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Nonlinear reversible hydrodynamics of the superfluid phases of ^{3}He

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The nonlinear reversible hydrodynamic equations for the superfluid phases of ^{3}He, i.e., ^{3}He-*A*, ^{3}He-A_1, ^{3}He-*B*, ^{3}He-*A* in high magnetic fields, and ^{3}He-*B* in high magnetic fields, are derived. The Mermin-Ho–type relations for ^{3}He-*A* in high magnetic fields and ^{3}He-*B* in high magnetic fields are given for the first time. The influence of higher-order gradient terms in all phases is discussed, and a new class of nonlinear terms containing the various kinds of velocities is given which have not been considered so far for any of the five superfluid phases. In addition we show that the hydrodynamic equations for ^{3}He-*A* in high magnetic fields contain as a special case the hydrodynamics of superfluid ^{3}He-A_1 and the orbit part of the hydrodynamic equations for ^{3}He without external field. Furthermore, we point out some structural similarities in the equations for ^{3}He-*A* in high magnetic fields and ^{3}He-*B* in high magnetic fields. As an additional effect we find that the higher-order gradient terms imply a preferred direction in the hydrodynamic equations for superfluid ^{3}He-*B*.

I. INTRODUCTION

In the last few years the study of the properties of the superfluid phases of ^{3}He has become, from the theoretical as well as from the experimental point of view, one of the most fascinating fields of low temperature physics.[1,2]

In the present paper we study in some detail the nonlinear reversible hydrodynamic equations for all superfluid phases of ^{3}He. The hydrodynamic theory, which is rigorous in the limit of small wave vectors (much smaller than any microscopic wave vector of the system) and sufficiently small frequencies (i.e. much smaller than any microscopic frequency), has been applied to many systems of condensed matter systems. In the linear domain the phenomenological hydrodynamic equations have been given, e.g., for crystals,[3,4] nematic liquid crystals,[5–7] cholesteric liquid crystals,[3,8,9] uniaxial discotic liquid crystals,[10,11] smectic-*A* and -*C* liquid crystals,[3] and spin-glasses[12,13]; the method using correlation functions has been applied to paramagnets and simple fluids,[14] superfluid ^{4}He,[15] and nematic liquid crystals.[16–18]

In the nonlinear domain, hydrodynamic equations have been presented, e.g., for superfluid ^{4}He,[19] superfluid solids,[20,21] magnetic systems,[22] and various types of liquid crystals.[6,7,23,24]

For the superfluid phases of ^{3}He the linearized phenomenological equations have been presented in Refs. 25–35 and the formulation in the framework of correlation functions has been given in Refs. 33–36. Nonlinear theories of the *A* phase have been considered by Hu and Saslow,[37] Lhuillier,[38] and Hall and Hook.[39] Corresponding work for the *B* phase is due to Liu and Cross[40]; and for the A_1 phase, due to Liu.[41]

For the *B* phase in high magnetic fields there seems to exist no nonlinear hydrodynamic theory, and for the *A* phase in high fields the authors know only of a preprint by Saslow and Hu dealing with the macroscopic dynamics incorporating explicitly nonhydrodynamic variables. Therefore it will be the aim of the present paper to derive the nonlinear hydrodynamic equations for the *A* phase in high magnetic fields and the *B* phase in high fields.

In addition to the appropriate linear equations[33–35] we give the generalized Mermin-Ho relations and introduce nonlinear terms on the static as well as on the reversible dynamic level. These nonlinear terms come roughly speaking in three classes: there are terms characterizing nonuniform textures in spin and orbit space; there are higher-order gradient terms including gradients of the conserved quantities; and there are terms quadratic in the various velocities present in the superfluid phases of ^{3}He in high magnetic fields. The latter kind of nonlinearities is also introduced in the nonlinear theories of ^{3}He-*A*, and -*B* (without fields) and of ^{3}He-A_1 thereby amending previous theories.

We will concentrate especially on a rigorous hydrodynamic approach to the problem which is not confined, e.g., to the Ginzburg-Landau regime and we leave open the microscopic calculation of the parameters occurring in our equations to future investigations (which are now in progress for some of the novel terms).

We restrict ourselves to the reversible part of the dynamic equation for practical reasons. Since the condition of positivity of entropy production is much less restrictive than that of vanishing entropy production, a systematic and complete nonlinear theory would bring about a host of irreversible terms with lots of phenomenological parameters.

The paper is organized as follows: In Sec. II we outline the general hydrodynamic properties of the superfluid phases. In Sec. III we give explicitly the hydrodynamic equations for ^{3}He-A in high magnetic fields. These results contain, as special cases, the hydrodynamic equations for ^{3}He-A_1, for ^{3}He-B in high magnetic fields, and (to a great extent) for ^{3}He-A without a magnetic field. Explicit formulas for the latter phases (and ^{3}He-B without magnetic fields) are given in Ref. 42. In Sec. IV the similarities in the structure of the hydrodynamic equations of the various phases of superfluid ^{3}He are examined. In Sec. V we give the conclusions summarizing the results which seem to be most important and discussing which of the nonlinearities will be the ones most easily accessible to experiments.

II. GENERAL PROPERTIES OF THE NONLINEAR HYDRODYNAMIC EQUATIONS OF SUPERFLUID ^{3}He

When the hydrodynamic equations for a special system under consideration are formulated, the first problem is the determination of the hydrodynamic variables. For the conserved quantities one has in the superfluid phases of ^{3}He, as in any hydrodynamic system, the density ρ, the density, $\vec{g}$, of linear momentum, the energy density ϵ (or the entropy density σ), and the magnetization density $\vec{M}$. In addition to these conserved quantities one must consider the variables characterizing the spontaneously broken symmetries. In all superfluid phases of ^{3}He one has broken gauge symmetry which leads to the phase deviation $\delta\varphi$ as a hydrodynamic variable. In addition to the broken gauge symmetry one has to face broken rotational symmetry (in the A phase, the A_1 phase, and the A phase in high magnetic fields) which gives rise to deviations, $\delta\hat{l}$, from the preferred direction $\hat{l}$ in real space. Furthermore one has to include variables which characterize spontaneously broken rotational symmetries in spin space. As will be discussed in more detail below one finds two additional variables in the A phase without external field, one additional variable in the A phase in high external magnetic field and in the B phase in high magnetic fields. For the B phase without external field one must keep three variables which characterize the spontaneously broken spin-orbit symmetry of this phase.

It should be kept in mind, however, that by the complicated structure of the order parameter in superfluid ^{3}He these variables are connected not only with one broken symmetry (by which they are defined) but with two or three more, e.g., the phase deviation $\delta\varphi$ characterizes partially broken rotational symmetry in orbit space and (for some phases) in spin space, too. We will discuss this point in detail below.

The starting point of the hydrodynamic description is the Gibbs relation. As usual, local thermodynamic equilibrium is assumed to hold.

If one chooses the energy density ϵ as a convenient thermodynamic potential, one finds

$$\epsilon V = E = E(V, \rho V, g_i V, M_i V, \sigma V, V\rho\nabla_i\phi^\alpha, V\rho\phi^\alpha, V\nabla_i\rho, V\nabla_i\sigma, V\nabla_i M_j) \ , \tag{2.1}$$

where V is the volume, ϕ^α are the variables characterizing the spontaneously broken symmetries. We have assumed $\nabla_i\phi^\alpha$ to be an intensive variable (i.e., a mass density). As usual for nonlinear theories the energy depends not only on gradients of the variables characterizing broken symmetries but on the variables themselves. The latter dependence has to vanish, however, in the homogeneous limit. As an example we mention the $\hat{l}$ dependence of ^{3}He-A (textures in orbital space).

