Structure of metastable states in a random Ising chain

Tiecheng Li*
Department of Physics, University of California, San Diego, La Jolla, California 92093
(Received 28 January 1981)

We investigate the structure of metastable states (states of local-energy minima) in a random Ising chain. It is shown that one can always achieve a unique reduction of a metastable state into a series of irreducible spin clusters. In a certain sense each irreducible spin cluster is equivalent to a two-level system. We present the exact degeneracy for a random Ising chain by a very simple argument. We obtain an analytical expression for the distribution function of the barrier heights r and the excitation energies ϵ of the clusters — $D(\epsilon,r)$. This includes contributions from all the possible spin clusters of various shapes and sizes. Even though the numerical results of this paper have been obtained for a Gaussian distribution, all the formulas of the distribution functions obtained are general, independent of the concrete form of any random continuous distribution. To illustrate how these results can be applied, we recover the "logarithmic law" for energy versus relaxation times at different temperatures, a well-known characteristic of spin-glasses.

I. INTRODUCTION

Spin-glasses have been under investigation for a long time. In spite of the progress made, many phenomena still await explanation.[1,2] The theoretical difficulties stem mainly from ambiguities inherent in the mechanism responsible for the spin-glass phase. Recently, there have been many experiments on the relaxation properties of spin-glasses at low temperatures, particularly on measurements of abnormally slow decay rates. These experiments have aroused considerable theoretical interest in the low-temperature properties of spin-glasses. One hopes to construct a simple, clear, and tractable model from which to obtain certain general analytical results free from ambiguous approximation or tedious numerical calculations. A one-dimensional (1D) model[3] does offer such advantages. While such a model lacks any real phase transition, it does help us to understand some of the important concepts; e.g., competition between ferromagnetism and antiferromagnetism, structure of spin clusters and the long-tail behavior of time-dependent phenomena, etc., which are all characteristics of the spin-glass phase. A one-dimensional model by its simple structure offers us certain exact analytical results, which form the subject of this paper.

For any equilibrium property, the contribution from metastable states can always be ignored because of its small occupation in the phase space (see Sec. II of this paper and Ref. 4). The metastable states,[5] corresponding to local minima in the energy, determine the various long time relaxation processes at low temperatures. Insight into the structure of these metastable states is needed.

It is easy to prove that the metastable state can always be resolved in a definite way into a series of irreducible spin clusters. The dynamics is adequately described by the flipping of spins of these irreducible spin clusters. This picture is very similar to that of the two-level system in the phenomenological theory of glasses.[6,7] As soon as one determines the distribution function of the barrier height r and the excitation energy ϵ of these irreducible spin clusters, $D(\epsilon,r)$, one obtains the tool for analyzing various relaxation processes. In this paper, we present an exact expression of the degeneracy and derive the distribution function $D(\epsilon,r)$ taking into consideration contributions from spin clusters of all shapes and sizes. As an example of its application, the logarithmic law in the energy relaxation process is obtained. It is well known that the logarithmic law is a peculiar time-dependent characteristic of spin-glasses.

II. STRUCTURE OF METASTABLE STATES

The Hamiltonian of the Ising model in one dimension with randomly distributed nearest-neighbor interactions is

$$\mathcal{H} = -\sum_{\langle i,j\rangle} \sigma_i \sigma_j \tilde{J}_{ij} \equiv -\sum_{\langle i,j\rangle} J_{ij} \quad . \tag{1}$$

Here the indices i and j denote the positions of the spins, the symbol $\langle i,j\rangle$ confines the summation to nearest neighbors only, σ_i is the Ising spin ($\sigma_i = \pm 1$), and $\tilde{J}_{ij}$ is the bond energy. A normalized Gaussian probability distribution for the exchange energy $\tilde{J}_{ij}$ for nearest neighbors is given by

$$P(\tilde{J}_{ij}) = \frac{1}{\sqrt{\pi}} \exp(-\tilde{J}_{ij}^2) = P(J_{ij}) \quad . \tag{2}$$

The metastable state[5] is defined as a state in which the total energy of the system cannot be decreased by flipping any single spin. One can plot the $|J_i| - i$ diagram (see Fig. 1), where $|J_i|$ is the absolute value of the bond energy and i is the position label.

In the following paragraphs we will summarize some essential characteristics of metastable states and express them in the form of lemmas for clearness.

Lemma 1. The necessary and sufficient condition for a metastable state is that any frustrated state[8] (or broken bond) may occur only in the valley of the $|J_i| - i$ plot (Fig. 1). This lemma is obvious and we will not present a proof.

