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Crossover in anisotropic Potts ϕ^3 field theory with quadratic symmetry breaking

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A continuous ϕ^3 field theory with quadratic symmetry breaking and isotropic trilinear interaction $g_0 \sum d_{ijk}\phi_i\phi_j\phi_k$ is studied, to one-loop order in dimension $d=6-\epsilon$, for an anisotropic $(n+1)$-state Potts model. Effective critical exponents for the crossover induced by quadratic symmetry breaking are calculated, and the results, in the limit $n=0$, are applicable to the thermally driven crossover in random bond-diluted ferromagnets near the percolation threshold. The limitation of a single renormalized g is explicitly pointed out.

Consider a generalized $(n+1)$-state Potts model on a lattice in which the pair interaction $K_i e_i(r) \times e_i(r')$, in the representation[1] where $e_i^\alpha(r)$ are $n+1$ n-dimensional vectors $(\alpha=1,\ldots,n+1;$ $i=1,\ldots,n)$ on the lattice site r, is not the same for all components i. This amounts to break the equivalence of the Potts states. In a continuum, field-theory version this leads to quadratic internal symmetry breaking in the effective Hamiltonian density. In the spirit of renormalization-group theory, Wallace and Young[2] argued that this is the most relevant perturbation and that there is a single trilinear coupling $g_0 d_{ijk}\phi_i\phi_j\phi_k$ (summation over repeated indices), with a tensorial coefficient $d_{ijk}=e_i^\alpha e_j^\alpha e_k^\alpha$, that is invariant under the *full* symmetry group of the usual Potts model.[3] Nevertheless, quadratic symmetry breaking favors a crossover to states of lower symmetry with a possible relevant change in trilinear couplings, which is of considerable interest to study. It has been proposed that the thermally driven transition from the percolation threshold to the critical line in randomly diluted magnets with nonmagnetic impurities[4] is a realization of crossover in an anisotropic one-state (the $n=0$ limit) Potts model.[5,6] The theoretical aspects of quadratic symmetry breaking have not been studied in detail so far. Neither the change of critical exponents under crossover nor the relevance of possible symmetry breaking in the trilinear couplings has been established.

In this work we report the calculation of effective critical crossover exponents in a continuous ϕ^3 field theory with quadratic symmetry breaking and trilinear isotropy. We point out that there is a natural and nontrivial symmetry breaking in the trilinear coupling which has not been noted before and that can have further consequences. We show that trilinear isotropy is only restored by taking the renormalized couplings at the isotropic Potts fixed-point value. The validity of the results is, as usual, subject to an analysis of the fully anisotropic theory, which will be studied separately.

Our Landau-Ginzburg-Wilson Hamiltonian that corresponds to the Hamiltonian of Domany[7] for the random bond-diluted Ising ferromagnet in the replica limit $m=0$ of the anisotropic $n+1=2^m$ state Potts model is

$$H=\int d^dx\left[\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}t\phi^2+\frac{1}{2}\tilde{m}^2\phi_2^2+\frac{1}{3!}g_0\kappa^{\epsilon/2}d_{ijk}\phi_i\phi_j\phi_k\right] \quad (1)$$

where ϕ is an n-component real field with longitudinal and transverse components, ϕ_μ and ϕ_p, respectively; $\phi^2=\phi_1^2+\phi_2^2$ with $\phi_1^2\equiv\phi_\mu\phi_\mu$ and $\phi_2^2\equiv\phi_p\phi_p$, summation over repeated indices (used in all that follows), g_0 is a dimensionless coupling constant, and κ is an arbitrary momentum-scale parameter. The tensorial coefficients are

$$d_{ijk}=\sum_{\alpha=1}^{n+1}e_i^\alpha e_j^\alpha e_k^\alpha,\quad i,j,k=1,\ldots,n \quad (2)$$

in the convenient representation[2]

$$e_\mu=\sigma_\mu,\quad \mu=1,\ldots,m\ , \quad (3a)$$

$$e_p=(\sigma_\mu\sigma_\nu),(\sigma_\mu\sigma_\nu\sigma_\eta),\ldots,(\sigma_1\sigma_2\cdots\sigma_m)\ , \quad (3b)$$

$p=m+1,\ldots,n$, where e_μ are "longitudinal" and e_p "transverse" components (formed by all possible distinct products of σ_μ) of the vectors e_i^α, in which α labels the $2^m=n+1$ "states" of the spin variables $\sigma_\mu=\pm1$. Moreover, the e_i^α are constrained by

$$e_i^\alpha e_i^\beta=(n+1)\delta_{\alpha\beta}-1,\quad e_i^\alpha e_j^\alpha=(n+1)\delta_{ij}\ ,$$
$$\sum_{\alpha=1}^{n+1}e_i^\alpha=0\ . \quad (4)$$

Our convention is that greek subindices run over longitudinal components, latin over transverse ones, *except* i,j,k, which stand for all components. It follows from (3a) that $d_{\mu\nu\eta}\equiv0$, which has relevant consequences.

