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Scaling studies of the resistance of the one-dimensional Anderson model with general disorder

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The resistance ρ of a one-dimensional Anderson model with both diagonal and off-diagonal disorder is studied by analytic and numerical techniques. A recursive method is developed and used to derive an exact scaling law for the average resistance at $E=0$ for arbitrary disorder, and for $E\neq 0$ in the limit of weak disorder. The average resistance grows exponentially with L, the length of the sample, in all cases. The typical resistance $\tilde{\rho}=\exp[\langle \ln(1+\rho)\rangle]-1$ is also found to grow exponentially with L in all cases, except for purely off-diagonal disorder at $E=0$, where $\langle \ln(1+\rho)\rangle \propto \sqrt{L}$. An explanation is given for the existence of this special case and it is shown that all our results are consistent with a lognormal probability distribution of the resistance for $\rho >> 1$. Quantitative estimates are made of the reliability of numerically performed averages which show that a numerical average will converge only very slowly to the analytic result. This provides a qualitative explanation of the slower than linear growth of $\ln\langle\rho\rangle$ with L found in several numerical calculations; its consequences for experiment are also explored.

I. INTRODUCTION

Recent theoretical work has shown an interesting difference in the behavior of the resistance of the one-dimensional (1D) Anderson model with off-diagonal as opposed to diagonal disorder at band center. Scaling studies of the Anderson model with diagonal disorder and of a number of other models of disordered 1D systems have found that the dimensionless resistance $\rho=R/T$ (Refs. 1 and 2) satisfies $\ln(\langle\rho\rangle)\propto L$ and $\langle\ln(1+\rho)\rangle\propto L$ as L, the length of the sample, goes to infinity.[3–8] In contrast, very recently analytic[9] and numerical[10] studies of the Anderson model with purely off-diagonal disorder at $E=0$ have found that while $\ln\langle\rho\rangle\propto L$,[9] $\langle\ln(1+\rho)\rangle\propto\sqrt{L}$.[9,10] The significance of this difference may be understood by a consideration of the rather unusual statistical properties of the distributions for ρ which arise from these models. There is mounting evidence that almost any model of a disordered one-dimensional solid will generate an approximately Gaussian distribution for $\ln\rho$ with a variance growing as L. This means that the distribution of ρ will have a very long tail which skews the arithmetic mean to be much greater than a typical value of ρ, i.e., $\mathrm{Prob}(\rho\geq\langle\rho\rangle)<<1$. However, the geometric mean of $1+\rho$ is much less affected by these large but improbable values of ρ, and in fact it is easy to show as we do in Sec. V that it always gives a result representative of the distribution of ρ if $\ln\rho$ is approximately normally distributed. Thus Anderson *et al.*[4] argued that one should study the scale resistance defined by $\tilde{\rho}\equiv\exp[\langle\ln(1+\rho)\rangle]-1$ instead of $\langle\rho\rangle$ or $\langle 1/\rho\rangle$. It is still a somewhat open question whether experimental results will be at all affected by the long tail of the distribution and thus measure $\langle\rho\rangle$ or whether they measure $\tilde{\rho}$.

This question is important and will be addressed below. If experiments are measuring $\tilde{\rho}$ then it is of course important to explore the question of whether the behavior of $\langle\ln(1+\rho)\rangle\propto\sqrt{L}$ can be found in a more general case than for purely off-diagonal disorder at $E=0$. Thus, we are motivated to study the resistance of the Anderson model with purely off-diagonal disorder and arbitrary energy, and the Anderson model with *both* diagonal and off-diagonal disorder in the work which follows.