Furthermore, it should be stressed that we have kept in Eq. (2.1) the gradients of the conserved quantities. The necessity to do this has become clear in connection with the so-called gauge wheel effect[43–46] and has been devised in a general framework in a recent paper by Combescot.[47] In the following we will count the order of our hydrodynamic contributions in the way proposed by Combescot, i.e., conserved variables and gradients of the variables characterizing broken symmetries will be considered to be first order whereas the gradients of the conserved variables will be treated as second-order quantities. We will take into account in the free-energy terms up to the third order and in the current terms up to the second order. While doing this, one should always keep in mind that it is necessary to keep fourth-order terms in the free energy to preserve positivity of the free energy, a fact which seems to have been overlooked so far. These fourth-order terms will, however, never occur in the final hydrodynamic equations, because they would be inconsistent there (cf. Ref. 23 for a discussion of an analogous situation in liquid crystals).

After these preliminaries, we are prepared to draw the first conclusions from Eq. (2.1). We obtain via Euler's relation for bulk hydrodynamics the Gibbs relation

$$d\epsilon=\mu\, d\rho+T\, d\sigma+\vec{v}^n\cdot d\vec{g}+\vec{H}\cdot d\vec{M}+\Pi_i^\alpha\, d\nabla_i\phi^\alpha+\Gamma^\alpha\, d\phi^\alpha+\tau_i\, d\nabla_i\rho+\theta_i\, d\nabla_i\sigma+\Lambda_{ij}\, d\nabla_i M_j \tag{2.2}$$

and

$$p=-\epsilon+\mu\rho+T\sigma+\vec{v}^n\cdot\vec{g}+\vec{H}\cdot\vec{M}+\tau_i\nabla_i\rho+\theta_i\nabla_i\sigma+\Lambda_{ij}\nabla_i M_j\ , \tag{2.3}$$

where $d\ldots$ refers as usual to distinct points in space *and* time. The thermodynamic conjugates are given by

$$\mu\equiv\left.\frac{\partial\epsilon}{\partial\rho}\right|_{\ldots},\quad v_i^n\equiv\left.\frac{\partial\epsilon}{\partial g_i}\right|_{\ldots},\quad \theta_i\equiv\left.\frac{\partial\epsilon}{\partial\nabla_i\sigma}\right|_{\ldots},$$

$$T\equiv\left.\frac{\partial\epsilon}{\partial\sigma}\right|_{\ldots},\quad H_i\equiv\left.\frac{\partial\epsilon}{\partial M_i}\right|_{\ldots},\quad \Lambda_{ij}\equiv\left.\frac{\partial\epsilon}{\partial\nabla_i M_j}\right|_{\ldots},$$

$$P\equiv-\frac{\partial E}{\partial V},\quad \tau_i\equiv\left.\frac{\partial\epsilon}{\partial\nabla_i\rho}\right|_{\ldots},\quad \Pi_i^\alpha=\left.\frac{\partial\epsilon}{\partial\nabla_i\phi^\alpha}\right|_{\ldots},$$

$$\Gamma^\alpha\equiv\left.\frac{\partial\epsilon}{\partial\phi^\alpha}\right|_{\ldots}, \tag{2.4}$$

where the ellipses in Eqs. (2.4) mean that during the differentiation with respect to the variable indicated all other variables are kept fixed.

For the equations of motion we have for the conserved quantities

$$\partial\rho/\partial t+\vec{\nabla}\cdot(\rho\vec{v}^n+\vec{g}')=0\ , \tag{2.5}$$

$$\partial\sigma/\partial t+\vec{\nabla}\cdot(\sigma\vec{v}^n+\vec{q}/T)=R/T\ , \tag{2.6}$$

$$\partial g_i/\partial t+\nabla_j(g_i v_j^n+p\delta_{ij}+\sigma_{ij})=0\ , \tag{2.7}$$

$$\partial\epsilon/\partial t+\vec{\nabla}\cdot[(\epsilon+p)\vec{v}^n+\vec{j}^\epsilon]+\nabla_j(\sigma_{ij}v_i^n)=0\ , \tag{2.8}$$

$$\partial M_i/\partial t+\nabla_j(M_i v_j+j_{ij}^M)=\gamma(\vec{H}_{\rm ext}\times\vec{M})_i\ , \tag{2.9}$$

and for the variables characterizing the broken symmetries there are the quasiconservation laws

$$\partial\phi^\alpha/\partial t+v_i\nabla_i\phi^\alpha+Y^\alpha=0\ . \tag{2.10}$$

In the most general case the currents q_i, σ_{ij}, j_i^ϵ, and Y^α consist of a reversible as well as of an irreversible part, whereas the dissipation function R is purely irreversible. In the present paper we concentrate on the nonlinear reversible hydrodynamic equations. Therefore one should add the irreversible terms from the linear theory (which can be made nonlinear in the usual trivial manner; e.g., $\hat{l}^0\rightarrow\hat{l}$) and then one has to add, for a specific experimental configuration, the other nonlinear irreversible terms which seem to be most important.

The explicit structure of q_i, σ_{ij}, g_i', Y^α, and j_{ik}^M is, of course, different for each superfluid phase and therefore the currents will be examined in detail in the following sections reflecting the different symmetries of all phases. Nevertheless it is possible to give constraints which have to be satisfied by the currents. From the Gibbs relation (2.2) and the conservation and quasiconservation laws (2.5)–(2.9), we have

$$-R\tilde{T}/T=\nabla_i Q_i-\tilde{\mu}\vec{\nabla}\cdot\vec{g}'-\Sigma_{ik}\nabla_i v_k-\tilde{H}_i\nabla_k j_{ki}^M+(q_i/T)\nabla_i\tilde{T}-\tilde{\Gamma}^\alpha Y^\alpha\ , \tag{2.11}$$

where

$$Q_i=j_i^\epsilon-(\tau_i\vec{\nabla}\cdot\vec{g}'+\theta_i\sigma\vec{\nabla}\cdot\vec{v}^n+\tau_i\rho\vec{\nabla}\cdot\vec{v}^n+\Lambda_{ij}M_j\vec{\nabla}\cdot\vec{v}^n+\Lambda_{ij}\nabla_k j_{kj}^M+q_i\tilde{T}/T-\theta_i R/T)\ , \tag{2.12}$$

$$\Sigma_{ik}=-\sigma_{ik}-\delta_{ik}(\nabla_j\rho\tau_j+\nabla_j\sigma\theta_j+\nabla_m M_j\Lambda_{mj})+\Pi_i^\alpha\nabla_k\phi^\alpha+\tau_i\nabla_k\rho+\theta_i\nabla_k\sigma+\Lambda_{ij}\nabla_k M_j\ , \tag{2.13}$$

$$\begin{aligned}\tilde{\Gamma}^\alpha&=\Gamma^\alpha-\nabla_i\Pi_i^\alpha-\Xi^\alpha\ ,\\ \tilde{\mu}&=\mu-\nabla_i\tau_i\ ,\\ \tilde{H}_i&=H_i-\nabla_j\Lambda_{ij}\ ,\\ \tilde{T}&=T-\nabla_i\theta_i\ ,\end{aligned} \tag{2.14}$$

where Ξ^α denotes the terms which come from the eventual anholonomity of the variables characterizing the broken symmetries. An example for this behavior is the phase deviation in the A phase. In addition we would like to stress that it becomes obvious from Eq. (2.10) that the currents should be expanded in terms of the "renormalized" quantities $\tilde{\mu}$, $\tilde{H}_i$, $\tilde{T}$, and $\tilde{\Gamma}^\alpha$.

