similar way

$$\psi_\chi(z,t) = \chi\tanh|z|\,\frac{\cos\omega_0 t-\cos t}{1-\omega_0^2}. \qquad (42)$$

We note that in the limit $\omega_0\to 1$, Eq. (42) yields

$$\psi_\chi(z,t) \simeq \tfrac{1}{2}\chi t\,\sin t\,\tanh|z|\,. \qquad (43)$$

This formula, which is verified with quite acceptable accuracy by experimental simulations performed in the case $\chi=0.3$ (see Fig. 1), shows a resonance effect which symmetrically amplifies the oscillations on both wings of the kink.

This nonrelativistic description leads to an important restriction. It does not include the critical value[23]

$$\chi_c = 0.72461\ldots \qquad (44)$$

which determines the sudden transition (or bifurcation) between the regime of an oscillating kink amplitude and a monotonously increasing one. Value (44) may be easily obtained when considering the limit $x\to\pm\infty$ of Eq. (9), including a constant torque χ at its rhs, $(\partial^2/\partial t^2)u+\sin u=\chi$, and when solving it by quadratures with the consideration of the initial conditions $u(0)=(\partial/\partial t)u|_0=0$. Then one has to study the motion of a fictitious particle, the trajectory of which is described by

$$\tfrac{1}{2}\dot{u}^2 - \cos u - \chi u + 1 = 0\,. \qquad (45)$$

It is clear that when χ is greater than the critical value (44) determined by

$$1+(1-\chi_c^2)^{1/2} - \pi\chi_c + \chi_c\sin^{-1}\chi_c = 0\,, \qquad (46)$$

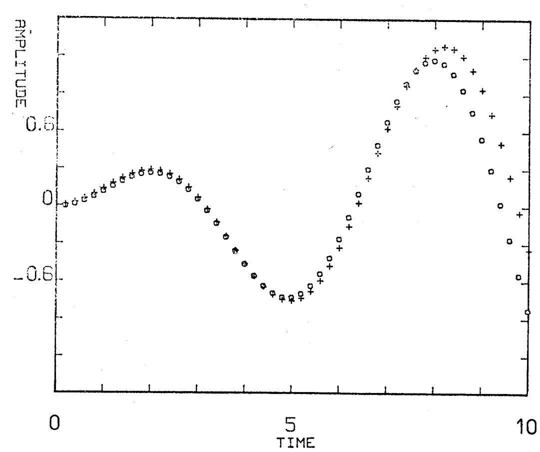

FIG. 1. Amplitude ψ_χ of the perturbation function at asymptotic abcissas $z\to+\infty$ versus time in the case of an SG antikink perturbed by an oscillating field $\chi(t)=0.3\cos t$. The resonance effect predicted by formula (43) is clearly seen through both theoretical (o line) and numerical (+ line) curves. The agreement between these curves is excellent up to $t\simeq 8$, which is rather unexpected since the theory predicted a good agreement only for $t\lesssim 1$.

the particle becomes unbounded in the potential defined by Eq. (45), and the corresponding soliton amplitude increases monotonously.[33] The reason for the impossibility of the present theory to recover this phenomenon is due to assumption (30), which means that the above fictitious particle is not allowed to oscillate—even once—in the potential well $(1-\cos u-\chi u)$ defined by Eq. (45). Its trajectory is restricted to the close vicinity of the potential as the particle "takes off" at $u=0$. It is by no means allowed to reach the point of abscissa $u\approx\pi$ where it would—or would not, depending on the relative values of χ and χ_c—undergo a reflection.

IV. SINE-GORDON SOLITON DYNAMICS IN THE PRESENCE OF A STATIC FORCE AND A WEAK DAMPING

For the sake of simplicity, we now assume $\chi(t)=\chi=\text{const}$. Therefore the results of the above section yield, with $\omega_0=0$:

$$F_k(t)=\frac{\chi}{\omega_k^2}(1-\cos\omega_k t)\,. \tag{47}$$

In order to recover the result of paper I concerning the next nonzero term in the large parentheses in (32), we set $\omega_k\sim 1$, which means that we consider the phonon wave packet, excited