We define a two-spin cluster[5] as two neighboring spins in metastable state that remain metastable after both spins are flipped. It is obvious that the necessary and sufficient condition for a two-spin cluster is that the interaction energy between these two spins be greater than the absolute values of their two nearest-neighbor bonds,

$$J_1 > |J'| \ , \tag{3}$$

and

$$J_1 > |J''| \ ,$$

i.e., a platform in Fig. 1. From Lemma 1 one can get the number of metastable states in an N-spin system, $N >> 1$, as follows. For any random continuous distribution (for example, the Gaussian distribution) the probability of a bond being a valley is $\frac{1}{3}$. Only the valleys have two states to choose from: broken or unbroken. Thus the number of metastable states in an N-spin system with a random continuous distribution is

$$f(N) \sim 2^{N/3} \quad (N >> 1) \tag{4}$$

where $f(N)$ is the number of metastable states in the N-spin system.

For a discrete random distribution with an infinite number of elements Eq. (4) is obvious whenever one

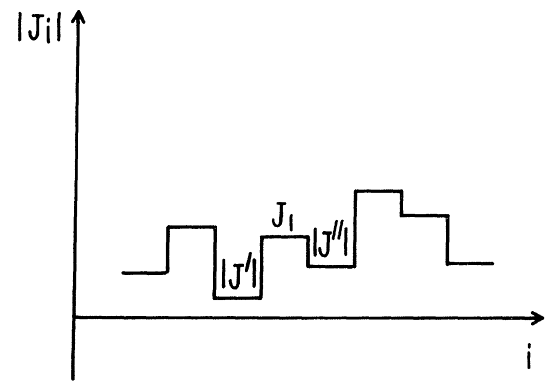

FIG. 1. $|J_i|$-i plot.

can neglect the probability of having the same bond values for neighbors. But, for a discrete random distribution with a finite number of elements, the situation becomes more complex. In this case, one has to take into account the probability of having the same bond values for neighbors. As a simple and useful example, we let the probability distribution function be given by

$$P(J) = \tfrac{1}{2}(\delta_{J,-J_0} + \delta_{J,J_0}) \ , \tag{5}$$

where J_0 is a given constant and δ the Kronecher symbol. We test the bonds one by one, e.g., from left to right. It the ith bond is a broken bond, then the $(i+1)$th one should be an unbroken bond. If the ith bond is an unbroken bond, the $(i+1)$th bond can have two choices: the broken and unbroken state. From this simple argument one has the following relation for the number of metastable states:

$$f(N+1) = f(N) + f(N-1) \ , \tag{6}$$

$f(N)$ here is the Fibonacci number. If $N >> 1$, the solution of Eq. (6) is

$$f(N) \approx 2^{\alpha N} \ , \quad \alpha = \frac{\ln(\sqrt{5}+1)}{\ln 2} - 1 \ . \tag{7}$$

Now we return to the study of the system described by Eq. (1) and Eq. (2), i.e., a random Ising chain with Gaussian probability distribution for the bond energy.

If one relaxes the definition of a metastable state, for example, to a state in which the total energy of the system cannot be decreased by flipping i spins ($i = 1, 2, \ldots, n_0$), n_0 being a given natural number, how many metastable states will the N-spin system have? Since i can take the value 1, Lemma 1 stands. It is not sufficient in this situation; one further demands that the values of any valley's $2n_0$ left and right neighboring bonds be greater than the absolute value of the valley bond. In other words, one must revise the definition of the valley bond as a minimum relative to its $2n_0$ neighbors in this situation. The probability of being a valley for any bond is

$$\begin{aligned} P_v(n_0) &= 2\int_0^\infty dJ\, p(J)\left[2\int_J^\infty p(t)\,dt\right]^{2n_0} \\ &= \frac{1}{2n_0+1}\left[2\int_0^\infty p(t)\,dt\right]^{2n_0+1} \\ &= \frac{1}{2n_0+1} \ . \end{aligned} \tag{8}$$

Using an argument similar to the above, the number of metastable states becomes

$$f_{n_0}(N) = 2^{N/(2n_0+1)} \ . \tag{9}$$

In fact, Eqs. (8) and (9) are valid for any continuous random distribution. Equations (8) and (9) are also valid for any discrete random distribution whenever one can neglect the probability of having the same bond values for neighbors. The reason is very simple, since for any bond the probability of being a minimum among randomly distributed $(2n_0+1)$ connecting bonds is just $1/2n_0+1$.

It is easy to extend the definition of a two-spin cluster to the m-spin cluster such that after flipping all of the connecting m spins in a metastable state one still gets a metastable state. We define the irreducible m-spin cluster as follows: It is an m-spin cluster, and it cannot be reduced into some sub-spin-cluster other than itself.

Lemma 2. Any spin cluster is filled with one or more irreducible spin clusters.