Since we are dealing with reversible hydrodynamics, we put $R=0$ in the remainder of this paper. We neglect the tiny magnetic dipole energy in the following, since there is a well known procedure[26,36] by which it can be incorporated into hydrodynamics, if necessary. The conservation of angular momentum density will be taken into account by an appropriate choice of the stress tensor. From the three components of the magnetization density M_i only the longitudinal one $m\equiv\vec{M}\cdot\hat{H}_{\rm ext}$ survives as a true conserved quantity in a strong external magnetic field $\vec{H}_{\rm ext}$. Of course, one may also set up a "macroscopic dynamics" taking into account the transverse components of the magnetization density and this path of thought has been pursued for the A_1 phase,[30] and for the A phase in high magnetic fields,[31,48] yielding gaps in some of the normal modes occurring in such a theory.

In the following we concentrate on a rigorous hydrodynamic theory and therefore we have to consider only the longitudinal magnetization density for the A_1 phase, the A phase in high magnetic fields, and the B phase in high magnetic fields. In the equations for the conserved densities we have already displayed explicitly the terms which ensure Galilean invariance and this implies that σ_{ij}, q_i, j_{ik}^M, and Y^α are not allowed to contain contributions proportional to the velocity whereas terms $\sim \nabla_j v_i$ are still possible.

III. NONLINEAR HYDRODYNAMICS OF THE A PHASE IN HIGH MAGNETIC FIELDS

In this section we present the nonlinear reversible hydrodynamic equations for the A phase in high magnetic fields. As is well known[1] these phases are characterized by the following order parameter

$$T_{ij}(\vec{r},\vec{X},t)=F(|\vec{r}|,\vec{X},t)A_{ij}(\vec{X},t)e^{i\varphi(\vec{X},t)} \quad , \tag{3.1}$$

where

$$A_{ij}(\vec{X},t)=V_i(\vec{X},t)\Delta_j(\vec{X},t) \quad . \tag{3.2}$$

For the orbital part we have the complex vector $\hat{\Delta}$, with

$$\hat{\Delta}\cdot\hat{\Delta}^*=1 \text{ and } \hat{\Delta}\cdot\hat{\Delta}=0 \quad ,$$

and the preferred axis in real space is therefore

$$\vec{l}=i(\hat{\Delta}\times\hat{\Delta}^*) \quad . \tag{3.3}$$

The spin part $V_i(\vec{X},t)$ can generally be written as

$$\vec{V}=(\Delta_\uparrow-\Delta_\downarrow)\hat{d}+\Delta_\uparrow\hat{n} \tag{3.4}$$

or

$$\vec{V}=(\Delta_\downarrow-\Delta_\uparrow)\hat{d}+\Delta_\downarrow\hat{n} \quad , \tag{3.5}$$

with the complex spin vector $\hat{d}$, for which

$$\hat{d}\cdot\hat{d}^*=1 \quad \text{and } \hat{d}\cdot\hat{d}=0 \quad , \tag{3.6}$$

and the real spin vector $\hat{n}$. In the A phase without a magnetic field or with a vanishing one the gaps for spin-up pairs, $\Delta_\uparrow$, and spin-down pairs, $\Delta_\downarrow$, are equal and $\hat{n}$ is the only preferred direction in spin space. In the A_1 phase, there are only spin-up pairs ($\Delta_\downarrow=0$) or only spin-down one pairs ($\Delta_\uparrow=0$) and from (3.4) or (3.5) we can construct the preferred direction in spin space,

$$\hat{W}=i(\hat{d}\times\hat{d}^*) \tag{3.7}$$

or

$$\hat{W}=i(\hat{d}^*\times\hat{d}) \tag{3.8}$$

In the A phase in strong magnetic fields, there are two preferred directions, $\hat{W}$ [defined by (3.7) or (3.8)], which is parallel or antiparallel to the external magnetic field, and $\hat{n}$, which is orthogonal to $\hat{W}$.

To set up the hydrodynamic equations one has to establish first the hydrodynamic variables. This has already been done in the literature (Refs. 25, 26, 30, 31, 33, 34, and 36), and in Ref. 42 we have listed the variables and their behavior under various symmetry transformations.

For the A phase in high magnetic fields we have, for the conservation and quasiconservation laws (2.5)–(2.10),

$$\begin{aligned}
&\dot{\rho}+\vec{\nabla}\cdot(\rho\vec{v}^n+\vec{g}')=0 \quad ,\\
&\dot{m}+\vec{\nabla}\cdot(m\vec{v}+\vec{j}^m)=0 \quad ,\\
&\dot{g}_i+\nabla_j(v_i g_j+p\delta_{ij}+\sigma_{ij})=0 \quad ,\\
&\dot{\sigma}+\nabla_i(\sigma v_i+q_i/T)=0 \quad ,\\
&\dot{\varphi}+v_i\nabla_i\varphi+I_\varphi=0 \quad ,\\
&\dot{l}_i+v_j\nabla_j l_i+X_i=0 \quad ,\\
&\dot{n}+v_j\nabla_j n+Y=0 \quad ,
\end{aligned} \tag{3.9}$$

where $m=\vec{M}\cdot\vec{H}$ is the longitudinal magnetization which is now a variable being even under parity and time reversal. Thus m has the same symmetry properties as σ and ρ, and $\vec{j}^m$ the same as $\vec{q}$ and $\vec{g}'$. We disregard the transverse components of $\vec{M}$, $\vec{H}\times\vec{M}$ and the transverse components of $\hat{W}$, $\vec{H}\times\hat{W}$ (where $\hat{n}\cdot\delta\hat{W}=-\hat{W}\cdot\delta\hat{n}$), since they are neither conserved nor connected with a spontaneously broken symmetry (there are theories which deal with these variables[31,48]). Therefore, $\delta n=\epsilon_{ijk}n_jW_i\delta n_k$, which is transverse to both $\hat{n}$ and $\hat{W}$, is the only hydrodynamic variable characterizing a spontaneously broken symmetry in spin space. δn is odd under time reversal and even under spatial inversion. With respect to these symmetry transformations δn (Y) can be treated on equal footing with $\delta\varphi$ (I_φ). However, with respect to gauge transformations, and rotations of spin and orbit space, both variables have different (and quite unusual) behavior.