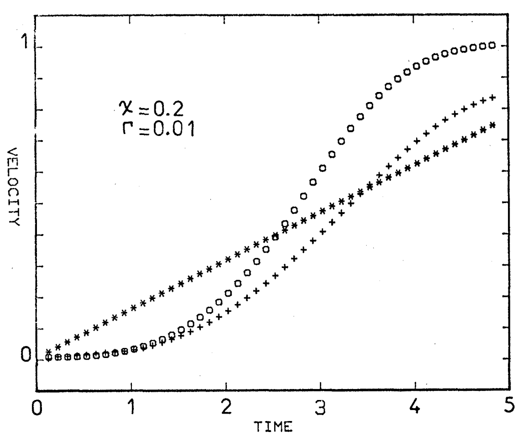

FIG. 2. Soliton velocity V versus time t for the force χ equal to 0.2 and the damping coefficient Γ equal to 0.01. o line: the theoretical curve (54). + line: the numerical values obtained by numerical simulations of Eq. (51) as described in paper I. Note that the part of the numerical curve which approaches—and even exceeds—$V=1$ is meaningless as explained in I. * line: the Newtonian soliton velocity as predicted by all other theories, i.e., $V(t)=(\pi\chi/4\Gamma)(1-e^{-\Gamma t})$. Mind that we neglect, in the theoretical formula (54), the relativistic corrections and approximate, therefore, the soliton rest frame time with the laboratory one. These corrections would lead to a better fitting of both o and + curves, as explained in I.

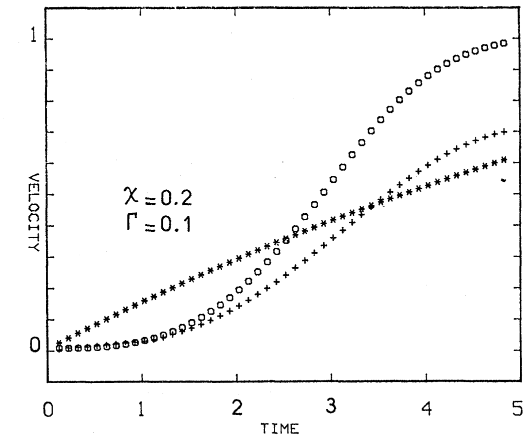

FIG. 3. Same as Fig. 2 with $\chi=0.2$ and $\Gamma=0.1$.

by the perturbation, as a quasimonochromatic one centered around the wave vector $k=0$. This is indeed suggested by formula (35). Then

$$F_k(t)\sim F_0(t)=\chi(1-\cos t)\simeq\chi\left(\frac{t^2}{2}-\frac{t^4}{24}+\cdots\right)\,. \tag{48}$$

Equations (29)–(32) and (48) imply, up to order t^4,

$$\psi_\chi(z,t)\simeq\frac{\pi\chi t^4}{24\sqrt{2}}f_b(z)+\chi\left(\frac{t^2}{2}-\frac{t^4}{24}\right). \tag{49}$$

Therefore the corresponding kink velocity is [see definition (27)]

$$v(t)=\mp\frac{\pi\chi t^3}{24} \tag{50}$$

(antikink: + sign; kink: − sign),
and we recover the basic result of paper I, which has been numerically checked with quite acceptable accuracy. We shall discuss in the Conclusion of this paper the physical meaning of the assump-

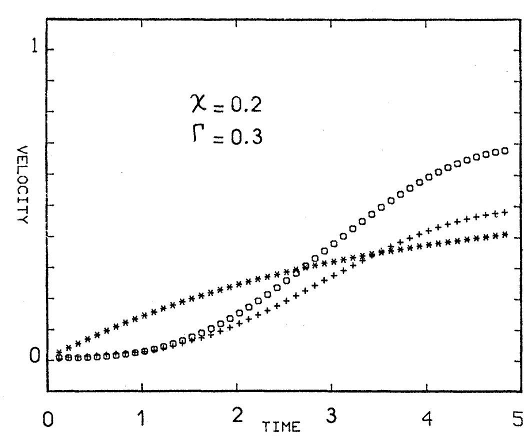

FIG. 4. Same as Fig. 2 with $\chi=0.2$ and $\Gamma=0.3$.

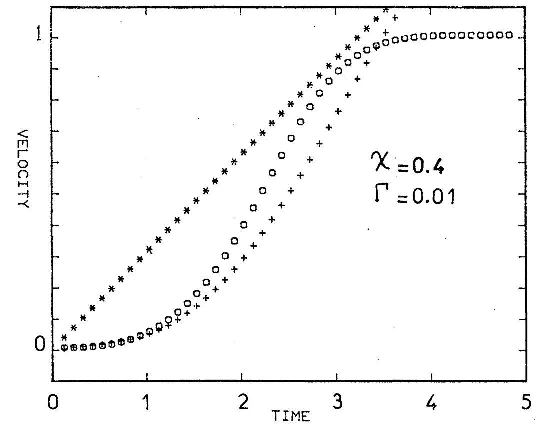

FIG. 5. Same as Fig. 2 with $\chi = 0.4$ and $\Gamma = 0.01$.

