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# Soliton damping and energy loss in the classical continuum Heisenberg spin chain

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We consider the effect of damping on the evolution of a classical continuum one-dimensional isotropic Heisenberg ferromagnetic spin chain due to relativistic interaction. The corresponding Landau-Lifshitz equation is shown to be identifiable with a damped nonlinear Schrödinger equation. By obtaining an explicit decaying solitary wave solution for the energy density and magnetization density, we demonstrate the damping of the soliton solution with an associated loss of the total energy.

In the phenomenological description of ferromagnetic systems, the magnetization (spin) density $\vec{S}(\vec{r},t)$ is assumed to be a continuous function of position $\vec{r}$, with an associated energy density involving suitable magnetic interactions. Classifying such interactions as (i) exchange interaction and (ii) relativistic interaction, Landau and Lifshitz[1] derived the following phenomenological equation of motion[2–5] for the spin:

$$\frac{\partial \vec{S}(\vec{r},t)}{\partial t} = \vec{S} \times \nabla^2 \vec{S} + \lambda[\nabla^2 \vec{S} - (\vec{S}\cdot\nabla^2\vec{S})\vec{S}] \quad . \quad (1)$$

Here we have not included the effects of anisotropy and external field. The first term on the right-hand side of Eq. (1) corresponds to exchange interaction, while the second term represents suitable relativistic interaction. The effect of the latter is usually assumed to be insignificant represented by the smallness of the constant $\lambda$. Conventionally $\lambda$ is identified[4] as $\alpha\gamma$ where $\alpha$ is the dimensionless Gilbert damping parameter, and $\gamma$ is the gyromagnetic ratio. The term proportional to the constant $\lambda$ may be considered as a loss (dissipative) term in the equation of motion which causes least analytical distress[2] and which also preserves the constancy of the magnetization density.

It is also well known that Eq. (1) with $\lambda = 0$ corresponds to the continuum limit of the equation of motion of the classical isotropic Heisenberg ferromagnetic spin chain[6] where the exchange constant and lattice parameters have been absorbed after appropriate scaling of space and time variables. In one-space one-time dimensions Eq. (1), for the lossless ($\lambda = 0$) case, is a completely integrable solitonic system and is equivalent to a nonlinear Schrödinger equation.[7–9] In higher dimensions also it possesses interesting solutions.[10–12] In practice the system is not completely lossless and it is of importance to consider the effect of dissipation ($\lambda > 0$) on the spins and the nature of energy loss in the system. In this paper we consider the one-space one-time dimensional analog of Eq. (1):

$$\frac{\partial \vec{S}(x,t)}{\partial t} = \vec{S} \times \vec{S}_{xx} + \lambda[\vec{S}_{xx} - (\vec{S}\cdot\vec{S}_{xx})\vec{S}] \ ,$$
$$\vec{S}^2(x,t) = 1, \ (\vec{S} \equiv S_1, S_2, S_3) \ , \quad (2)$$

and derive appropriate evolution equations for energy and current densities, including dissipation, in the form of a damped nonlinear Schrödinger equation. Considering a damped analog of the static single soliton solution for the energy density, it is demonstrated explicitly how the energy decays and the spins get oriented towards a single direction as time passes on.

The nonlinear dynamics characterized by the equation of motion (2) could be understood easily by identifying the underlying geometry of the system. For this purpose we map the evolution equation (2) onto a *moving* space curve in $E^3$ of given curvature $\kappa$ and torsion $\tau$ characterized by the Serret-Frenet equation[7,8,13]

$$\vec{e}_{ix} = \vec{d} \times \vec{e}_i, \quad \vec{d} = \kappa \vec{e}_3 + \tau \vec{e}_1 \quad . \quad (3)$$

The $\vec{e}_i$, $i = 1,2,3$ are the usual unit tangent, principal normal, and binormal vectors and define a local coordinate system with an origin $O'$ on the space curve. Then a position vector $\vec{R}$ could be ascribed to $O'$ with respect to a fixed frame with origin $O$ and then a complete description of the space curve may be given.