In two recent papers[33,34] by the present authors, a linearized theory of ^{3}He-A in high magnetic fields was given and the properties of $\delta\varphi$ and δn were derived; the main features are listed in Ref. 42. Since $\delta\varphi$ is connected with rotations in orbit *and* spin space, it is affected by the anholonomity of finite rotations in orbit *and* spin space. Thus, the Mermin-Ho relation is generalized to

$$\begin{aligned}(\partial_1\partial_2-\partial_2\partial_1)\varphi=\hat{l}\cdot[(\partial_1\vec{l})\times(\partial_2\vec{l})]\\
+\beta_1\hat{W}\cdot[(\partial_1\vec{W})\times(\partial_2\vec{W})]\end{aligned} \tag{3.10}$$

with $\beta_1=|\Delta_\uparrow^2-\Delta_\downarrow^2|/(\Delta_\uparrow^2+\Delta_\downarrow^2)$. In the same way one finds, for δn,

$$(\partial_1\partial_2-\partial_2\partial_1)n=\hat{W}\cdot[(\partial_1\vec{W})\times(\partial_2\vec{W})] \quad , \tag{3.11}$$

reflecting the fact that the various rotations in spin space ($\delta n, \delta W_i$) do not commute. Of course, there is a corresponding relation for δW_i, which we will mention here for completeness only:

$$(\partial_1\partial_2-\partial_2\partial_1)\vec{W}=\hat{W}\times[(\partial_1\vec{W})(\partial_2 n)-(\partial_2\vec{W})(\partial_1 n)] \quad . \tag{3.12}$$

The term β_1 in (3.10) arises from the fact that the components of V_i, the spin part of the order parameter in Eq. (3.4), carry prefactors $\sim(\Delta_\uparrow-\Delta_\downarrow)$ and $\sim(\Delta_\uparrow+\Delta_\downarrow)$. It is easily checked that Eq. (3.10) contains as a special case the Mermin-Ho–type relation for the A_1 phase which has been derived by Liu[41]; because $\Delta_\uparrow$ or $\Delta_\downarrow$ equal to zero implies $\beta_1=1$. Furthermore Eq. (3.10) reduces continuously (as it should, because there is no phase transition between the A phase in high external fields and the A phase without external field, contrary to the case $A_1\rightarrow A$ phase in high fields which is accompanied by a phase transition) to the Mermin-Ho relation of the A phase without external field.[49] Eq. (3.11) does not exist for the A_1 phase because δn does not exist in ^{3}He-A_1 and in addition Eq. (3.11) is satisfied in a trivial manner (with the right-hand side identically zero) in the A phase without external field because $\hat{W}$ does not exist when the external field is switched off, i.e., we have

$$\epsilon_{ijk}\partial_j\partial_k n\equiv 0 \quad , \tag{3.13}$$

a relation which is well known for the A phase.

On the other hand, Eq. (3.12) does not exist for the A phase in a vanishing magnetic field, since $\hat{W}$ does not exist there, and Eq. (3.12) is always satisfied in the A_1 phase, since $\delta n\equiv 0$ there.

The Gibbs relation (2.2) can now be specified:

$$\begin{aligned} d\epsilon=&\mu\, d\rho+T\, d\sigma+\vec{v}^n\cdot d\vec{g}+H\, dm\\ &+\vec{\lambda}^s\cdot d\vec{\nabla}\varphi+\phi_{ij}\, d(\nabla_j l_i)\\ &+\Gamma_i\, dl_i+\psi_i\, d\nabla_i n+\tau_i\, d\nabla_i\rho\\ &+\Theta_i\, d\nabla_i\sigma+\Lambda_i\, d\nabla_i m \quad . \end{aligned}\tag{3.14}$$

In addition to the gradients of σ and ρ we have also kept in (3.14) the gradients of m, which can now be treated on an equal footing as the density and entropy.

For the pressure (2.3) we have, for the A phase in high magnetic fields,

$$\begin{aligned} p=&-\epsilon+\mu\rho+T\sigma+Hm+\tau_i\nabla_i\rho\\ &+\Theta_i\nabla_i\sigma+\Lambda_i\nabla_i m \quad . \end{aligned}\tag{3.15}$$

For the energy density (expressed in $\vec{v}^n$) $F\equiv \epsilon-\vec{g}\cdot\vec{v}^n$ we find, up to third-order terms (cf. Sec. II for a discussion of this terminology),

$$F=F^{\rm lin}+F^{\tilde{C}}+F^{PB} \quad , \tag{3.16}$$

where $F^{\rm lin}$ is the expression from the linear theory,[33,34]

$$\begin{aligned} F^{\rm lin}=&\tfrac{1}{2}\chi_{11}^{-1}(\delta m)^2+\tfrac{1}{2}T_0C_V^{-1}(\delta\sigma)^2+\zeta_4(\delta\rho)^2+\zeta_1(\delta m)(\delta\sigma)+\zeta_3(\delta\rho)(\delta\sigma)+\zeta_2(\delta m)(\delta\rho)-H_{\rm ext}m\\ &-\tfrac{1}{2}\rho_{ij}^n v_i^n v_j^n-\rho_{ij}^s(\nabla_j\varphi)v_i^n-\rho_{ij}^W(\nabla_j n)v_i^n-C_{ij}^{(1)}v_i^n(\vec{\nabla}\times\vec{l})_j+\tfrac{1}{2}\rho_{ij}^s(\nabla_i\varphi)(\nabla_j\varphi)\\ &+(\rho_{ij}^W-\beta_1\tilde{M}_{ij})(\nabla_j n)(\nabla_i\varphi)-\tfrac{1}{2}\beta_1(\rho_{ij}^W-\beta_1 M_{ij})(\nabla_j\varphi)(\nabla_i\varphi)+C_{ij}^{(2)}(\vec{\nabla}\times\vec{l})_j(\nabla_i\varphi)+\tfrac{1}{2}\tilde{M}_{ij}(\nabla_i n)(\nabla_j n)\\ &+\beta_1^{-1}(C_{ij}^{(1)}-C_{ij}^{(2)})(\nabla_i n)(\vec{\nabla}\times\vec{l})_j+\tfrac{1}{2}K_{ijkl}(\nabla_l l_k)(\nabla_j l_i) \quad . \end{aligned}\tag{3.17}$$

The tensors ρ_{ij}^s,ρ_{ij}^n, etc., are of the axial form $\rho_{ij}=\rho_\parallel l_i l_j+\rho_\perp(\delta_{ij}-l_i l_j)$ and K_{ijkl} is the same as in the A phase without magnetic fields. For details, especially the restrictions due to positivity, we refer to the linear theory.[34,50]

For the contribution $F^{\tilde{C}}$ we have (the terms which are similar in structure to those which have been given by Combescot for the A phase without external field; however, Combescot has not considered the A phase in high magnetic fields)

$$\begin{aligned} F^{\tilde{C}}=&(\nabla_i\varphi-v_i^n)\epsilon_{ijk}l_j(\alpha^C\nabla_k\rho+\beta^C\nabla_k\sigma+\gamma^C\nabla_k m)+(\nabla_i n-\beta_1\nabla_i\varphi)\epsilon_{ijk}l_j(\alpha^n\nabla_k\rho+\beta^n\nabla_k\sigma+\gamma^n\nabla_k m)\\ &+a_{1\rho}(\vec{\nabla}\cdot\vec{l})(l_i\partial_i\rho)+a_{2\sigma}l_i(\partial_i l_j)(\partial_j\sigma)+a_{1\sigma}(\vec{\nabla}\cdot\vec{l})(l_i\partial_i\sigma)+a_{2\rho}l_i(\partial_i l_j)(\partial_j\rho)\\ &+a_{1m}(\vec{\nabla}\cdot\vec{l})(l_i\partial_i m)+a_{2m}l_i(\partial_i l_j)(\partial_j m) \quad . \end{aligned}\tag{3.18}$$