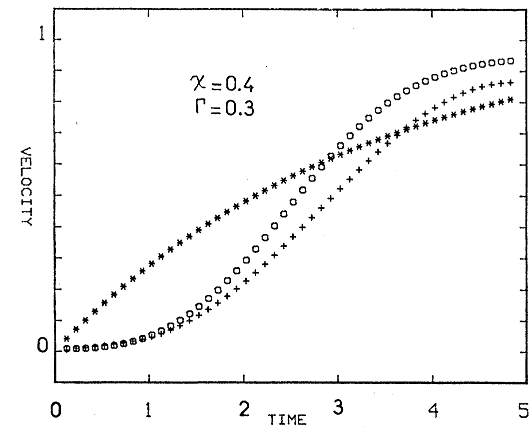

FIG. 7. Same as Fig. 2 with $\chi = 0.4$ and $\Gamma = 0.3$.

tion $\omega_k \sim 1$.

The presence of a weak damping term in the "forced" SG equation

$$\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + \sin u + \Gamma \frac{\partial}{\partial t}u = \chi\,, \tag{51}$$

where

$$0 < \Gamma \ll 1\,, \tag{52}$$

does not qualitatively change the results of the above parts as long as we assume a nonrelativistic soliton dynamics. Indeed, in the approximations (18) and (52), Eq. (51) simply leads to the addition of $\Gamma(\partial/\partial t)\psi_b(t)$ and $\Gamma(\partial/\partial t)\psi(k,t)$ at the lhs of Eqs. (20) and (21), respectively.[19,20] As a consequence, the integration of these equations will lead to the existence of a terminal velocity $(1 - V_{\text{term}}^2)^{-1/2} V_{\text{term}} = \mp \pi\chi/4\Gamma$,[19-30] and by either the method described in this paper or by the method of the associated fictitious particle described in paper I, one obtains the correction to formula (50) due to the weak damping:

$$v(t) = \mp \frac{\pi\chi t^3}{24}\left(1 - \frac{\Gamma}{2}t\right). \tag{53}$$

Using similar relativistic arguments as those in I, we obtain the SG soliton dynamics in the presence of a static force and a weak damping [see condition (52)]:

$$V(t) = \tanh\left[\mp \frac{\pi\chi t^3}{24}\left(1 - \frac{\Gamma}{2}t\right)\right] \tag{54}$$

(+ and − signs, respectively, for antikinks and kinks). Formula (54) has been numerically checked for various values of the force χ ($\chi = 0.2, 0.4, 0.6$) and the damping Γ ($\Gamma = 0.01, 0.1, 0.3$). The results are shown on Figs. 2–10. The agreement with the theoretical formula (54) is excellent for small time values [when formula (54) reduces to (53)] and acceptable for larger times [note that the terminal velocity remains equal to unity within approximation (52), as shown by Eq. (54)]. Moreover, we point out that the relativistic correction

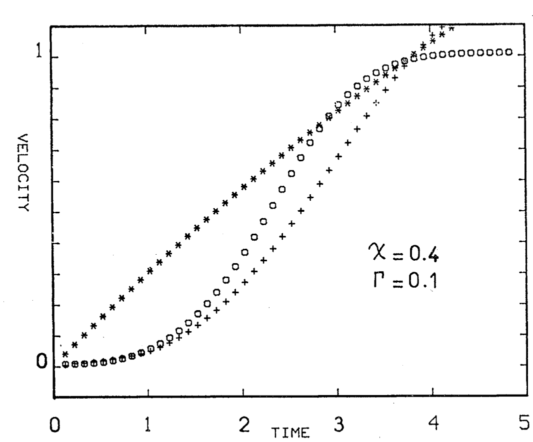

FIG. 6. Same as Fig. 2 with $\chi = 0.4$ and $\Gamma = 0.1$.

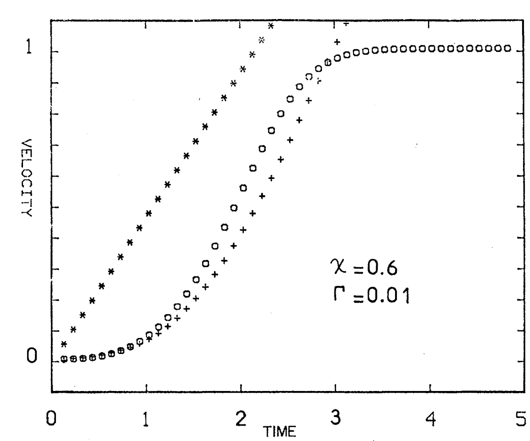

FIG. 8. Same as Fig. 2 with $\chi = 0.6$ and $\Gamma = 0.01$.