In Sec. II of this paper we derive an entirely general expression for the dimensionless resistance $\rho=R/T$ for the Anderson model with one scattering channel which is convenient for both numerical

and analytic calculations. In Sec. III we develop a recursive method which enables us to calculate an exact scaling law for the arithmetic mean of the resistance of an Anderson model with any combination of diagonal and off-diagonal disorder. An explicit closed form is given for $\langle\rho\rangle$ at band center and its extension to $E\neq 0$ is discussed. In contrast to some numerical results,[10] we find $\ln\langle\rho\rangle \propto L$ in all cases, even for off-diagonal disorder at $E=0$. (An explanation for this disagreement is given in Sec. V.) In Sec. IV we examine numerically the behavior of $\tilde{\rho}$ in two cases.[11] First, we consider the case of purely off-diagonal disorder and $E\neq 0$. Second, we consider the case of both diagonal and off-diagonal disorder at $E=0$. In both cases we find that the asymptotic behavior as $L\rightarrow\infty$ appears to be $\langle\ln(1+\rho)\rangle \propto L$ even for small energies or a small degree of diagonal disorder. Thus, at least in one dimension, off-diagonal disorder at $E=0$ does turn out to be a special case in terms of the behavior of $\tilde{\rho}$. In Sec. V we interpret our results and show that they are explained by the hypothesis that in all cases $\ln\rho$ is normally distributed for $\rho >> 1$ with $\langle\ln\rho\rangle = N\gamma$, and off-diagonal disorder at $E=0$ corresponds to the special case $\gamma=0$. This implies that the geometric mean of ρ will always be representative of the distribution of ρ and that the arithmetic mean will not. It also suggests that numerical studies of the average resistance and conductance will converge only very slowly to their limiting values as calculated analytically. Finally, in Sec. VI we discuss the relation of these results to experiment and make an argument that experiments could measure an ill-converged approximation to $\ln(\langle\rho\rangle)$. As we discuss below, such an approximation would not give a clear linear dependence of the logarithm of the resistance on sample length, and would tend to give a value closer to $\ln\tilde{\rho}$ than to $\ln(\langle\rho\rangle)$ as $L\rightarrow\infty$.

II. DIMENSIONLESS RESISTANCE IN THE ANDERSON MODEL

We consider the generalized one-dimensional Anderson model described by the Hamiltonian

$$H=\sum_{n=-\infty}^{\infty}\epsilon_n|n\rangle\langle n| + \sum_{n=-\infty}^{\infty}V_{n,n+1}(|n\rangle\langle n+1|+|n+1\rangle\langle n|), \tag{1}$$

where $\{|n\rangle\}$ form a tight-binding basis set. The site energies $\{\epsilon_n\}$ are assumed to be uncorrelated random variables distributed with an arbitrary probability density $P(\epsilon_n)$, symmetric around zero for $1\leq n\leq N+1$, and to be identically zero outside this region. Similarly, $\{V_{n,n+1}\}$ are taken to be uncorrelated random variables for $1\leq n\leq N$, and simply equal to a constant V_0 outside this region. Thus the system consists of a disordered segment containing $N+1$ atoms and of length Nd (where d is the lattice spacing), embedded in an infinite, perfectly conducting, ordered chain. If we write the solutions of the time-independent Schrödinger equation in the tight-binding basis, the eigenvalue equation for the wave-function amplitudes a_n may be summarized in terms of a 2×2 matrix which we call the promotion matrix, as follows:

$$P^{(n)}\begin{bmatrix}a_n\\ a_{n-1}\end{bmatrix}=\begin{bmatrix}\frac{E-\epsilon_n}{V_{n,n+1}} & \frac{-V_{n-1,n}}{V_{n,n+1}}\\ 1 & 0\end{bmatrix}\begin{bmatrix}a_n\\ a_{n-1}\end{bmatrix}=\begin{bmatrix}a_{n+1}\\ a_n\end{bmatrix}. \tag{2}$$

Therefore the real-space wave-function amplitudes at each end of the disordered segment are related by

$$\begin{bmatrix}a_{N+2}\\ a_{N+1}\end{bmatrix}=\left[\prod_{i=1}^{N+1}P^{(i)}\right]\begin{bmatrix}a_1\\ a_0\end{bmatrix}\equiv P_N\begin{bmatrix}a_1\\ a_0\end{bmatrix}. \tag{3}$$

The Landauer formula for the resistance of a finite one-dimensional system embedded in a perfect conductor is

$$r=(2\pi\hbar/e^2)R/T\equiv(2\pi\hbar/e^2)\rho,$$

where R and T are the total reflection and transmission coefficients of the disordered region.[1,2] We now derive a useful expression for ρ in terms of the matrix elements of P_N by relating P_N to the transfer matrix T_N.