Another presentation of this lemma is that any spin cluster can be reduced into a set of irreducible clusters in a definite way.

$$m = m_1 + m_2 + \cdots + m_l \ . \tag{10}$$

Here m_i is the number of spins of the ith irreducible spin cluster, and l is the number of irreducible spin clusters, $1 \leq l \leq m-1$,

From Lemma 2, and Eq. (4), the average size of an irreducible cluster is three spins, a small cluster. Thus one expects that small spin clusters $(m=2,3,4)$ could play a major role in some relaxation processes. This expectation was one of the assumptions confirmed by a Monte Carlo simulation for a 2D random Ising spin system.[5] It may be valid even in real spin-glasses (Ref. 2, Binder). For magnetic relaxation the larger cluster will become important because of the larger number of spins involved.

It is clear that only the neighboring geometric relation exists among clusters, and no containing and/or overlapping geometric relation can be found for a 1D system. It would be interesting to study the topological relations among irreducible clusters in higher dimensional systems. This is not an academic subject. From Lemma 1 and Lemma 2, we can actually study spin clusters under a simpler alternative. Consider only the properties of the boundaries of the irreducible spin clusters.

We define the valley bond in Fig. 1 as a weak bond. The absolute average value of weak bonds is

$$\begin{aligned}\bar{J}_w &= \int_0^\infty J\tilde{P}_v(J)\,dJ \\ &= \int_0^\infty J_* 2p(J)\left[2\int_0^\infty p(t)\,dt\right]^2 \\ &= \frac{8}{3}\int_0^\infty \phi^3(x)\,dx = 0.2376 \ ,\end{aligned}$$

where $\tilde{P}_v(J)$ is the probability distribution of valley with bond energy J and $\phi(x) \equiv \int_x^\infty p(t)\,dt$. The distribution of the absolute values of weak bonds is shown in Fig. 2. The probability decreases very fast when $|J_i|$ increases. From Lemma 1, one can say that for the metastable state, the broken bonds may only exist on the weak bonds. From Lemma 2, the so-called irreducible spin cluster is nothing but a spin assembly surrounded by weak bonds. We think this description is meanginful even for real spin-glasses. When $m=2,3$, the spin cluster itself is an irreducible one. Since the reduction (into irreducible clusters) is definite and time independent, according to Lemma 2 (see Fig. 1) there is no transition between clusters. This leads to great simplification for dynamical problems, i.e., one can take these irreducible clusters as an assembly of two-level systems in a certain sense, and statistics on this assembly becomes the only information one needs.

For the definition of the barrier height r and the excited energy ϵ, see Ref. 5. We will not reproduce it here.

Lemma 3. The bond values J_i in the irreducible spin cluster are positive

$$J_i > 0 \ , \tag{11}$$

and if one arranges them in order, there is no minimum in the J_i-i plot; the number of maxima, should they exist, are equal to one or zero according to L'Hôpital's rule.

From Lemmas 2 and 3 the necessary and sufficient condition for an m-spin irreducible cluster is as follows:

$$J_1 > |J'| \ , \quad J_{m-1} > |J''| \ , \quad J_i > 0 \ , \tag{12}$$

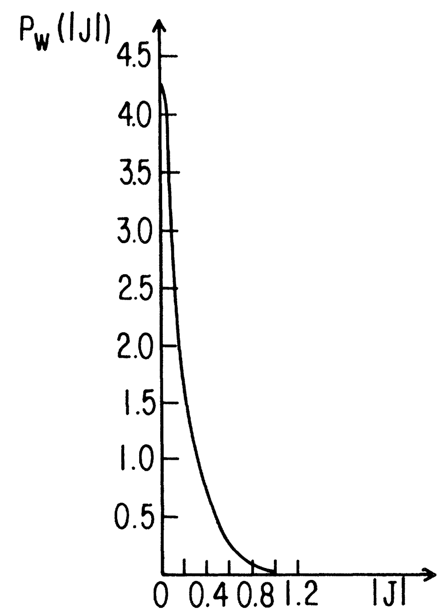

FIG. 2. The distribution of the absolute values of weak bonds.

and there be no minimum in the $J_1 - i$ plot.