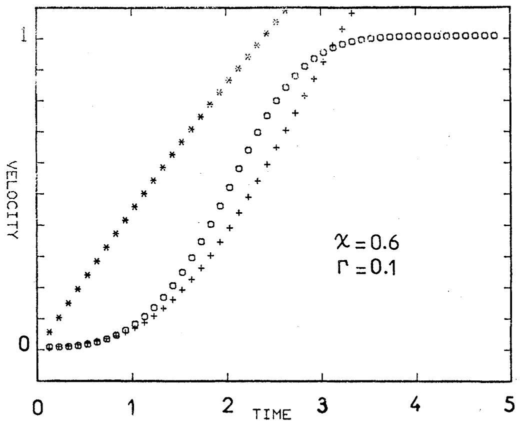

FIG. 9. Same as Fig. 2 with $\chi = 0.6$ and $\Gamma = 0.1$.

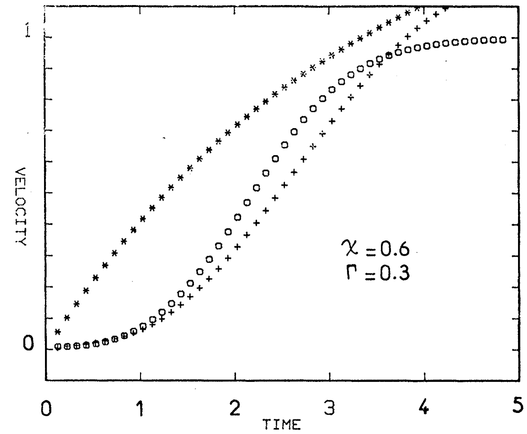

FIG. 10. Same as Fig. 2 with $\chi = 0.6$ and $\Gamma = 0.3$.

related to the Lorentz contraction of time $\tau = (1-V^2)^{1/2}t$ leads to the corrected formula [see paper I]

$$V(t) = \tanh\left[\mp \frac{\pi\chi t^3}{24\gamma^5}\left(1-\frac{\Gamma}{2}t\right)\right] \qquad (55)$$

with $\gamma \approx (1-V^2)^{-1/2}$, which fits the experimental curves better in Figs. 2–10, due to the weak dependence of γ over the time t; see paper I.

CONCLUSION

In the basic situation of the acceleration of a single SG kink by an external (time-dependent) force—which is actually the exact translation to soliton quasiparticles of Newton's well-known falling-apple experiment—we have demonstrated that the theoretically predicted Newtonian kink acceleration does not occur (see *Note added in proof*), due to the dynamical reaction—or screen—effect upon the moving soliton of the "cloud" of phonon waves excited by the external force. This last effect rigorously balances the Newtonian acceleration. It has been confirmed by numerical experiments (I and see Figs. 2–10). We understand this reaction of the excited phonon spectrum upon the soliton wave as a kind of resonant effect of the coherent (almost monochromatic) phonon wave packet defined by $k \sim 0$ with the moving kink. Indeed, these zero wave-number phonons have a vanishing group velocity $V_{gr} = \omega^{-1}k \simeq k = 0$ in the soliton rest frame. Therefore their energy may be regarded as "trapped" by the moving kink. We note, in relation to the above statement, that the resonance effect which appears between the oscillations of the soliton far wings and a forced external oscillating field $\chi(t) = \chi\cos t$ [which is described by formula (43) and checked in Fig. 1 of the present paper] is actually due to the contribution of the pole $k = 0$ to the integral (36) describing via the method of contours the continuum contribution $\psi_\chi^c(z,t)$. This suggests that the modes $k \sim 0$ are excited predominantly because they fit the physically reasonable boundary conditions at the far wings ($z = \pm\infty$) of the soliton: $(\partial/\partial z)\psi_\chi = 0$. We similarly note that the resonant "dynamical" interaction between this excited monochromatic wave packet and the moving kink mathematically results in a pole $k = +i$ in the integral giving the expression of the kink state $\psi_\chi(z,t)$. This pole yields a corresponding zero-frequency mode $\omega_k^2 = 1 + k^2 = 0$, which is of the same physical nature as the (Goldstone) translation mode f_b.