For F^{PB} we obtain the rather awful looking expression

$$\begin{aligned} F^{PB}=&(\delta\rho)A_{ij}(v_i^n-\nabla_i\varphi)(v_j^n-\nabla_j\varphi)+(\delta\sigma)B_{ij}(v_i^n-\nabla_i\phi)(v_j^n-\nabla_j\varphi)+(\delta m)D_{ij}(v_i^n-\nabla_i\varphi)(v_j^n-\nabla_j\varphi)\\ &+(\delta\rho)E_{ij}(\nabla_i n-\beta_1\nabla_i\varphi)(\nabla_j n-\beta_1\nabla_j\varphi)+(\delta\sigma)F_{ij}(\nabla_i n-\beta_1\nabla_i\varphi)(\nabla_j n-\beta_1\nabla_j\varphi)\\ &+(\delta m)G_{ij}(\nabla_i n-\beta_1\nabla_i\varphi)(\nabla_j n-\beta_1\nabla_j\varphi)+(\delta\rho)H_{ij}(\nabla_i n-\beta_1\nabla_i\varphi)(\vec{\nabla}\times\vec{l})_j\\ &+(\delta\sigma)K_{ij}(\nabla_i n-\beta_1\nabla_i\varphi)(\vec{\nabla}\times\vec{l})_j+(\delta m)L_{ij}(\nabla_i n-\beta_1\nabla_i\varphi)(\vec{\nabla}\times\vec{l})_j+(\delta\rho)N_{ij}(\nabla_i n-\beta_1 v_i^n)(\vec{\nabla}\times\vec{l})_j\\ &+(\delta\sigma)R_{ij}(\nabla_i n-\beta_1 v_i^n)(\vec{\nabla}\times\vec{l})_j+(\delta m)S_{ij}(\nabla_i n-\beta_1 v_i^n)(\vec{\nabla}\times\vec{l})_j \quad . \end{aligned}\tag{3.19}$$

The tensors $A_{ij} \dots S_{ij}$ in Eq. (3.19) are of the usual axial form. Note that not all combinations of "velocities" which one can think of are allowed in (3.19) because of Galilean invariance.

The equations of state can be derived from Eqs. (3.16)–(3.19) by differention:

$$\begin{aligned}
\tilde{\mu} &= \frac{\partial F}{\partial \rho} - \nabla_i \frac{\partial F}{\partial(\nabla_i \rho)} = \mu - \nabla_i \tau_i \ , \\
\tilde{T} &= \frac{\partial F}{\partial \sigma} - \nabla_i \frac{\partial F}{\partial(\nabla_i \sigma)} = T - \nabla_i \Theta_i \ , \\
\tilde{H} &= \frac{\partial F}{\partial m} - \nabla_i \frac{\partial F}{\partial(\nabla_i m)} = H - \nabla_i \Lambda_i \ , \\
\tilde{\Gamma}_i &= \frac{\partial F}{\partial l_i} - \nabla_j \frac{\partial F}{\partial(\nabla_j l_i)} + \lambda_i^s \epsilon_{klp} l_k \nabla_j l_p \\
&= \Gamma_i - \nabla_j \phi_{ij} + \lambda_i^s \epsilon_{klp} l_k \nabla_j l_p \ , \\
\vec{g} &= \frac{\partial F}{\partial \vec{v}^n} = \rho \vec{v}^n + \vec{g}' = \rho \vec{v}^n + \vec{\lambda}^s + \beta_1 \vec{\psi} \ , \\
\vec{\lambda}^s &= \frac{\partial F}{\partial(\vec{\nabla}\varphi)}, \quad \psi_i = \frac{\partial F}{\partial(\nabla_i n)} \ .
\end{aligned} \tag{3.20}$$

When Eqs. (3.9), (3.14), and (3.15) are taken into account, the yet unknown reversible currents are restricted by

$$0 = \nabla_i Q_i - \mu \vec{\nabla} \cdot \vec{g}' + (\nabla_i \tilde{T}) q_i / T - \vec{\lambda}^s \cdot \vec{\nabla} I_\varphi - \tilde{H} \nabla_i j_i^m - \Gamma_i X_i - \psi_i \nabla_i Y + \Sigma_{ik} \nabla_i v_k^n \ , \tag{3.21}$$

where

$$\begin{aligned}
\Sigma_{ik} &= \lambda_i^s \nabla_k \varphi + \phi_{ji} \nabla_k l_j + \psi_i \nabla_k n + \Theta_i \nabla_k \sigma \\
&\quad + \tau_i \nabla_k \rho + \Lambda_i \nabla_k m - \sigma_{ik} \\
&\quad - \delta_{ik} [\nabla_j(\rho \tau_j) + \nabla_j(\sigma \Theta_j) + \nabla_j(m \Lambda_j)] \ ,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
Q_i &= j_i^\epsilon - \tau_i \vec{\nabla} \cdot \vec{g}' - \Theta_i \nabla_k \left(\frac{q_k}{T} \right) - \rho \tau_i \vec{\nabla} \cdot \vec{v}^n \\
&\quad - \Theta_i \sigma \vec{\nabla} \cdot \vec{v}^n - \phi_{ij} X_j - q_i \tilde{T}/T - \Lambda_i m \vec{\nabla} \cdot \vec{v}^n \\
&\quad - \Lambda_i \nabla_k j_k^m \ .
\end{aligned}$$

To close the set of our nonlinear reversible hydrodynamic equations of ^{3}He-A in high magnetic fields we expand the currents $(q_i, I_\varphi, j_i^m, X_i, Y, \Sigma_{ik})$ into the "renormalized" thermodynamic conjugate quantities $(\tilde{\mu}, \tilde{T}, \vec{\lambda}^s, \tilde{H}, \tilde{\Gamma}_i, \psi_i, \nabla_i v_k^n)$. At first we obtain, in generalization of the linear terms,

$$\begin{aligned}
g_i &= \rho v_i^n + \lambda_i^s + \beta_1 \psi_i \ , \\
Y^{(1)} &= \beta_1 \tilde{\mu} + \beta_2 \tilde{T} + \beta_3 \vec{l} \cdot (\vec{\nabla} \times \vec{v}^n) + \gamma \tilde{H} \ , \\
I_\varphi^{(1)} &= \tilde{\mu} + \gamma \vec{l} \cdot (\vec{\nabla} \times \vec{v}^n) - \gamma \beta_1 \tilde{H} \ , \\
X_i^{(1)} &= -\beta \epsilon_{ipk} l_k \tilde{\Gamma}_p \\
&\quad - [\alpha_1(\delta_{ij} - l_i l_j) l_k + \alpha_2(\delta_{ik} - l_i l_k) l_j] \nabla_j v_k^n \\
&\quad + \Pi_3 l_k l_p \epsilon_{pij} \nabla_k \nabla_j \tilde{T} + \Pi_s l_p l_k \epsilon_{pij} \nabla_k \nabla_j \tilde{H} \ , \\
q_i^{(1)} &= T \beta_2 \psi_i - \Pi_3 T \epsilon_{pji} \nabla_k l_k l_p \tilde{\Gamma}_j \\
&\quad + \zeta_1 \epsilon_{ijk} l_j \nabla_k \tilde{T} + \zeta_2 \epsilon_{ijk} l_j \nabla_k \tilde{H} \ , \\
j_k^{m(1)} &= \gamma \psi_k - \beta_1 \gamma \lambda_k^s - \Pi_5 \epsilon_{pji} \nabla_k l_k l_p \tilde{\Gamma}_j \\
&\quad + \zeta_3 \epsilon_{ijk} l_j \nabla_k \tilde{H} - (\zeta_2/T) \epsilon_{ijk} l_j \nabla_k \tilde{T} \ , \\
\Sigma_{ij}^{(1)} &= \beta_3 l_p \epsilon_{pjl} \nabla_l \psi_i - \gamma l_p \epsilon_{pij} \nabla_k \lambda_k^s \\
&\quad - [\alpha_1(\delta_{jl} - l_j l_l) l_i + \alpha_2(\delta_{il} - l_i l_l) l_j] \tilde{\Gamma}_l \\
&\quad + (\gamma_{ik}^{(1)} \epsilon_{jpq} + \gamma_{jq}^{(2)} \epsilon_{ipk} \\
&\quad + \gamma_{kj}^{(3)} \epsilon_{ipq} + \gamma_{iq}^{(3)} \epsilon_{jpk}) l_p \nabla_q V_k^n \ .
\end{aligned} \tag{3.23}$$