The square of the critical mass, t, and the noncritical mass $\tilde{m}$ are related to the scaling fields of the random bond-diluted Ising model,[8] $\mu_1 \simeq p - p_c$ and $\mu_2 \simeq e^{-2K}$ in the low-temperature limit, through $t \simeq \mu_1 - A\tilde{m}^{2/\phi}$ and $\tilde{m}^2 = \mu_2$, p being the bond concentration with mean-field critical value p_c and $K = J/T$ for an Ising pair interaction J.

We renormalize H by dimensional regularization[9,10] with generalized minimal subtraction (GMS)[11] and the calculations are checked against renormalized perturbation theory regularized with a cutoff.[12] The renormalization is done at the critical theory for fixed $\tilde{m}$. For the random ferromagnet this means renormalization at a point on the critical line. There are no critical mass subtraction terms in dimensional regularization[13] and we can proceed as usual by taking the critical line as given by mean-field theory. The reason for going beyond minimal subtraction of dimensional poles is that there are logarithmic terms in the dimensionless mass $\tilde{\mu} \equiv \tilde{m}/\kappa$ which appear behind the poles in an expansion in $\epsilon = 6 - d$ that have to be subtracted in the limit of large $\tilde{\mu}$, in order to make the theory finite, in complete analogy with ϕ^4 field theory with quadratic symmetry breaking.[11] Although the massive propagators vanish in the limit of large $\tilde{\mu}$, they lead to singular (in $\tilde{\mu}$) contributions to the diagrams. We are interested in allowing $\tilde{\mu}$ to vary over an infinite range in order to study the global features of the crossover. This amounts to a rescaling of $\tilde{\mu}$ in the flow equations for the renormalized parameters.

Let us first assume that, instead of the single isotropic trilinear coupling in Eq. (1), we have the four $u_0\kappa^{\epsilon/2}d_{\mu\nu\eta}\phi_\mu\phi_\nu\phi_\eta$, $u_1\kappa^{\epsilon/2}d_{\mu\nu m}\phi_\mu\phi_\nu\phi_m$, $u_2\kappa^{\epsilon/2}d_{\mu mn} \times \phi_\mu\phi_m\phi_n$, and $u_3\kappa^{\epsilon/2}d_{mnr}\phi_m\phi_n\phi_r$. Even if such couplings are not present in the original Hamiltonian they will be generated in the renormalization-group equations by the quadratic symmetry breaking for small but finite $\tilde{m}$. Although $d_{\mu\nu\eta} \equiv 0$ the u_0 coupling should not be left aside. Otherwise the isotropic Potts model behavior will not be recovered in the $\tilde{m} = 0$ limit. Moreover, an internal u_0 vertex connected to other internal vertices can lead to nonzero contributions to some of the irreducible three-point vertex functions $\Gamma^{(3)}_{ijk}$ which are needed in the renormalization scheme. Indeed, $\Gamma^{(3)}_{\mu mn}$ involves a term proportional to $e^\alpha_m e^\alpha_n e^\alpha_p e^\alpha_\sigma d_{\mu\rho\sigma}$—summation over repeated indices—which is $(n+1)d_{\mu mn}$ if $e_\rho e_\sigma = e_m e_n$, that is $e_m e_\rho e_\sigma = e_n$, for $n \neq m$, where use is being made of $e_i^2 = 1$ for any i.