The transfer matrix is defined so as to relate the amplitudes of the solutions of the Schrödinger equation in the asymptotic (ordered) regions on either side of the disordered segment. These solutions are

$$a_n=Ae^{ikdn}+Be^{-ikdn},\quad -\infty<n\leq 1 \tag{4a}$$

$$a_n = Ce^{ikdn} + De^{-ikdn}, \quad n \geq N+1 \tag{4b}$$

with

$$E = 2V_0 \cos(kd)$$

and we define T_N by

$$T_N \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} D \\ C \end{bmatrix} . \tag{5}$$

By relating T_N to the scattering matrix one finds that, by definition

$$| T_N^{12} |^2 = \rho , \tag{6}$$

so we have merely to relate T_N to P_N to get the desired result. Equations (4) give us a relation between a_{N+2}, a_{N+1} and C,D of the form

$$\begin{bmatrix} a_{N+2} \\ a_{N+1} \end{bmatrix} = \begin{bmatrix} e^{-ikd} & e^{ikd} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-ik(N+1)d} & 0 \\ 0 & e^{ik(N+1)d} \end{bmatrix} \begin{bmatrix} D \\ C \end{bmatrix}$$

$$\equiv S\theta^{-1} \begin{bmatrix} D \\ C \end{bmatrix} \tag{7}$$

and a similar relation for a_1, a_0 and A,B with the phase matrix θ^{-1} simply replaced by the unit matrix. Substitution of these relations into (3) gives the result

$$T_N = \theta S^{-1} P_N S \tag{8}$$

where we note that $S^{-1} \neq S^+$ in general. From (8) and (6) we find, using the relation $\det(P_N) = 1$, the general expression for the resistance of any one-dimensional Anderson model

$$\rho = \frac{1}{4 - (E^2/V_0^2)} \left[(P_N^{11})^2 + (P_N^{12})^2 + (P_N^{21})^2 + (P_N^{22})^2 + \frac{E}{V_0}(P_N^{11} - P_N^{22})(P_N^{12} - P_N^{21}) - \left[\frac{E^2}{V_0^2} \right] P_N^{21} P_N^{12} - 2 \right] . \tag{9}$$

This expression is very convenient for performing both analytic and numerical calculations on the Anderson model which we proceed to do in the following sections.

III. ARITHMETIC MEAN OF ρ IN THE GENERAL CASE

A. $E = 0$

Averaging Eq. (9) gives an expression for the arithmetic mean, $\langle \rho \rangle$, in terms of the quantities $\langle P_N^{ij} P_N^{kl} \rangle$, which can be determined by a recursive method based on Eqs. (2) and (3). We first consider the average of Eq. (9) for arbitrary diagonal and off-diagonal disorder at $E = 0$. Equation (3) implies the recursion relations

$$P_n^{11} = \frac{(E - \epsilon_n)}{V_{n,n+1}} P_{n-1}^{11} - \frac{V_{n-1,n}}{V_{n,n+1}} P_{n-2}^{11} , \tag{10a}$$

$$P_n^{21} = P_{n-1}^{11} , \tag{10b}$$

$$P_n^{12} = \frac{(E - \epsilon_n)}{V_{n,n+1}} P_{n-1}^{12} - \frac{V_{n-1,n}}{V_{n,n+1}} P_{n-2}^{12} , \tag{10c}$$

$$P_n^{22} = P_{n-1}^{12} . \tag{10d}$$

If we define

$$F_n = \langle (P_n^{11})^2 \rangle, \quad f_n = \langle (P_n^{12})^2 \rangle , \tag{11}$$

and set $E = 0$, then (9) takes the form

$$\langle \rho \rangle = \tfrac{1}{4}[F_N + F_{N-1} + f_N + f_{N-1}] - \tfrac{1}{2} . \tag{12}$$

From squaring Eqs. (10) and averaging we obtain the same recursion relation for F_n and f_n which can be written in matrix form as