For a given bond energy J_1, the probability distribution for two-spin clusters is

$$\tilde{P}^{(2)}(J_1) = N_2 p(J_1)\left[\int_{-J_1}^{J_1} dx\, p(x) \int_{|x|}^{\infty} p(y)\, dy\right]^2$$

$$\equiv N_2 p(J_1)\psi^2(J_1) \ , \tag{13}$$

where

$$\psi(x) \equiv [\mathrm{erfc}^2(0) - \mathrm{erfc}^2(x)]/2 \ , \tag{14}$$

with the complementary error function

$$erf(x) \equiv 2\phi(x) = 2\int_x^{\infty} p(t)\, dt, \quad \mathrm{erf}(0) = 1 \ . \tag{15}$$

And N_2 is a normalization constant. The average bond value for two-spin clusters is

$$\bar{J}^{(2)} = \int x\tilde{P}^{(2)}(x)\, dx = \frac{\int_0^{\infty} xp(x)\psi^2(x)\, dx}{\int_0^{\infty} p(x)\psi^2(x)\, dx} = 0.8192 \ . \tag{16}$$

The average bond value for three-spin clusters is

$$\bar{J}^{(3)} = \frac{1}{2}\int (x+y)\tilde{P}^{(3)}(x,y)\, dx\, dy = \frac{\int_0^{\infty} xp(x)\psi(x)\, dx}{\int_0^{\infty} p(x)\psi(x)\, dx} = 0.7283 \ , \tag{17}$$

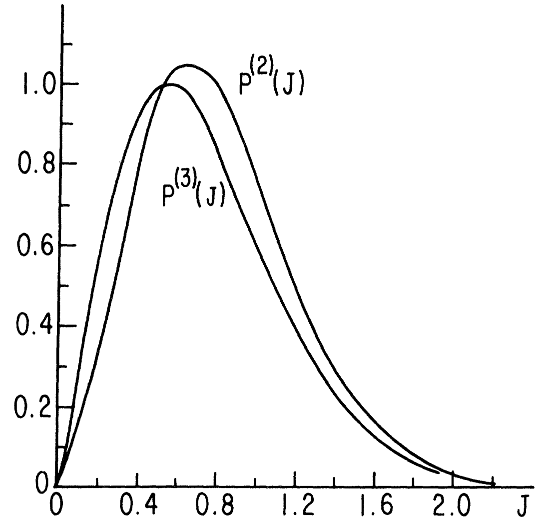

FIG. 3. The probability distribution of bond values for two- and three-spin clusters.

where $\tilde{P}^{(3)}(x,y)$ is the probability distribution for three-spin clusters with bond energy values of x and y. The probability distribution of bond values for two- and three-spin clusters is shown in Fig. 3.

For the given bond values of $J_1, J_2, \ldots, J_{m-1}$, the probability distribution of m-spin irreducible clusters is

$$\tilde{P}^{(m)}(J',J_1,J_2,\ldots,J_{m-1},J'') \sim p(J')p(J'')\Theta(J_1-|J'|)\Theta(J_{m-1}-|J''|)\phi(J')\phi(J'')*2^{m+1}\prod_{i=1}^{m-1}p(J_i) \ , \tag{18}$$

where the Lemma 3 must be valid for these $(m-1)$ bonds.

Lemma 4. Flipping in order (from left to right or inversely) is the only way in which the irreducible spin cluster experiences the lowest barrier.

This is the preferred flipping method in practice. It is easy to check out this lemma. According to the above lemma, one can write the barrier height $r(>0)$ and the excitation energy $\epsilon(>0)$ of the irreducible spin cluster as

$$r = 2(J_{\max} - |J',J''|_{\max}) \ , \tag{19}$$

and

$$\epsilon = 2|J'+J''| \ , \tag{20}$$

where $J_{\max}$ is the largest bond value among $(m-1)$ bonds, and the symbol $|J',J''|_{\max}$ is the value of the greater of $|J'|$ and $|J''|$. The distribution function of the barrier height and the excitation energy for a m-spin irreducible cluster is

$$D^{(m)}(\epsilon,r) \sim \int \delta(\epsilon - 2|J'+J''|)\delta(r - 2(J_{\max} - |J',J''|_{\max})\, \tilde{P}^{(m)}(J',J_1,J_2,\ldots,J_{m-1},J'')\, dJ'\, dJ''\, dJ_1, \ldots, dJ_{m-1} \ . \tag{21}$$

From Eqs. (19) and (20),

$$\begin{aligned} D^{(m)}(\epsilon,r) \sim \int \tilde{P}^{(m)}(J,J_1,J_2,\ldots,J'')[&\Theta(J'-|J''|)\delta(\epsilon-2(J'+J''))\delta(r-2(J_{\max}-J')) \\ &+\Theta(J''-|J'|)\delta(\epsilon-2(J'+J''))\delta(r-2(J_{\max}-J'')) \\ &+\Theta(-J'-|J''|)\delta(\epsilon+2(J'+J''))\delta(r-2(J_{\max}-J'')) \\ &+\Theta(-J''-|J'|)\delta(\epsilon+2(J'+J''))\delta(r-2(J_{\max}+J''))]\, dJ'\, dJ'' \prod_{l=1}^{m-1} dJ_l \ . \end{aligned} \tag{22}$$