We are primarily interested in the critical behavior of the longitudinal two-point vertex $\Gamma^{(2)}_{\mu\nu}$. To one-loop order, the bare vertex is

$$\Gamma^{(2)}_{\mu\nu}(\tilde{k}, u_i, \tilde{\mu}) = k^2\delta_{\mu\nu}\{1 - (n+1)^2[(m-1)u_1^2 J_1 + \tfrac{1}{2}(n+1-2m)u_2^2 J_2]\} \;, \tag{5}$$

where $J_1(\tilde{k}, \tilde{\mu})$ and $J_2(\tilde{k}, \tilde{\mu})$ are dimensionless integrals with one and two internal transverse propagators $(q^2 + \tilde{\mu}^2)^{-1}$, and one or no longitudinal propagator q^{-2}, and subtracted as usual at $\tilde{k} \equiv k/\kappa = 0$. Although a coupling constant renormalization is not needed to obtain a renormalized $\Gamma^{(2)}_{R\mu\nu}$ to this order, it will be needed for the fixed-point equations. In the fully anisotropic theory, that will be reported elsewhere,[14] the renormalization of a given u_i is related to that of all others. In the case of isotropic trilinear couplings that we are concerned with in this work we note first that, even if all u_i are taken to be the same, there are two distinct three-point vertex functions and to keep track of this we let $u_2 \neq u_1$ in

$$(d_{\mu\nu m})^{-1}\Gamma^{(3)}_{\mu\nu m}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3; u_1, \tilde{\mu}) = u_1\kappa^{\epsilon/2}\{1 + u_1^2(n+1)^2[3L_1 + (3m-10)L_2 + (n+5-3m)L_3]\} \;, \tag{6}$$

$$(d_{mnr})^{-1}\Gamma^{(3)}_{mnr}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3; u_2, \tilde{\mu}) = u_2\kappa^{\epsilon/2}\{1 + u_2^2(n+1)^2[L_0 + 3(m-1)L_2 + (n-3m)L_3]\} \;, \tag{7}$$

no index summation, with $\tilde{k}_3 = -\tilde{k}_1 - \tilde{k}_2$, in which $L_1(\tilde{k}_1, \tilde{k}_2, \tilde{\mu})$, L_2, and L_3 are given by triangular diagrams with one, two, and three $(q^2 + \tilde{\mu}^2)^{-1}$ propagators and L_0 with three longitudinal propagators q^{-2}. The symmetry of the model yields a tensorial coefficient for L_0 that vanishes in the first vertex and a vanishing coefficient for L_1 in the second one. We note also that in the small and large $\tilde{\mu}$ limits, $d^{-1}_{\mu\nu\eta}\Gamma^{(3)}_{\mu\nu\eta}$ reduces to Eq. (6) whereas $d^{-1}_{\mu mn}\Gamma^{(3)}_{\mu mn}$ behaves as Eq. (7). All the L_i, $i = 0, \ldots, 3$, have the same dimensional pole in $1/\epsilon$ and, although the detailed μ dependences of L_1, L_2, and L_3 are not the same, they have the same behavior for large $\tilde{\mu}$ that is relevant to make the theory finite by GMS and this is $[1 - \frac{1}{2}\epsilon\ln(1+\tilde{\mu}^2)]/\epsilon$. The next-to-leading terms in Eqs. (6) and (7) become $(n+1)^2(n-2)u^3\kappa^{\epsilon/2}/\epsilon$, the isotropic result for $\tilde{\mu} = 0$, but they differ for any other $\tilde{\mu}$.

To remove the dimensional poles and the large-$\tilde{\mu}$ behavior of the relevant vertex functions we renormalize as

$$Z_\phi\Gamma^{(2)}_{\mu\nu} = \Gamma^{(2)}_{R\mu\nu} \;, \quad Z_{\phi 2}\Gamma^{(2,1)}_{\mu\nu,i} = \Gamma^{(2,1)}_{R\mu\nu,i} \;,$$
$$Z_\phi^{3/2}\Gamma^{(3)}_{\mu\nu m} = \Gamma^{(3)}_{R\mu\nu m}, \quad Z_\phi^{3/2}\Gamma^{(3)}_{mnr} = \Gamma^{(3)}_{Rmnr} \;, \tag{8}$$

by means of dimensionless functions $u_1(u, \epsilon; \tilde{\mu})$, $u_2(v, \epsilon; \tilde{\mu})$, $Z_\phi(u, \epsilon; \tilde{\mu})$, and $Z_{\phi 2}(u, \epsilon; \tilde{\mu})$, where u and v are renormalized couplings, u_2 is set equal to u_1 in Eq. (5), and $\Gamma^{(2,1)}_{\mu\nu,i}$ is the longitudinal two-point vertex with one ϕ^2 insertion, summed over all the i