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} \langle \epsilon^2 \rangle \left\langle \frac{1}{V^2} \right\rangle & \langle V^2 \rangle \left\langle \frac{1}{V^2} \right\rangle \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} , \tag{13}$$

where we have used $\langle \epsilon \rangle = 0$ and the fact that P_{n-1}^{11} depends only on $\epsilon_{n-1} \cdots \epsilon_1$ and $V_{n-1,n} \cdots V_{1,2}, V_0$. Denoting the 2×2 recursion matrix in (13) by R, Eq. (3) implies that

$$\begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix} = R^{N-2} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix} . \tag{14}$$

Some care must be taken in the last iteration of the recursion relation since the promotion matrix at the boundary, $P^{(N+1)}$, depends on a nonrandom variable $V_{N+1,N+2} = V_0$ which is not to be averaged. The eigenvalues and eigenvectors of R are found to be

$$\lambda_\pm = \tfrac{1}{2} \langle \epsilon^2 \rangle \left\langle \frac{1}{V^2} \right\rangle \pm \left[\left[\frac{\langle \epsilon^2 \rangle}{2} \left\langle \frac{1}{V^2} \right\rangle \right]^2 + \langle V^2 \rangle \left\langle \frac{1}{V^2} \right\rangle \right]^{1/2} \tag{15a}$$

$$U_+ = \begin{bmatrix} 1 \\ 1/\lambda_+ \end{bmatrix}, \quad U_- = \begin{bmatrix} 1 \\ 1/\lambda_- \end{bmatrix}. \tag{15b}$$

By expanding the initial vector of (14) as

$$\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \alpha U_+ + \beta U_- , \tag{16}$$

we see that

$$\begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix} = \alpha(\lambda_+)^{N-2} U_+ + \beta(\lambda_-)^{N-2} U_- , \tag{17}$$

where f_{N-1}, f_{N-2} may be determined in an exactly analogous manner. Equations (15)–(17) together with a careful treatment of the averages of promotion matrices at the boundaries which contain nonrandom variables, yield the desired exact general scaling law for $\langle \rho \rangle$:

$$\langle \rho \rangle = \frac{1}{4(\lambda_+ - \lambda_-)} \left[\left[1 + \frac{\langle \epsilon^2 \rangle}{V_0^2} \right] (\lambda_+^{N+1} - \lambda_-^{N+1}) + \left[\frac{\langle V^2 \rangle}{V_0^2} + V_0^2 \left\langle \frac{1}{V^2} \right\rangle + \langle \epsilon^2 \rangle \left\langle \frac{1}{V^2} \right\rangle \right] (\lambda_+^N - \lambda_-^N) + \langle V^2 \rangle \left\langle \frac{1}{V^2} \right\rangle (\lambda_+^{N-1} - \lambda_-^{N-1}) \right] - \frac{1}{2} . \tag{18}$$

The salient features of this result are as follows. First, the eigenvalue λ_+ is always greater than or equal to the quantity $\langle V^2 \rangle \langle 1/V^2 \rangle$, which in turn is greater than unity for any normalizable probability distribution of finite width by the Schwartz inequality. Since $\langle \epsilon^2 \rangle > 0$ for any degree of diagonal disorder, we see that $\lambda_+ > 1$ for any type or degree of disorder. Also λ_- is always negative and $|\lambda_-| \leq \lambda_+$. Therefore, in the limit $N \to \infty$ Eq. (18) gives the result that in all cases the mean resistance grows exponentially with $\langle \rho \rangle \propto \exp(N \ln \lambda_+)$. Second, $\langle \rho \rangle$ depends only on the moments $\langle \epsilon^2 \rangle$, $\langle V^2 \rangle$, and $\langle V^{-2} \rangle$ of $P(\epsilon)$ and $Q(V)$ so we see that our result is relatively insensitive to the precise nature of these distributions. Third, in the limit of weak disorder as $N \to \infty$, Eq. (18) reduces to

$$\langle \rho \rangle = \tfrac{1}{2}(e^{N \ln \lambda_+} - 1) \tag{19}$$

which is the form of the result which Landauer[1] and Anderson *et al.*[4] derived for the independent scattering model which is a continuum model. That the result of Landauer and Anderson *et al.* should correspond to the limit of weak disorder was already noted in Ref. 4 since their approach requires that the elastic mean free path (which is the wave-function localization length in one dimension) be much greater than the length at which the relative phases of successive scatterers are randomized. A related point is that the continuum limit of our result is obtained by letting $N \to \infty$, $d \to 0$ while the physical lengths $L = Nd$ and $\xi = d/\ln \lambda_+$ remain constant; but we see that for ξ to remain constant as $d \to 0$, λ_+ must go to unity as it does in the limit of weak disorder, so the continuum limit of this discrete model is the limit in which the Landauer result holds, as one might expect.