Here Eq. (12) should be fulfilled. Using the following symmetries,

$$\tilde{P}^{(m)}(J',J_1,\ldots,J_{m-1},J'')=\tilde{P}^{(m)}(J'',J_{m-1},\ldots,J_1,J') \tag{23}$$

and

$$\tilde{P}^{(m)}(J',J_1,\ldots,J_{m-1},J'')=\tilde{P}^{(m)}(-J',J_1,\ldots,J_{m-1},-J'') \ , \tag{24}$$

Eq. (22) becomes

$$D^{(m)}(\epsilon,r)\sim 4\int \tilde{P}^{(m)}(J',J_1,J_2,\ldots,J_{m-1},J'')\Theta(J'-|J''|)\delta(\epsilon-2(J'+J''))\delta(r-2(J_{\max}-J'))\,dJ'dJ''\prod_{l=1}^{m-1}dJ_l \ . \tag{22$'$}$$

Suppose $J_{\max}=J_i$, $1\leq i\leq m-1$, one can rewrite Eq. (22′) as follows:

$$\begin{aligned}D^{(m)}(\epsilon,r)\sim 4\int\int dJ'dJ''\Theta(J'-|J''|)\delta(\epsilon-2(J'+J''))2\phi(|J'|)2\phi(|J''|)2p(J')2p(J'')\\ \times\int_{|J'|}^{\infty}dJ_i\delta(r-2(J_i-|J'|))2p(J_i)\left[\int_{|J'|}^{J_i}dJ_{i-1}\int_{|J'|}^{J_{i-1}}dJ_{i-2}\cdots\int_{|J'|}^{J_2}dJ_1\prod_{k=1}^{i-1}2p(J_k)\right]\\ \times\left[\int_{|J''|}^{J_i}dJ_{i+1}\cdots\int_{|J''|}^{J_{m-2}}dJ_{m-1}\prod_{l=i+1}^{m-1}2p(J_l)\right] \ .\end{aligned} \tag{22$''$}$$

From Lemma 2, one must perform statistical analysis over the assembly consisting of all irreducible clusters with various shapes and sizes. Thus the distribution function is

$$\begin{aligned}D(\epsilon,r)\sim\sum_{m=2}^{\infty}\sum_{i=1}^{m-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dJ'dJ''\Theta(J'-|J''|)\delta(\epsilon-2(J'+J''))\,\phi(|J'|)\phi(|J''|)p(J')p(J'')\\ \times\int_{|J'|}^{\infty}dJ_ip(J_i)\delta(r-2(J_i-J'))\\ \times\left[\int_{J'}^{J_i}dJ_{i-1}\int_{J'}^{J_{i-1}}dJ_{i-2}\cdots\int_{J'}^{J_2}dJ_1\prod_{k=1}^{i-1}2p(J_k)\right]\\ \times\left[\int_{|J''|}^{J_i}dJ_{i+1}\cdots\int_{|J''|}^{J_{m-2}}dJ_{m-1}\prod_{l=1}^{m-1}2p(J_l)\right]\\ \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dJ'dJ''\Theta(J'-|J''|)\delta(\epsilon-2(J'+J''))\phi(|J'|)\phi(|J''|)p(J')p(J'')\\ \times\int_{J'}^{\infty}dxp(x)\delta(r-2(x-J'))\left[\sum_{m=0}^{\infty}u_m(J',x)\right]\left[\sum_{m=0}^{\infty}u_m(|J''|,x)\right] \ ,\end{aligned} \tag{25}$$

where

$$\begin{cases}u_0\equiv 1,\\ u_m(x,y)\equiv\int_x^y dx_1\int_x^{x_1}dx_2\cdots\int_x^{x_{m-1}}dx_m\prod_{l=1}^{m}2p(x_l),m=1,2,\ldots,\infty \ .\end{cases} \tag{26}$$

Defining

$$v_m(x,y)\equiv\sum_{k=0}^{m}u_k(x,y) \ , \tag{27}$$

we get the following integral equation chain:

$$v_m(x,y)=1+2\int_x^y dzp(z)v_{m-1}(x,z),m=1,2,\ldots,\infty \ . \tag{28}$$

If $v_m(x,y)$ is convergent,

$$\lim_{m\to\infty}v_m(x,y)=v(x,y) \tag{29}$$

from Eqs. (27) and (28), we have

$$v(x,y)=1+2\int_x^y dzp(z)v(x,z) \ . \tag{30}$$

Alternatively, one can write the corresponding differential equations

$$\frac{\partial v(x,y)}{\partial y}=2p(y)v(x,y) \ , \tag{31}$$

$$\frac{\partial v(x,y)}{\partial x}=-2p(x)v(x,y) \ , \tag{32}$$

and the boundary condition

$$v(x,x)=1 \ . \tag{33}$$

The solution of Eqs. (31)–(33) is

$$v(x,y)=\exp\left[2\int_x^y p(z)\,dz\right] . \tag{34}$$

In fact from the definition of $v(x,y)$, Eqs. (27) and (29), one can write down the expression for the function $v(x,y)$ directly,