components of the insertion. Actually, the renormalization of $\Gamma^{(2,1)}_{\mu\nu,i}$ is not an independent one because the diagrams are those of $\Gamma^{(3)}_{\mu\nu\eta}$ with the tensorial coefficients of $\Gamma^{(2)}_{\mu\nu}$. The expansion coefficients in $u_1 \simeq u + a_1 u^3$, $u_2 \simeq v + a_2 v^3$, $Z_\phi \simeq 1 + b_1 u^2$, and $Z_{\phi 2} \simeq 1 + c_1 u^2$, and with u^2 replaced by v^2, that make the vertex functions finite are found to be

$$a_1(\tilde{\mu}) = \frac{1}{4\epsilon}(n+1)^2(7-3n)[1-\tfrac{1}{2}\epsilon\ln(1+\tilde{\mu}^2)] \ , \tag{9}$$

$$a_2(\tilde{\mu}) = a_1(\tilde{\mu}) - \tfrac{1}{2}(n+1)^2\ln(1+\tilde{\mu}^2) \ , \tag{10}$$

$$b_1(\tilde{\mu}) = -\frac{1}{6\epsilon}(n+1)^2(n-1)[1-\tfrac{1}{2}\epsilon\ln(1+\tilde{\mu}^2)] \ , \tag{11}$$

$$c_1(\tilde{\mu}) = -\frac{1}{\epsilon}(n+1)^2(n-1)[1-\tfrac{1}{2}\epsilon\ln(1+\tilde{\mu}^2)] \ . \tag{12}$$

The Wilson β functions are then obtained as[13,15]

$$\beta_1(u,\epsilon;\tilde{\mu}) = \left(\kappa\frac{\partial u}{\partial\kappa}\right)_{\lambda_1} = -\tfrac{1}{2}\epsilon u + \tfrac{1}{4}(n+1)^2(7-3n) \times (1+\tilde{\mu}^2)^{-1}u^3 \ , \tag{13}$$

$$\beta_2(v,\epsilon;\tilde{\mu}) = \left(\kappa\frac{\partial v}{\partial\kappa}\right)_{\lambda_2} = -\tfrac{1}{2}\epsilon v + \tfrac{1}{4}(n+1)^2(7-3n) \times (1-\tilde{\mu}^2/\tilde{\mu}_c^2)(1+\tilde{\mu}^2)^{-1}v^3 \ , \tag{14}$$

for fixed dimensional couplings $\lambda_1 = \kappa^{\epsilon/2}u_1$ and $\lambda_2 = \kappa^{\epsilon/2}u_2$, with $4\tilde{\mu}_c^2 \equiv 7-3n$. The nontrivial solutions to the fixed-point equations $\beta_1(u^*,\epsilon;\tilde{\mu})=0$ and $\beta_2(v^*,\epsilon;\tilde{\mu})=0$ are then

$$u^{*2}(\tilde{\mu}) = 2\epsilon(1+\tilde{\mu}^2)[(n+1)^2(7-3n)]^{-1} \ , \tag{15}$$

$$v^{*2}(\tilde{\mu}) = 2\epsilon(1+\tilde{\mu}^2)[(n+1)^2(7-3n)(1-\tilde{\mu}^2/\tilde{\mu}_c^2)]^{-1} \ . \tag{16}$$

Our equations are consistent with an isotropic trilinear coupling only if we set $\tilde{\mu}=0$ in u^{*2} and v^{*2}, and we are thus left with the fixed-point coupling $u^{*2}(0) = 2\epsilon/(n+1)^2(7-3n)$ of the isotropic Potts model. Equations (15) and (16) are discussed further below.

The other Wilson functions

$$\gamma_\phi(u,\epsilon,\tilde{\mu}) = \beta(u,\epsilon)\left(\frac{\partial \ln Z_\phi}{\partial u}\right) = \tfrac{1}{6}(n+1)^2(n-1)(1+\tilde{\mu}^2)^{-1}u^2 \ , \tag{17}$$

$$\gamma_{\phi^2}(u,\epsilon,\tilde{\mu}) = \beta(u,\epsilon)\left(\frac{\partial \ln Z_{\phi^2}}{\partial u}\right) = (n+1)^2(n-1)(1+\tilde{\mu}^2)^{-1}u^2 \ , \tag{18}$$

in which $\beta(u,\epsilon) \equiv \beta_1(u,\epsilon;0) = \beta_2(u,\epsilon;0)$ yield, when $u = u^*(0)$, the effective crossover exponents

$$\eta(\tilde{\mu}) = \gamma_\phi(u^*,\epsilon,\tilde{\mu}) = (n-1)\epsilon/3(7-3n)(1+\tilde{\mu}^2) \ , \tag{19}$$