For purely diagonal disorder which has been considered by Abrahams and Stephen,[5] $|\lambda_-| < 1$ and so terms proportional to $(\lambda_-)^N$ decay exponentially. Dropping these terms and setting $\langle V^2 \rangle \times \langle V^{-2} \rangle = 1$ gives

$$\langle \rho \rangle_{\text{diag}} = \frac{1 + [(\langle \epsilon^2 \rangle / 2V_0)^2 + 1]^{1/2}}{\{4[(\langle \epsilon^2 \rangle / 2V_0)^2 + 1]^{1/2}\}} \exp\left((N+1) \ln \left\{ \frac{\langle \epsilon^2 \rangle}{2V_0} + \left[1 + \left[\frac{\langle \epsilon^2 \rangle}{2V_0} \right]^2 \right]^{1/2} \right\} \right) - \frac{1}{2} . \tag{20}$$

For purely off-diagonal disorder $-\lambda_- = \lambda_+ = [\langle V^2 \rangle \langle (1/V^2) \rangle]^{1/2}$ and (18) gives the result

$$\langle \rho \rangle = \frac{1}{2} \left[\left[\langle V^2 \rangle \left\langle \frac{1}{V^2} \right\rangle \right]^{N/2} - 1 \right], \quad N \text{ even} \tag{21a}$$

$$\langle \rho \rangle = \frac{1}{4} \left[\left[\frac{\langle (1/V^2) \rangle^{-1}}{V_0^2} + \frac{V_0^2}{\langle V^2 \rangle} \right] \left[\langle V^2 \rangle \left\langle \frac{1}{V^2} \right\rangle \right]^{N/2} \right] - \frac{1}{2}, \quad N \text{ odd} . \tag{21b}$$

The difference between chains with an even or odd number of atoms arises from different boundary scattering in the two cases. Again, we note that $\ln\langle\rho\rangle \propto N$ as $N\to\infty$, even for purely off-diagonal disorder at $E=0$ where numerical results[10] have indicated $\ln\langle\rho\rangle \propto N^{1/2}$. We suggest that this disagreement is due to the fact that $\langle\rho\rangle$ is skewed by a few very large very improbable values which will not show up at all in numerical averaging over an ensemble of reasonable size. This point will be discussed in detail in Sec. V.

B. $E\neq 0$

Next we consider the case $E\neq 0$ which requires treating Eq. (9) in its full generality. Besides generalizing the recursion relation for $\langle(P_n^{ij})^2\rangle$, there are in principle five new quantities $\langle P_n^{ij}P_n^{kl}\rangle$ for which recursion relations may be derived. However, this treatment may be simplified in several ways. First, since P_n^{12} satisfies the same recursion relation as P_n^{11}, and $P_n^{21}=P_{n-1}^{11}$ and $P_n^{22}=P_{n-1}^{12}$, the only difference between $\langle(P_n^{11})^2\rangle$ and $\langle(P_n^{12})^2\rangle$, $\langle(P_{n+1}^{22})^2\rangle$, $\langle(P_{n+1}^{21})^2\rangle$ consists in different initial values $\langle(P_n^{ij})^2\rangle$, which is a boundary effect. Since we will only concern ourselves with the rate of exponential growth of $\langle\rho\rangle$ we neglect this difference and simply consider $\langle(P_n^{11})^2\rangle$. Also, neglecting boundary effects, $\langle P_n^{12}P_n^{22}\rangle=\langle P_n^{11}P_n^{21}\rangle$ and $\langle P_n^{21}P_n^{22}\rangle=\langle P_{n-1}^{11}P_{n-1}^{21}\rangle$. Finally, again since P_n^{11} and P_n^{12} satisfy the same recursion relations, neglecting boundary effects, $\langle P_n^{11}P_n^{12}\rangle$ satisfies the same recursion relations as $\langle(P_n^{11})^2\rangle$ and $\langle P_n^{11}P_n^{22}\rangle$ the same as $\langle P_n^{11}P_n^{21}\rangle=\langle P_n^{11}P_{n-1}^{11}\rangle$. Thus the only two quantities required to get the growth rate of $\langle\rho\rangle$ are $\langle(P_n^{11})^2\rangle$ and $\langle P_n^{11}P_{n-1}^{11}\rangle$ (so at this point we suppress the matrix indices):