$$\begin{aligned} v(x,y) &= \sum_{m=0}^{\infty}\int_x^y dx_1\int_x^{x_1} dx_2\cdots\int_x^{x_{m-1}} dx_m \prod_{l=1}^{m} 2p(J_l) \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\int_x^y dx_1\int_x^y dx_2\cdots\int_x^y dx_m \prod_{l=1}^{m} 2p(J_l)=\exp\int_x^y 2p(z)\,dz . \end{aligned}$$

The function $v(x,y)$ has the same form as the U matrix in quantum-mechanical perturbation theory, and thus the same properties such as the group character

$$v(x,z)\,v(z,y)=v(x,y) \tag{35}$$

and the unitary property — Eq. (33). But here all of the quantities are real and classical, and there is no problem with commutation, i.e., they are simpler than the U matrix in quantum mechanics. From Eqs. (27)–(35), we have

$$D(\epsilon,r)=\frac{1}{N_D}\int_{-\infty}^{\epsilon/4} dx\,\phi(|x|)p(x)\phi\left[\frac{\epsilon}{2}-x\right]p\left[\frac{\epsilon}{2}-x\right]p\left[\frac{\epsilon+r}{2}-x\right]v\left[\frac{\epsilon}{2}-x,\frac{\epsilon+r}{2}-x\right]v\left[|x|,\frac{\epsilon+r}{2}-x\right] , \tag{36}$$

where N_D is a normalization constant. From the above expression for $D(\epsilon,r)$, the strong correlation between the barrier height (r) distribution and the excitation energy (ϵ) distribution is obvious. From Eqs. (31)–(34), after several straightforward algebraic steps, we get the distribution function of the excited energy $\epsilon(>0)$

$$E(\epsilon)\sim\int_0^{\infty} D(\epsilon,r)\,dr ;$$

thus,

$$E(\epsilon)=\frac{1}{N_E}\int_{-\infty}^{\epsilon/4} dx\,\phi(|x|)p(x)\phi\left[\frac{\epsilon}{2}-x\right]p\left[\frac{\epsilon}{2}-x\right]v(|x|,\infty)\left[v\left[\frac{\epsilon}{2}-x,\infty\right]-v\left[\infty,\frac{\epsilon}{2}-x\right]\right] . \tag{37}$$

Here N_E is a normalization constant. From Eqs. (25) and (34), the distribution function for the barrier height r is found

$$\begin{aligned} R(r)\sim\int_0^{\infty} D(\epsilon,r)\,d\epsilon &\sim\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dJ'dJ''\Theta(J'-|J''|)\phi(|J'|)\phi(|J''|) \\ &\qquad\times p(J')p(J'')p\left[\frac{r}{2}+J'\right]v\left[J',\frac{r}{2}+J'\right]v\left[|J''|,\frac{r}{2}+J'\right] . \end{aligned} \tag{38}$$

The integrand is an even function of J''. One thus has

$$R(r)\sim\int_0^{\infty} dJ'p(J')\phi(J')p\left[\frac{r}{2}+J'\right]v\left[J',\frac{r}{2}+J'\right]\int_0^{J'} dJ''p(J'')\phi(J'')v\left[J'',\frac{r}{2}+J'\right] .$$

For the integral over J'', using Eqs. (31)–(33),

$$\begin{aligned} \int_0^{J'} dJ''\phi(J'')p(J'')v\left[J'',\frac{r}{2}+J'\right] &= -\frac{1}{2}\int_0^{J'}\phi(J'')\,dv_{J''}\left[J'',\frac{r}{2}+J'\right] \\ &= \frac{1}{2}\phi(J'')v\left[J'',\frac{r}{2}+J'\right]\Bigg|_{J''=J'}^{0}-\frac{1}{2}\int_0^{J'} dJ''v\left[J'',\frac{r}{2}+J'\right]p(J'') \\ &= \frac{1}{2}\left[\phi(J'')v\left[J'',\frac{r}{2}+J'\right]-v\left[J'',\frac{r}{2}+J'\right]\right]\Bigg|_{J''=J'}^{0} . \end{aligned}$$