Note added in proof. The authors were recently informed by Professor A. Newell of numerical experiments performed in the case of a small static force $\chi \ll 0.5$. Newell obtained a quasi-Newtonian part of the kink trajectory $V(t)$ for intermediate time values roughly defined by $t_N < t < t_R$, in which $t_N = \sqrt{2}$ is the time value for which Eq. (50) yields a Newtonian acceleration $\pi\chi/4$, and $t_R \simeq (4/\pi\chi)(2/3)^{1/2}$ is of the same order of magnitude as the time for which relativistic effects described by the equation $\dot{V} = (1-V^2)^{3/2}(\pi\chi/4)$ become important. Thus, the smaller the value of χ, the larger this quasi-Newtonian time interval. Actually the threshold value χ_{th} needed for this interval to exist is $\chi_{th} = 4/\pi\sqrt{3}$. Hence we did not choose small enough χ values compared with χ_{th} in the present paper to obtain Newell's quasi-Newtonian trajectory sections. Finally, it must be remembered that even when $t_N < t < t_R$, the soliton dynamics is never exactly Newtonian, since the interference effect between the continuum perturbation spectrum and the discrete one, as described in this paper, still does exist, although progressively reduced by a factor $1/t$ as $t \to t_R$ (cf. I). The authors are grateful to

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[1]J. K. Perring and T. H. R. Skyrme, Nucl. Phys. 31, 550 (1962).
[2]A. C. Scott, Am. J. Phys. 37, 52 (1969).
[3]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262 (1973).
[4]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass., 1965), Vol. III.
[5]A. C. Scott, *Active and Nonlinear Wave Propagation in Electronics* (Wiley-Interscience, New York, 1970).
[6]K. Nakajima, T. Yamashita, and Y. Onodera, J. Appl. Phys. 45, 3141 (1974).
[7]R. Rajaraman, Phys. Rep. C 21, 229 (1975).
[8]U. Enz, Hel. Phys. Acta 37, 245 (1964).
[9]J. A. Krumhansl and J. R. Schrieffer, Phys. Rev. B 11, 3535 (1975).
[10]T. R. Koehler, A. R. Bishop, J. A. Krumhansl, and J. R. Schrieffer, Solid State Commun. 15, 1515 (1975).
[11]K. Maki and H. Ebisawa, J. Low. Temp. Phys. 23, 351 (1976).
[12]K. Maki and P. Kumar, Phys. Rev. B 14, 118 (1976); 14, 3928 (1976).
[13]B. D. Josephson, Adv. Phys. 14, 419 (1965).
[14]A. C. Scott, Nuovo Cimento B 69, 241 (1970).
[15]A. Barone, F. Esposito, C. J. Magee, and A. C. Scott, Riv. Nuovo Cimento 1, 227 (1971).
[16]J. Rubinstein, J. Math. Phys. 11, 258 (1970).
[17]K. Nakajima, Y. Onodera, T. Nakamura, and R. Sato, J. Appl. Phys. 45, 4095 (1974).
[18]M. J. Rice, A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Phys. Rev. Lett. 36, 432 (1976).
[19]M. B. Fogel, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, Phys. Rev. Lett. 36, 1411 (1976); Phys. Rev. B 15, 1578 (1977).
[20]J. F. Currie, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, Phys. Rev. B 15, 5567 (1977).
[21]D. J. Kaup and A. C. Newell, Proc. R. Soc. London Ser. A 361, 413 (1978).
[22]K. H. Spatschek, Z. Phys. B 32, 425 (1979).
[23]M. Inoue and S. G. Chung, J. Phys. Soc. Jpn. 46, 1594 (1979).
[24]M. Inoue, J. Phys. Soc. Jpn. 47, 1723 (1979).
[25]L. W. Adams, Jr., Phys. Rev. A 21, 1648 (1980).
[26]K. Nakajima, Y. Sawada, and Y. Onodera, J. Appl. Phys. 46, 5272 (1975).
[27]J. P. Keener and D. W. McLaughlin, Phys. Rev. A 16, 777 (1977); J. Math. Phys. 18, 2008 (1977).
[28]A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Physica D 1, 1 (1980), and references therein.
[29]*Solitons in Action*, edited by K. Lonngren and A. Scott (Academic, New York, 1978); see particularly the chapters by A. R. Bishop, R. D. Parmentier, and D. W. Mc Laughlin and A. Scott; see also D. W. Mc Laughlin and A. C. Scott, Phys. Rev. A 18, 1652 (1978).
[30]*Solitons*, edited by R. K. Bullough and P. J. Caudrey (Springer, New York, 1980).
[31]J. C. Fernandez, J. M. Gambaudo, S. Gauthier, and G. Reinisch, Phys. Rev. Lett. 46, 753 (1981).
[32]About the physical nature of this mode f_b, see R. Jackiw, Rev. Mod. Phys. 43, 3, 681 (1977).
[33]We are indebted to Dr. M. Inoue for an interesting private discussion of this point.