$$\langle(P_n)^2\rangle=\left\langle\left[\frac{(E-\epsilon_n)}{V_{n,n+1}}P_{n-1}-\frac{V_{n-1,n}}{V_{n,n+1}}P_{n-2}\right]^2\right\rangle$$
$$=(E^2+\langle\epsilon^2\rangle)\left\langle\frac{1}{V^2}\right\rangle\langle(P_{n-1})^2\rangle+\langle V^2\rangle\left\langle\frac{1}{V^2}\right\rangle\langle(P_{n-2})^2\rangle-2E\left\langle\frac{1}{V^2}\right\rangle\langle V_{n-1,n}P_{n-1}P_{n-2}\rangle\ . \quad (22)$$

Since P_{n-1} depends on $V_{n-1,n}$ we cannot immediately factor the last average, i.e.,

$$\langle V_{n-1,n}P_{n-1}P_{n-2}\rangle\neq\langle V\rangle\langle P_{n-1}P_{n-2}\rangle\ .$$

However, calculation of the relevant recursion relations shows that $\langle V_nP_nP_{n-1}\rangle=\langle 1/V\rangle^{-1}\langle P_nP_{n-1}\rangle$ and $\langle P_nP_{n-1}\rangle$ satisfies the recursion relation

$$\langle P_nP_{n-1}\rangle=E\left\langle\frac{1}{V}\right\rangle\langle(P_{n-1})^2\rangle-\langle P_{n-1}P_{n-2}\rangle \quad (23)$$

so we can write the recursion relations (22) and (23) in the form

$$\begin{bmatrix}\langle(P_n)^2\rangle\\ \langle(P_{n-1})^2\rangle\\ \langle P_nP_{n-1}\rangle\end{bmatrix}=\begin{bmatrix}(E^2+\langle\epsilon^2\rangle)\left\langle\frac{1}{V^2}\right\rangle & \langle V^2\rangle\left\langle\frac{1}{V^2}\right\rangle & -2E\left\langle\frac{1}{V}\right\rangle^{-1}\left\langle\frac{1}{V^2}\right\rangle\\ 1 & 0 & 0\\ E\left\langle\frac{1}{V}\right\rangle & 0 & -1\end{bmatrix}\begin{bmatrix}\langle(P_{n-1})^2\rangle\\ \langle(P_{n-2})^2\rangle\\ \langle P_{n-1}P_{n-2}\rangle\end{bmatrix}\ . \quad (24)$$

The eigenvalues of this recursion matrix are the roots of the cubic equation

$$C(\lambda)=\left[\lambda^3+\left[1-(E^2+\langle\epsilon^2\rangle)\left\langle\frac{1}{V^2}\right\rangle\right]\lambda^2+\left[\left\langle\frac{1}{V^2}\right\rangle(E^2-\langle\epsilon^2\rangle-\langle V^2\rangle)\right]\lambda-\langle V^2\rangle\left\langle\frac{1}{V^2}\right\rangle\right]=0\ . \quad (25)$$