The classical magnetization or spin vector $\vec{S}(x,t)$ corresponding to (2) is now identified with the unit tangent vector $\vec{e}_1(x,t)$ of the space curve and thereby we associate the energy density and current density of the undamped ($\lambda = 0$) system (2) in terms of the curvature and torsion as

$$\mathcal{E}(x,t) = \frac{1}{2}\left(\frac{\partial \vec{S}}{\partial x}\right)\cdot\left(\frac{\partial \vec{S}}{\partial x}\right) = \tfrac{1}{2}\kappa^2(x,t) \ , \quad (4a)$$

$$\mathcal{J}(x,t) = \vec{S}\cdot\left(\frac{\partial \vec{S}}{\partial x} \times \frac{\partial^2 \vec{S}}{\partial x^2}\right) = \kappa^2(x,t)\,\tau(x,t) \quad . \quad (4b)$$

 

Correspondingly the equation of motion (2) becomes

$$\vec{e}_{1t}=\vec{e}_1\times\vec{e}_{1xx}+\lambda[\vec{e}_{1xx}-(\vec{e}_1\cdot\vec{e}_{1xx})\vec{e}_1]$$

and so using Eq. (3) it can be written as $\vec{e}_{1t}=(-\kappa\tau+\lambda\kappa_x)\vec{e}_2+(\kappa_x+\lambda\kappa\tau)\vec{e}_3$. Again using Eqs. (2) and (3) the time evolution of the other two members of the trihedral may be found straightaway. Thus the evolution of the trihedral $\vec{e}_i$, $i=1,2,3$, at $O'$ of the space curve associated with (2) takes the rigid-body form

$$\vec{e}_{it}=\vec{\omega}\times\vec{e}_i,\quad \omega=\omega_1\vec{e}_1+\omega_2\vec{e}_2+\omega_3\vec{e}_3\ , \tag{5}$$

where

$$\omega_1=\left[\frac{\kappa_{xx}}{\kappa}-\tau^2\right]+\frac{\lambda}{\kappa}(2\kappa_x\tau+\kappa\tau_x)\ , \tag{6a}$$

$$\omega_2=-(\kappa_x+\lambda\kappa\tau)\ , \tag{6b}$$

$$\omega_3=(-\kappa\tau+\lambda\kappa_x)\ . \tag{6c}$$

In order that Eqs. (3) and (5) are compatible $(\vec{e}_{it})_x=(\vec{e}_{ix})_t$. We thus obtain

$$\kappa_t=-2\kappa_x\tau-\kappa\tau_x+\lambda(\kappa_{xx}-\kappa\tau^2)\ , \tag{7a}$$

$$\tau_t=\left[\frac{\kappa_{xx}}{\kappa}-\tau^2\right]_x+\kappa\kappa_x+\lambda\left[\left[\frac{1}{\kappa^2}(\kappa^2\tau)_x\right]_x+\kappa^2\tau\right]\ , \tag{7b}$$

and therefore the evolution of the energy and current densities is expressed as

$$\mathcal{E}_t=-\mathcal{J}_x+\lambda\left[\mathcal{E}_{xx}-\frac{\mathcal{E}_x^2}{2\mathcal{E}}-\frac{\mathcal{J}^2}{2\mathcal{E}}\right]\ , \tag{8a}$$

$$\mathcal{J}_t=\mathcal{E}_{xxx}-\frac{2\mathcal{E}_x}{\mathcal{E}}\left[\mathcal{E}_{xx}-\frac{\mathcal{E}_x^2}{2\mathcal{E}}\right]+\left[\mathcal{E}^2-\frac{\mathcal{J}^2}{\mathcal{E}}\right]_x$$
$$+2\mathcal{E}\mathcal{J}+\lambda\left[\frac{\mathcal{J}\mathcal{E}_{xx}}{\mathcal{E}}-\frac{\mathcal{E}_x^2\mathcal{J}}{2\mathcal{E}^2}-\frac{\mathcal{J}^3}{2\mathcal{E}^2}+\mathcal{J}_{xx}-\frac{\mathcal{J}_x\mathcal{E}_x}{\mathcal{E}}\right]\ . \tag{8b}$$

Now considering the total energy of the magnetic chain

$$E(t)=\int_{\infty}^{\infty}\mathcal{E}(x,t)\,dx=\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial\vec{S}}{\partial x}\right]^2=\frac{1}{2}\int_{-\infty}^{\infty}\kappa^2(x,t)\,dx \tag{9}$$

and using (7) and (8), we obtain the following expression for the rate of change of $E(t)$:

$$\frac{dE(t)}{dt}=\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}\kappa^2dx$$
$$=-\lambda\int_{-\infty}^{\infty}[(\kappa_x)^2+\kappa^2\tau^2]dx\le 0\quad\text{if }\lambda\ge 0\ . \tag{10}$$

It is evident from Eq. (10) that there is an energy loss in the system when the relativistic interaction is included ($\lambda>0$), as the rate of change of total energy is negative, and thereby showing the depletion of energy in the system.