So

$$R(r)=\frac{1}{2N_R}\int_0^\infty dJ'\phi(J')p(J')p(\tfrac{1}{2}r+J')v^2(J',\tfrac{1}{2}r+J')[\phi(0)-\phi(J')]\ , \tag{39}$$

where N_R is a normalization constant. If we expand the exponential function $v(x,y)$ in Eq. (36), and carry out similar manipulations for $R(r)$ and $E(\epsilon)$, from the leading term of expansion we get the distribution function of the barrier height r and that of the excitation energy ϵ for two-spin clusters

$$R^{(2)}(r)=\frac{1}{2N_R}\int_0^\infty dJ\phi(J)p(J)p(\tfrac{1}{2}r+J)[\phi^2(0)-\phi^2(j)]\ , \tag{40}$$

and

$$E^{(2)}(\epsilon)=\frac{4}{N_E}\int_{-\infty}^{\epsilon/4} dx\,\phi(|x|)p(x)\phi^2(\tfrac{1}{2}\epsilon-x)p(\tfrac{1}{2}\epsilon-x)\ . \tag{41}$$

From the second term in the expansion we get the distribution function for three-spin clusters

$$\begin{aligned}R^{(3)}(r)&=\frac{1}{N_R}\int_0^\infty dJ\phi(J)p(J)p(\tfrac{1}{2}r+J)\{\tfrac{2}{3}[\phi^2(0)-\phi^3(J)]+[\phi^2(0)-\phi^2(J)][\phi(J)-2\phi(\tfrac{1}{2}r+J)]\}\\&=\frac{1}{N_R}\int_0^\infty dJ\phi(J)p(J)p(\tfrac{1}{2}r+J)[\tfrac{1}{12}+\tfrac{1}{6}\phi(J)-\tfrac{1}{2}\phi(\tfrac{1}{2}r+J)\\&\qquad+\tfrac{1}{2}\phi(J)\phi(\tfrac{1}{2}r+J)+\phi^2(J)\phi(\tfrac{1}{2}r+J)-\tfrac{4}{3}\phi^3(J)]\end{aligned} \tag{42}$$

and

$$E^{(3)}(\epsilon)=\frac{4}{N_E}\int_{-\infty}^{\epsilon/4} dx\,\phi^2(|x|)p(x)\phi^2(\tfrac{1}{2}\epsilon-x)p(\tfrac{1}{2}\epsilon-x)\ . \tag{43}$$

Numerical results of these distributions are shown in Figs. 4 and 5.

A feature worth noting in Fig. 4 is the existence of the nonmonotonic behavior. The average excited energies are

$$\bar{\epsilon}\equiv\int_0^\infty \epsilon E(\epsilon)\,d\epsilon=0.6772\ ,$$

$$\bar{\epsilon}^{(2)}\equiv\int_0^\infty \epsilon E^{(2)}(\epsilon)\,d\epsilon=0.7220\ ,$$

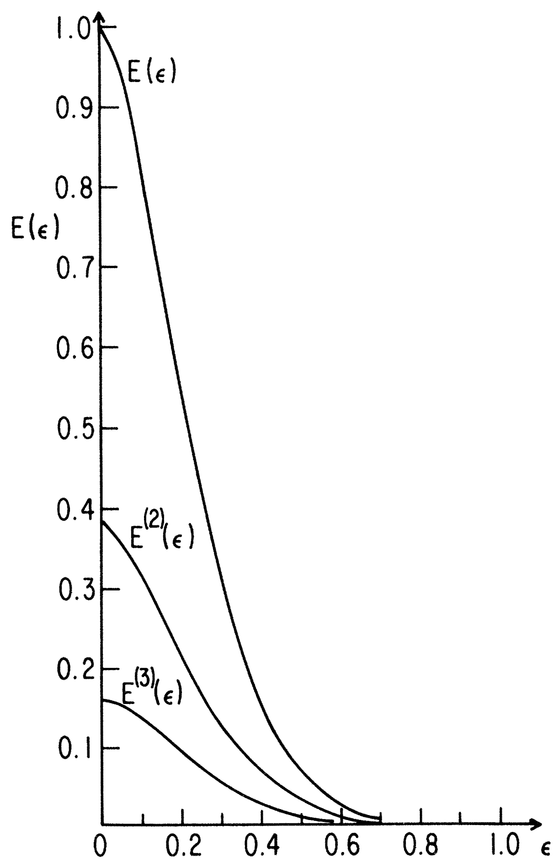

FIG. 4. The distribution of the excited energies of irreducible spin clusters.