The higher-order gradient terms Π_3 and Π_5 have been given very recently by the present authors.[50] In addition to those terms given in Eqs. (3.23) which are generalizations of the linear terms to the nonlinear domain we find some novel terms which exist only in a nonlinear theory, and it seems important to notice that none of them exists in ^{3}He-A without external field! These terms are

$$\begin{aligned}
q_i^{(2)} &= \zeta_{ij}(\beta_1 \delta \varphi - \delta n) \nabla_j \tilde{H} \ , \\
j_i^{m(2)} &= -\zeta_{ij}(\beta_1 \delta \varphi - \delta n) \nabla_j \tilde{T} \ , \\
I_\varphi^{(2)} &= \eta_{ij}^{(1)}(\beta_1 \delta \varphi - \delta n) \nabla_i v_j^n \\
&\quad + \eta^{(4)}(\beta_1 \delta \varphi - \delta n) \nabla_i \psi_i \ , \\
Y^{(2)} &= \eta_{ij}^{(2)}(\beta_1 \delta \varphi - \delta n) \nabla_i v_j^n \\
&\quad - \eta^{(4)}(\beta_1 \delta \varphi - \delta n) \nabla_i \lambda_i^s \ , \\
X_i^{(2)} &= \xi_{ijk}(\beta_1 \delta \varphi - \delta n) \nabla_j v_k^n \ , \\
\Sigma_{ij}^{(2)} &= (\beta_1 \delta \varphi - \delta n) \eta_{ij}^{(1)} \nabla_k \lambda_k^s \\
&\quad + \eta_{ij}^{(2)}(\beta_1 \delta \varphi - \delta n) \nabla_k \psi_k \\
&\quad - \xi_{kij}(\beta_1 \delta \varphi - \delta n) \tilde{\Gamma}_k \ ,
\end{aligned} \tag{3.24}$$

where

$$\xi_{ijk} = \xi \hat{l}_p (\hat{l}_k \epsilon_{ijp} + \hat{l}_i \epsilon_{kjp}) \ .$$

The last point to show is the conservation of angular momentum. To reach this aim we have to bring the stress tensor $\tilde{\sigma}_{ij}$ to the form

$$\tilde{\sigma}_{ij} = \tfrac{1}{2}(\tilde{\sigma}_{ij} + \tilde{\sigma}_{ji}) + \nabla_k \Pi_{ijk} \tag{3.25}$$

with $\Pi_{ijk} = -\Pi_{jik}$.[3,32,51] We now investigate the behavior under rigid rotations of the hydrodynamic variables. All scalar quantities remain unchanged, i.e., $0 = \delta\epsilon = \delta\rho = \delta\sigma = \delta M$, while vectors (in real space) transform like $\delta g_i = \Omega_{ij} g_j$, $\delta\nabla_i = \Omega_{ij}\nabla_j$ (where Ω_{ij} is a constant antisymmetric rotation matrix). The transformation behavior of δl_i, $\delta\varphi$, and δn can be extracted from the exact commutator relations and we obtain

$$\delta n = \tfrac{1}{2}\beta_1 \epsilon_{pjk} l_p \Omega_{jk} \ , \quad \delta l_i = \Omega_{ij} l_j \ , \quad \delta\varphi = \tfrac{1}{2}\epsilon_{pjk} l_p \Omega_{jk} \ , \tag{3.26}$$

and, with (3.10),

$$\delta\nabla_i\varphi = \Omega_{ij}\nabla_j\varphi - \epsilon_{pjk}\nabla_i l_p \Omega_{jk} + \epsilon_{pjk} l_p l_m \Omega_{jm}\nabla_i l_k \ .$$

Therefore, for rigid rotations the Gibbs relation (2.2) takes the form

$$0 = v_i^n \Omega_{ij} g_j + \lambda_i^s(\Omega_{ij}\nabla_j\varphi - \Omega_{jk}\epsilon_{pjk}\nabla_i l_p + \epsilon_{pjk} l_p l_m \Omega_{jm}\nabla_i l_k) + \phi_{ij}(\Omega_{jk}\nabla_k l_i + \nabla_j\Omega_{ik} l_k) + \Gamma_i\Omega_{ij} l_j$$
$$+ \psi_j \Omega_{jk}\nabla_k n + \tau_i\Omega_{ij}\nabla_j\rho + \Theta_i\Omega_{ij}\nabla_j\sigma + \Lambda_i\Omega_{ik}\nabla_k m + \tfrac{1}{2}\beta_1\psi_i\Omega_{ik}\epsilon_{pjk}\nabla_j l_p \ . \tag{3.27}$$

With the help of (3.27) the antisymmetric part of $\tilde{\sigma}_{ij}$ (3.25) can be brought into the required form (3.25) with

$$\Pi_{ijk} = l_i\phi_{jk} - l_j\phi_{ik} - \tfrac{1}{2}\epsilon_{pij} l_p \lambda_k^s - \tfrac{1}{2}\beta_1\epsilon_{pji} l_p \psi_k \tag{3.28}$$

if we postulate $\gamma = \frac{1}{2}$, $\alpha_2 - \alpha_1 = 1$, $\gamma_\parallel^1 = \gamma_\parallel^2 = \gamma_\parallel^3$, $\gamma_\perp^1 = \gamma_\perp^2$, and $\beta_3 = \frac{1}{2}\beta_1$.

IV. NONLINEAR HYDRODYNAMICS OF THE OTHER SUPERFLUID PHASES OF ^{3}He

In this section we will not give the explicit formulas in order to conserve space. The hydrodynamic equations are listed in Ref. 42. We carry out here a comparison with the results for the A phase in high magnetic fields and show similarities and differences.

A. A_1 phase

In the A_1 phase one of the two gaps $\Delta_\uparrow$ or $\Delta_\downarrow$ is zero. Therefore $\beta_1 = 1$ and there is no hydrodynamic variable characterizing a broken symmetry in spin space. Thus, deleting all expressions of Sec. III involving δn or its thermodynamic conjugate and putting $\beta_1 = 1$ lead, in a straightforward manner, to the nonlinear hydrodynamic equations for the A_1 phase. The equations for ^{3}He-A_1 derived in that manner present a generalization of the work by Liu[41]; e.g., Liu's generalized Mermin-Ho–type relation emerges naturally from our Eq. (3.10). We find in the equation for the current of the magnetization density a term ($\sim \zeta_3$), which has not been considered previously. In addition we have found various novel terms of higher gradient order in the static as well as in the dynamic equations; some of these terms, which are already present in the linear domain, have been given by the present authors recently.[50] Although the hydrodynamics of the A phase in high magnetic fields contains that of the A_1 phase as a special case, there is, of course, a phase transition between the two phases, which is—on the hydrodynamic level—manifest by loosing one degree of freedom (δn).