To identify the evolution equations (7) or (8) with more standard nonlinear partial differential equations, we make the following complex transformation for the dependent variables:

$$\psi(x,t)=\kappa(x,t)\exp\left[i\int\tau(x,t)\,dx\right]$$
$$\equiv[2E(x,t)]^{1/2}\exp\left[i\int\left[\frac{\mathcal{J}(x,t)}{2\mathcal{E}(x,t)}\right]dx\right]\ . \tag{11}$$

Then Eqs. (7) become the damped nonlinear Schrödinger equation

$$i\psi_t+(1-i\lambda)\psi_{xx}+\frac{1}{2}|\psi|^2\psi$$
$$+\frac{i\lambda\psi}{2}\int(\psi\psi_x^*-\psi^*\psi_x)\,dx=0\ . \tag{12}$$

We might mention that Eq. (12) closely resembles the damped nonlinear Schrödinger equation discussed by Pereira and Stenflo[14] except for the nonlocal term in (12). In the undamped case ($\lambda=0$) Eq. (12) is the well-known completely integrable nonlinear Schrödinger equation[15] with associated soliton solutions. This has the further consequence that the corresponding isotropic Heisenberg ferromagnetic spin chain in its continuum limit is a completely integrable dynamical system and hence nonergodic. Then the question immediately arises as to how the nonlinear normal modes interact to lose energy in the presence of the dissipative term ($\lambda>0$) here. While a complete answer for the general $N$-soliton solution for the energy or the spin is not yet known, in the present article we confine our attention to the one-soliton case. To consider this, we proceed as below.

From Eq. (12) and its complex conjugate, we can show that

$$i\frac{d}{dt}(\psi\psi^*)+(1-i\lambda)\psi^*\psi_{xx}-(1+i\lambda)\psi_{xx}^*\psi=0\ . \tag{13}$$

Making use of Eq. (11) in Eq. (13) and intergrating between the limits $-\infty$ and $+\infty$, we arrive at an equation for the rate of change of total energy

$$2i\frac{dE}{dt}=(1+i\lambda)\int_{-\infty}^{\infty}\psi_{xx}^*\psi\,dx-(1-i\lambda)\int_{-\infty}^{\infty}\psi^*\psi_{xx}dx \tag{14}$$

which is identical to Eq. (10). Now a static form (corresponding to an observer in suitably moving coordinate system) for the one-soliton solution of the undamped case of Eq. (12) is known to be of the form[8,15]

$$\psi(x,t)=c\left[\operatorname{sech}\frac{c}{2}x\right]\left[\exp i\frac{c}{2}x\right] \tag{15}$$

so that

$$\kappa(x,t)=c\,\operatorname{sech}\frac{c}{2}x,\quad \tau(x,t)=\frac{c}{2}\ , \tag{16}$$

where $c$ is a constant.

In the damped case we allow $c$ to be a function of time and then using Eq. (15) in (14) the appropriate expression for it can be obtained. Making use of Eq. (15) in Eq. (14) we have

$$\frac{dc}{dt} = -\tfrac{2}{3}\lambda c^3 \tag{17}$$

and so

$$c = (\sqrt{3}/2)(\lambda t + \delta)^{-1/2} , \tag{18}$$

where $\delta$ is the integration constant. Thus in the damped case, the energy and current densities are given by

$$\begin{aligned}\mathcal{E}(x,t) &= \tfrac{1}{2}\kappa^2 \\ &= (\tfrac{3}{8})(\lambda t + \delta)^{-1}\operatorname{sech}^2(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}] ,\end{aligned} \tag{19a}$$

$$\begin{aligned}\mathcal{J}(x,t) &= \kappa^2\tau \\ &= (3\sqrt{3}/16)(\lambda t + \delta)^{-3/2} \\ &\quad \times \operatorname{sech}^2(\sqrt{3}x/4)^{1/2}(\lambda t + \delta)^{-1/2}] .\end{aligned} \tag{19b}$$

The form of the energy density is depicted in Fig. 1(a) for different times, showing the damping of energy density in the system for the one-soliton solution, while Fig. 1(b) demonstrates the loss of total energy in the system.