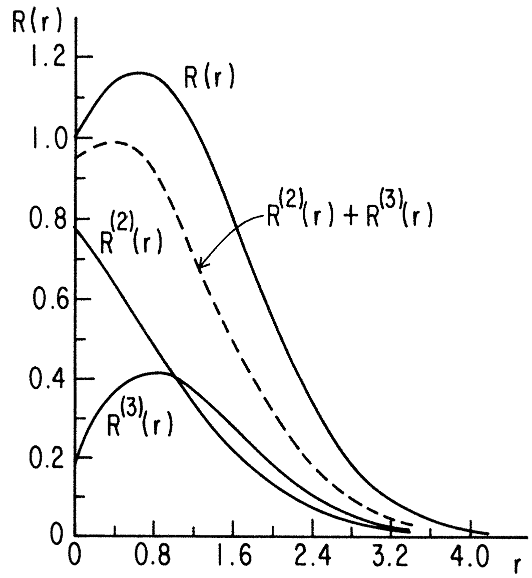

FIG. 5. The distribution of the barrier heights of irreducible spin clusters.

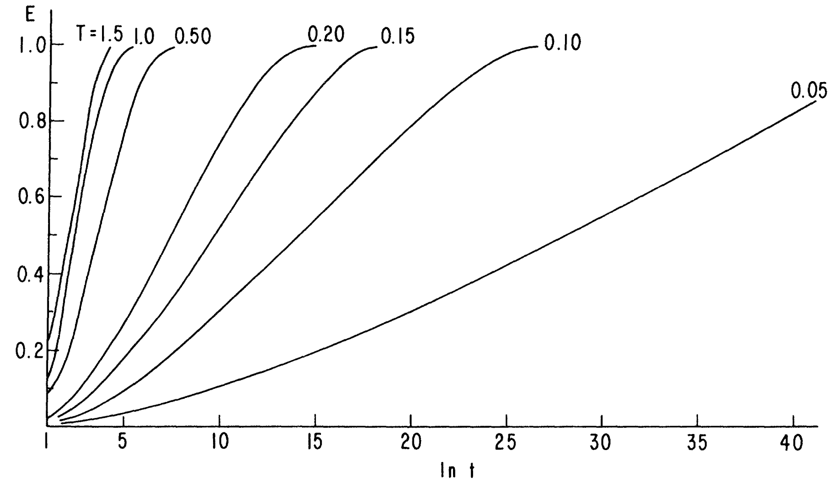

FIG. 6. The energy relaxation for different temperatures.

and

$$\bar{\epsilon}^{(3)} \equiv \int_0^\infty \epsilon E^{(3)}(\epsilon)\, d\epsilon = 0.6840 \ .$$

The effective temperature for overcoming the barrier heights are

$$\bar{T} \equiv \int_0^\infty r R(r)\, dr = 1.142 \ ,$$

$$\bar{T}^{(2)} \equiv \int_0^\infty r R^{(2)}(r)\, dr = 0.8378 \ ,$$

and

$$\bar{T}^{(3)} \equiv \int_0^\infty r R^{(3)}(r)\, dr = 1.199 \ .$$

The distribution function $D(\epsilon,r)$ can be used as input to kinetic equations for calculating various properties. It is able to reproduce a wealth of results observed in Monte Carlo calculations. We will do it elsewhere. Here as a simple example of application of the distribution function $D(\epsilon,r)$, we let all the irreducible states be at the lower state of the two-level system at $t=0$. Regard the spins overcoming the barrier heights as a rate process of relaxation. By adopting Boltzmann statistics at different temperatures, the results for this energy relaxation process for different temperatures are shown in Fig. 6. A nearly logarithmic time dependence is obtained.

In Figs. 4 and 5, note that even though the numerical results have been obtained for a Gaussian distribution, all the distribution functions obtained are general, independent of the actual form of the random distribution. Since the distribution function $D(\epsilon,r)$ includes contributions from all the possible spin clusters of various shapes and sizes, the scale of temperature and time is larger than that of previous work. One can consider the probable events leading to climbing over higher barriers and visiting distant regions of the phase space. In all of the previous practical Monte Carlo calculations for higher dimensionality ($d \geq 2$), only very small spin clusters (the number of spins in the cluster being 2 or 3)[5] were involved.

Similar conclusions of this paper would be very attractive for higher dimensions. Of course, exact description and proof of the corresponding lemmas will not be easy to state in those cases.

ACKNOWLEDGMENTS

I would like to thank Professor Shang-keng Ma for some valuable and stimulating discussions that led me to study this problem. I am also grateful to Professor Chia-Wei Woo for his support and for reading this manuscript. Thanks are also due to Professor S. F. Edwards and Dr. F. Tanaka for sending me reports of their work prior to publication. This work was supported primarily by the Institute of Physics of the Chinese Academy of Sciences. Partial support by the U.S. National Science Foundation while the author was a visitor to the University of California, San Diego through a grant (No. DMR80-08816) is also acknowledged.

*Permanent address: Institute of Physics, Chinese Academy of Sciences, China.

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