B. A phase (without magnetic field)

Since the A phase in high magnetic fields reduces continuously into the A phase by switching off the magnetic field, the hydrodynamics derived in Sec. 3 is applicable to the A phase without magnetic fields, if $\vec{H} \to 0$. However, one has to add the additional degree of freedom δn_i (with $n_i\,\delta n_i = 0$ and $H_i\,\delta n_i = 0$), which regains its hydrodynamic character in the limit $\vec{H} \to 0$. By doing so, one obtains nonlinear hydrodynamic equations (cf. Ref. 42) which generalize previous results (Refs. 37–39 and 47) with respect to higher-order gradient terms. Especially we find additional contributions to the pressure (3.15) and some additional terms to the free energy F (3.16):

$$F_{\rm add} = \tilde{t}(\delta_{ip} - \hat{l}_i\hat{l}_p)\hat{l}_k M_l \epsilon_{pjk}\epsilon_{lmn}(\nabla_j n_m)(\nabla_i n_n) + (\delta\rho)A_{ij}(v_i^n - \nabla_i\varphi)(v_j^n - \nabla_j\varphi)$$
$$+ (\delta\sigma)B_{ij}(v_i^n - \nabla_i\varphi)(v_j^n - \nabla_j\varphi) + [(\delta\rho)D_{ij} + (\delta\sigma)E_{ij}](v_i^n - \nabla_i\varphi)(\vec{\nabla}\times\vec{l})_j \ , \tag{4.1}$$

where $A_{ij} \ldots E_{ij}$ are of the axial form. The four last terms can be viewed as generalizations of Khalatnikov's $(\vec{v}^n - \vec{v}^s)^2$ terms.[19]

C. B phase in high magnetic fields

The linear hydrodynamics of ^{3}He-B in high magnetic fields was recently given by the present authors.[35] In high magnetic fields the gaps for spin-up pairs, $\Delta_\uparrow$, spin-down pairs, $\Delta_\downarrow$, and symmetrically mixed pairs Δ_3 are different from each other. The equilibrium order-parameter matrix, therefore, takes the form[35]

$$n_{i\alpha}^0 = \tfrac{1}{2}\Delta_\uparrow \hat{e}_\alpha \hat{f}_i^* + \tfrac{1}{2}\Delta_\downarrow \hat{e}_\alpha^* \hat{f}_i + \tfrac{1}{4}\Delta_3(\hat{e}\times\hat{e}^*)_\alpha(\hat{f}\times\hat{f}^*)_i \tag{4.2}$$

with

$$\hat{e}\cdot\hat{e}^*=\hat{f}\cdot\hat{f}^*=1,\quad \hat{e}^2=0=\hat{f}^2\ ,$$

and the gap $\Delta(k)$ is neither isotropic nor unitary.[35] The existence of the external magnetic field $\vec{H}^{\rm ext}$ defines a preferred direction in spin space $\hat{H}_\alpha=\vec{H}^{\rm ext}/|\vec{H}^{\rm ext}|$.

By Eq. (4.2) this implies the existence of a preferred direction in orbit space, too, which is given by the vector[35]

$$\hat{H}_i=\tfrac{1}{2}\epsilon_{ijk}n^0_{j\alpha}n^0_{k\alpha}\ . \tag{4.3}$$

Since $\hat{H}_i$ is parallel to $\hat{H}_\alpha$, we will not discriminate between the two unit vectors and between spin and orbit indices in the future. This preferred direction is identical with the quantization axis $\hat{d}$.

Apart from the conserved quantities there are two hydrodynamic variables connected with spontaneously broken symmetries, the phase deviation $\delta\varphi$ and rotation angle (about $\hat{H}$) $\delta\Theta$. Thereby $\delta\varphi$ (characterizing broken gauge symmetry) is affected by rotations in spin space and orbit space, while $\delta\theta$ (characterizing broken rotational symmetry) is affected by gauge transformation, too. Thus, we are left with a set of hydrodynamical variables, which have the identical symmetry properties than those in the A phase in high magnetic fields. In the B phase in high magnetic fields, however, the rotational symmetry in orbit space is not spontaneously broken, but only externally (by $\hat{H}_i$) and, therefore, there are no variables like δl_i in the A phase. Thus, deleting all expressions containing δl_i (or its thermodynamic force or its current) from Sec. III and identifying δn with $\delta\theta$, δW_i with δd_i, and $\hat{l}_i$ with $\hat{H}_i$, we obtain the nonlinear hydrodynamic of the B phase in high magnetic fields, which is, therefore very similar in structure to that of the A phase in high magnetic fields (cf. Ref. 42).

There are, however, some differences in detail, which we will mention here. Because of the different structure of the equilibrium order parameter, the quotient $\beta_1=|\Delta_\downarrow^2-\Delta_\uparrow^2|/(\Delta_\downarrow^2+\Delta_\uparrow^2)$ of the A phase in high magnetic fields, has to be replaced in the B phase in high magnetic fields either by

$$\beta_1=\tfrac{3}{4}|\Delta_\uparrow^2-\Delta_\downarrow^2|/(\Delta_3^2+\Delta_\uparrow^2+\Delta_\downarrow^2) \tag{4.4a}$$

(if the formulas are connected with $\delta\varphi$) or by

$$\tilde{\beta}_1=\tfrac{1}{4}|\Delta_\uparrow^2-\Delta_\downarrow^2|/(\Delta_\uparrow^2+\Delta_\downarrow^2) \tag{4.4b}$$

(if the formulas are connected with $\delta\theta$). In particular, the "Mermin-Ho" relation (3.10) reads, for the B phase in high magnetic fields,

$$(\partial_1\partial_2-\partial_2\partial_1)\varphi=\beta_1\hat{H}[(\partial_1\vec{d})\times(\partial_2\vec{d})]\ . \tag{4.5}$$

For the behavior under rigid rotations one obtains [instead of Eq. (3.26)]

$$\delta\varphi=\tfrac{1}{2}\beta_1\epsilon_{jkp}\hat{H}_p\Omega_{jk}\ ,\qquad \delta\theta=\tfrac{1}{2}\epsilon_{pjk}\hat{H}_p\Omega_{jk}\ , \tag{4.6}$$

and the antisymmetric part of the stress tensor reads

$$\nabla_k\Pi_{ijk}=-\tfrac{1}{2}\nabla_k(\beta_1\epsilon_{pij}\hat{H}_p\lambda_k^s+\epsilon_{pji}\hat{H}_p\psi_k) \tag{4.7}$$

with $\gamma=\frac{1}{2}$, $\beta_3=-\frac{1}{2}\beta_1$.

D. *B* phase (without magnetic field)

The linearized hydrodynamic theory of ^{3}He-B was given by Graham and Pleiner[27,29] and generalized to the nonlinear domain by Liu and Cross.[40] In this section we will show how the expressions of the preceding subsection change in the limit $H^{\rm ext}\to 0$; thereby we will give some novel higher-order terms in the statics, which may have experimental consequences.

For $H^{\rm ext}\to 0$ the gap becomes isotropic $(\Delta_\uparrow\to\Delta_\downarrow\to\Delta_3\to\Delta_0)$ and the order parameter matrix $R_{i\alpha}$ is a real rotation matrix. Therefore, the parameter β_1 and $\tilde{\beta}_1$ (4.4) tends to zero and $\delta\varphi$ is a true scalar quantity and θ_i describes simply rotations of spin space against real space. The preferred direction in spin and orbit space due to the external magnetic field vanishes, of course.