Having known the curvature and torsion of the space curve it is possible to construct[16] the orthogonal trihedral uniquely (up to rigid motions) from known procedures in classical differential geometry. In other words, using the solutions (19) of the damped one-soliton case for the energy density and current density, the associated solutions for the spins may be obtained by noting that Eqs. (3) and (5) are equivalent to a set of two Riccati equations in terms of the Darboux vector[7,16] $z_l$:

$$z_{lx} = -i\kappa z_l + \tfrac{1}{2}i\tau(z_l^2 - 1) , \tag{20a}$$

$$\begin{aligned} z_{lt} = &-i\left[\left(\frac{\kappa_{xx}}{\kappa} - \tau^2\right) + \frac{\lambda}{\kappa}(2\kappa_x\tau + \kappa\tau_x)\right]z_l \\ &- \tfrac{1}{2}[(\kappa\tau - \lambda\kappa_x) + i(\kappa_x + \lambda\kappa\tau)]z_l^2 \\ &- \tfrac{1}{2}[(\kappa\tau - \lambda\kappa_x) - i(\kappa_x + \lambda\kappa\tau)] ,\end{aligned} \tag{20b}$$

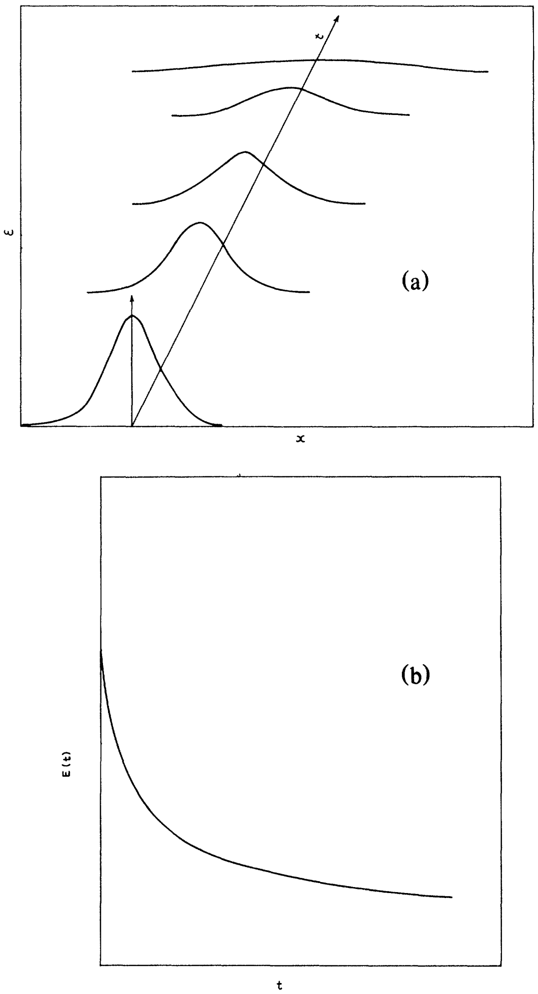


FIG. 1. (a) Damping of the static one-soliton solution for the energy density of the spin system as a function of time. (b) Dissipation of total energy $E(t)$ of the spin system.

where $z_l = (e_{1l} + ie_{2l})/(1 - e_{3l})$, $e_{1l}^2 + e_{2l}^2 + e_{3l}^2 = 1$, and $l = 1,2,3$. Solving the Riccati equations appropriately, for the assumed expressions for the curvature and torsion from (19), the expressions for the spins may be obtained as

$$\begin{aligned} S^x = \operatorname{sech}[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}]\{&\tanh[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}]\sin[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}] \\ &- \cos[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}]\} ,\end{aligned} \tag{21a}$$

$$\begin{aligned} S^y = -\operatorname{sech}[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}]\{&\tanh[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}]\cos[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}] \\ &+ \sin[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}]\} ,\end{aligned} \tag{21b}$$

$$S^z = \tanh^2[(\sqrt{3}x/4)(\lambda t + \delta)^{-1/2}] . \tag{21c}$$

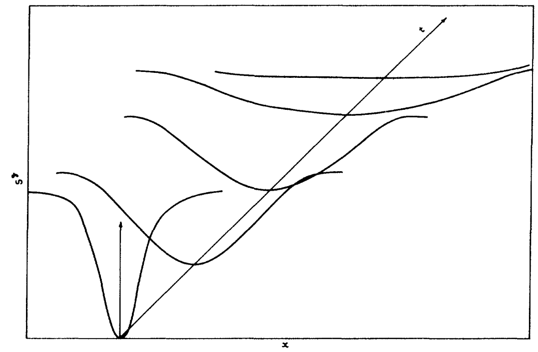

FIG. 2. Damping phenomenon of the $z$ component of the spin vector $S^z$.

The nature of the dependence of $c$ on $t$ in Eq. (19) causes the spins to loose their pulse nature. The damping aspect for the $z$ component of the spin vector is illustrated in Fig. 2. From the expression (19a) it is clear that as time passes the energy density decreases. Correspondingly, the magnitude of $S^y$ and $S^z$ components of spins start decreasing as per the Eqs. (21b) and (21c) and eventually they align parallel to the $x$ axis which may be seen from Eq. (21a). Similar analysis may be carried for multisoliton solutions, using the soliton perturbation theory.[17]

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