$$\nu^{-1}(\tilde{\mu}) - 2 = \gamma_{\phi^2}(u^*,\epsilon,\tilde{\mu}) - \gamma_\phi(u^*,\epsilon,\tilde{\mu}) = 5(n-1)\epsilon/3(7-3n)(1+\tilde{\mu}^2) \tag{20}$$

that describe the critical $\Gamma^{(2)}_{\mu\nu}(k,\tilde{\mu}) \propto \delta_{\mu\nu}k^{2-\eta(\tilde{\mu})}$ and the longitudinal correlation length $\xi_{\parallel}(t,\tilde{\mu})\ t^{-\nu(\tilde{\mu})}$. When $\tilde{\mu}=0$ we recover the exponents of the isotropic Potts model[10,16] and, for $\tilde{\mu}=\infty$ (extreme anisotropy), the mean-field values $\eta(\infty)=0$ and $\nu^{-1}(\infty)=2$ in $d=6-\epsilon$ dimensions are obtained, with a smooth change in between. We also studied the critical behavior of the transverse $\Gamma^{(2)}_{pq}$, but before discussing this point we consider further the fixed-point equations (15) and (16). They clearly indicate that trilinear anisotropy may become important for large $\tilde{\mu}$. We do not know, at present, if they are at all characteristic of the fully anisotropic theory, but we note that $u^*(\tilde{\mu})$ has a runaway for $\tilde{\mu}\to\infty$, while $v^*(\tilde{\mu})$ runs away from a finite $\tilde{\mu}=\tilde{\mu}_c$ and becomes imaginary for $\tilde{\mu} > \tilde{\mu}_c$ with a finite asymptotic fixed-point value $v^*(\infty) = i(n+1)^{-1}\sqrt{\epsilon}/2$. Other ϕ^3 field theories with an imaginary coupling have been considered in a different context before.[17]

When we consider $\Gamma^{(2)}_{pq}$ we find that, for any finite $\tilde{\mu}$, it is renormalized by the same Z_ϕ that makes finite the longitudinal $\Gamma^{(2)}_{\mu\nu}$. Despite the discrete symmetry group of the Potts model this is a check on the renormalizability of the model with quadratic symmetry breaking, in accordance with current expectations. The diagram expansion for $\Gamma^{(2)}_{pq}$ is, of course, not the same. It differs in the tensorial coefficients and there is an additional diagram in $\Gamma^{(2)}_{pq}$ that involves two longitudinal propagators q^{-2}. For small $\tilde{\mu}$ we find $\Gamma^{(2)}_{pq}(k,\tilde{\mu}) \propto \delta_{pq}k^{2-\eta(\tilde{\mu})}$, with the same $\eta(\tilde{\mu})$ as for the longitudinal fluctuations. Similarly, $\Gamma^{(2,1)}_{pq,i}$ is renormalized with the same Z_{ϕ^2} as for $\Gamma^{(2,1)}_{\mu\nu,i}$ and this yields $\xi_\perp \sim t^{-\nu(\tilde{\mu})}$, with the same $\nu(\tilde{\mu})$ as for $\xi_\parallel$. On the other hand, for asymptotically large $\tilde{\mu}$ our conclusions are more tentative, because of the isotropic fixed point $u^*(0)$. Nevertheless, we find, as one would expect, that $\Gamma^{(2)}_{pq} \sim (\tilde{m}^2 + Ct)\delta_{pq}$, for $k=0$ and nonzero t, C being a constant, and the second term is just the mean-field energy density $E \sim t^{1-\alpha}$, where $\alpha = 0$.

To comment on our results we have shown that, for the present problem, the choice of an isotropic trilinear coupling used in previous work[5] is not a straightforward one. The presence of a single isotropic fixed-point value is in sharp contrast to ϕ^4 theory with quadratic symmetry breaking where the initial coupling flows to an isotropic fixed point of lower symmetry. It is useful to recall that this follows by assuming an isotropic four-point coupling to start with. It is possible, although unlikely, that the study of the fully anisotropic theory for the present ϕ^3 field Potts model could reveal a further isotropic fixed point. This is currently being studied. Our results should apply to the percolation crossover in the limit $n=0$. With a proper interpretation of t and $\tilde{m}^2$ they should also be applicable to the second-order transition in the $(n+1 \leq 2)$-state Potts model. Because of our quadratic symmetry breaking it is not clear that they should apply to the recently studied metastable region for the Potts model with $1 < n < \frac{7}{3}$.[18]

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