Thus, the "transverse" components of the magnetization $\hat{d}\times\vec{M}$ and the rotation angles about the quantization axis $\hat{d},\hat{d}\times\delta\vec{d}$, become true hydrodynamic variables for $H^{\rm ext}\to 0$. Equation (4.5) reduces to $\partial_1\partial_2\varphi=\partial_2\partial_1\varphi$ and Eqs. (3.11) and (3.12) express the well-known B-phase Mermin-Ho relation[40] (replacing δn by $\delta\theta$ and δW_i by δd_i).

With that proviso we are able to transform the energy expression of Sec. III to the case of the B phase in vanishing magnetic field. Equation (3.17) reduces to the well-known expression[27,40]

$$\begin{aligned}F^{\rm lin}=&\tfrac{1}{2}\chi^{-1}m_\alpha^2+\tfrac{1}{2}T_0C_v^{-1}\,\delta\sigma^2\\&+\zeta_4\,\delta\rho^2+\zeta_3\,\delta\rho\,\delta\sigma\\&+\tfrac{1}{2}\rho^s(\vec{v}^n-\vec{\nabla}\varphi)^2-\tfrac{1}{2}\rho(\vec{v}^n)^2\\&+M_{ij\alpha kl\beta}(\nabla_iR_{j\alpha})(\nabla_kR_{l\beta})\end{aligned} \tag{4.8}$$

and the higher-order gradient terms of Combescot[47] are reobtained from (3.18)

$$\begin{aligned}F^{\tilde{C}}=&\gamma^C(\nabla_i\varphi-v_i^n)\epsilon_{ijk}R_{\alpha k}\nabla_jM_\alpha\\&+(\alpha^d\nabla_k\rho+\beta^d\nabla_k\sigma)\epsilon_{ijk}R_{\alpha j}\epsilon_{\alpha\beta\gamma}R_{\beta m}\nabla_iR_{\gamma m}\\&+(v_i^n-\nabla_i\varphi)DM_\alpha\epsilon_{\alpha\beta\gamma}R_{\beta k}\nabla_iR_{\gamma k}\ .\end{aligned} \tag{4.9}$$

There are novel third-order terms coming from F^{PB}

(3.19):

$$F^{PB}=(\vec{v}^n-\vec{\nabla}\varphi)^2(A\,\delta\rho+B\,\delta\sigma)\ . \qquad (4.10)$$

These terms imply a dependence of the chemical potential and of the temperature on the relative velocity

$$\delta\mu=A(\vec{v}^n-\vec{\nabla}\varphi)^2\ ,\quad \delta T=B(\vec{v}^n-\vec{\nabla}\varphi)^2\ , \qquad (4.11)$$

and a dependence of linear momentum and of $\vec{\lambda}^s$ on density and entropy

$$\vec{g}=2(\vec{v}^n-\vec{\nabla}\varphi)(A\,\delta\rho+B\,\delta\sigma)\ ,$$
$$\vec{\lambda}^s=2(\vec{v}^n-\vec{\nabla}\varphi)(A\,\delta\rho+B\,\delta\sigma)\ . \qquad (4.12)$$

These terms were—for He II—already discussed by Khalatnikov many years ago.[19] Although we do not know the magnitude of the parameters A and B, the effects described by (4.11) may be measurable.

As already stated in Sec. II, third-order terms in the energy expression destroy the positivity of that thermodynamic potential. From the fourth-order terms by which one can overcome this difficulty, we will pick out those, which have a chance to be experimentally accessible. Since ^{3}He-B is isotropic any effect (even a tiny one), which lifts this degeneracy of direction, may become important. There are fourth-order terms defining a preferred direction $\hat{W}\equiv(\vec{v}^n-\vec{\nabla}\varphi)/|\vec{v}^n-\vec{\nabla}\varphi|$ due to the relative velocity. Thereby the material parameters change into tensors, e.g.,

$$\delta_{ij}\rho^s\rightarrow\rho^s_{ij}=\rho^s_{\perp}(\delta_{ij}-\hat{W}_i\hat{W}_j)+\rho^s_{\parallel}\hat{W}_i\hat{W}_j \qquad (4.13\text{a})$$

or

$$\delta_{ij}\chi\rightarrow\chi_{ij}=\chi_{\perp}(\delta_{ij}-\hat{W}_i\hat{W}_j)+\chi_{\parallel}\hat{W}_i\hat{W}_j \qquad (4.13\text{b})$$

reflecting uniaxiality. This holds as well for the dissipative parameters, like viscosity, heat conduction, or order-parameter friction. Thus, second-sound velocity and damping, first-sound damping and spin-orbit wave dispersion relation become anisotropic even for ^{3}He-B without external field. The magnitude of these effects cannot be estimated, since the coefficients of the fourth-order terms are unknown. It should be mentioned here, that the flow-induced anisotropy discussed above is—in its origin but not in its phenomenology—quite different from the anisotropy due to flow-induced gap distortion.

V. CONCLUSIONS

In the present paper we present the nonlinear reversible hydrodynamic equations for all known superfluid phases of ^{3}He. From previous linear theories we take over as hydrodynamic variables the conserved quantities and the quantities describing spontaneously broken continuous symmetries. The energy density is expressed in terms of the hydrodynamic variables including third-order terms. The dynamics are obtained by expanding the currents or quasicurrents into partial derivatives of the energy density.

In order to show in a local description that angular momentum is conserved in the superfluid phases we use a technique which is different from that used previously in the field: we consider the behavior of the Gibbs relation under rigid rotations. This procedure is well known from the theory of liquid crystals and allows to state concisely the restrictions on some of the phenomenological parameters.

The nonlinearities introduced on the static as well as on the dynamical level contain higher-order gradient terms, contributions due to the inhomogeneous structure of the equilibrium order parameter, and, for the first time, terms quadratic in the velocity differences like, e.g., $(v_i^n-\nabla_i\varphi)^2$. Such terms have been considered in ^{4}He II a long time ago by Khalatnikov.[19] This type of terms becomes even more important if one takes into account fourth-order terms in the free-energy density (which are required in order to guarantee thermodynamic stability!). Thereby a preferred direction (induced by flow) is established and, e.g., the B phase without magnetic field becomes uniaxial in orbit space. All other phases, which are uniaxial in orbit space due to $\hat{l}^0$ (A or A_1 phase) or $\hat{H}^0$ (B phase in high magnetic fields), turn over to biaxial systems (in orbit space) if these flow-induced fourth-order terms are taken into account. It is difficult, of course, to give quantitative estimates of these new effects due to the lack of a microscopic calculation for the phenomenological parameters involved. In addition, the effects described above should be contrasted to recent theories which discuss flow-induced gap distortions via microscopic calculations.

Another general feature of the nonlinear terms is the mixing of spin-space and orbit-space variables. Thereby various couplings of spin space and orbit space are established. Especially in the A phase without magnetic fields, where in a linear theory only magnetic dipole forces provide a coupling of spin and orbit space, the nonlinear terms introduced in Sec. III can have measurable effect on inhomogeneous equilibrium structures, solitons, mode coupling, and spin echoes. In the nonlinear domain, the spin part of the A phase without magnetic fields is, therefore, not isomorphic to an antiferromagnet.

For the other phases the coupling of spin and orbit space due to nonlinearities has to be compared with the coupling of spin and orbit space due to those linear terms which owe their existence to the external magnetic field.

In conclusion, we state, that the A phase in high magnetic fields contains as special cases the A_1 phase, the A phase without magnetic fields (apart from degrees of freedom, which loose their hydrodynamic character in an external magnetic field), and the B phase in high magnetic fields (at least from a structural point of view).

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