Since

$$C(1)=2\left[1-\langle V^2\rangle\left\langle\frac{1}{V^2}\right\rangle-\langle\epsilon^2\rangle\left\langle\frac{1}{V^2}\right\rangle\right]<0$$

and $C(\lambda\to\infty)>0$, there must always exist a positive root greater than unity, λ_+, which leads to exponential growth with $\langle\rho\rangle\propto\exp[N\ln(\lambda_+)]$. A simple expression for λ_+ may be obtained in the

limit of weak disorder but for arbitrary energy by linearizing Eq. (25) around $\lambda=1$, which is the relevant root for zero disorder. We write $\lambda_+=1+\delta\lambda$, $V=V_0+\delta V$ and insert these expressions into (25), keeping lowest-order terms in $\delta\lambda$, $\langle(\delta V)^2\rangle$, and $\langle\epsilon^2\rangle$. Solving for $\delta\lambda$ gives

$$\lambda_+=1+\delta\lambda$$
$$=1+\frac{2}{4-E^2/V_0^2}\left[\frac{\langle\epsilon^2\rangle}{V_0^2}+\frac{4\langle(\delta V)^2\rangle}{V_0^2}\right], \tag{26}$$

where in this case we have assumed $\langle\delta V\rangle=0$. Thus, expanding $\ln(1+\delta\lambda)$,

$$\langle\rho\rangle\propto\exp\left[N\frac{2(\langle\epsilon^2\rangle/V_0^2+4\langle\delta V^2\rangle/V_0^2)}{4-E^2/V_0^2}\right]. \tag{27}$$

A more careful treatment of the case of weak diagonal disorder with $E\neq0$ analogous to our exact treatment of the $E=0$ case yields the result

$$\langle\rho(E)\rangle_{\text{diag}}=\tfrac{1}{2}\exp\left[N\ln\left[1+\frac{2\langle\epsilon^2\rangle/V_0^2}{4-E^2/V_0^2}\right]\right], \tag{28}$$

where we have neglected terms of order unity.

We see that the resistance rises more rapidly as we approach $E=2V_0$, reflecting the greater localization of the states near the band edge. At the band edge, R/T blows up since there exist no normalizable eigenstates with $E>2V_0$ which have finite amplitudes in both ordered regions.

We have seen that examination of the arithmetic mean indicates the existence of localization with $\ln\langle\rho\rangle\propto N$ in all cases, even for purely off-diagonal disorder at $E=0$ [see Eq. (21)]. However, for these systems as noted earlier, the arithmetic mean does *not* in general represent a "typical" value for the resistance. One indication of this behavior may be obtained by an examination of $\langle\rho^2\rangle$; if it grows more rapidly with N than $\langle\rho\rangle^2$ then the distribution becomes broader and broader relative to the mean as $N\to\infty$. We can calculate $\langle\rho^2\rangle$ by the recursion method, and at $E=0$ it is determined by the recursion relation

$$\begin{bmatrix}\langle(P_n^{11})^4\rangle\\ \langle(P_{n-1}^{11})^4\rangle\\ \langle(P_n^{11}P_{n-1}^{11})^2\rangle\end{bmatrix}=\begin{bmatrix}\langle\epsilon^4\rangle\left\langle\frac{1}{V^2}\right\rangle & \langle V^2\rangle\left\langle\frac{1}{V^4}\right\rangle & 6\langle\epsilon^2\rangle\left\langle\frac{1}{V^2}\right\rangle^{-1}\left\langle\frac{1}{V^2}\right\rangle\\ 1 & 0 & 0\\ \langle\epsilon^2\rangle\left\langle\frac{1}{V^2}\right\rangle & 0 & 1\end{bmatrix}\begin{bmatrix}\langle(P_{n-1}^{11})^4\rangle\\ \langle(P_{n-1}^{11})^4\rangle\\ \langle(P_{n-1}^{11}P_{n-2})^2\rangle\end{bmatrix}. \tag{29}$$

The eigenvalues are the roots of the cubic equation

$$0=\Gamma^3-\left[\langle\epsilon^4\rangle\left\langle\frac{1}{V^4}\right\rangle+1\right]\Gamma^2$$
$$+\left[\left\langle\frac{1}{V^4}\right\rangle(\langle\epsilon^4\rangle-\langle V^4\rangle-6\langle\epsilon^2\rangle^2)\right]\Gamma$$
$$+\langle V^4\rangle\left\langle\frac{1}{V^4}\right\rangle. \tag{30}$$

Solving for the largest root Γ_+ in the limit of weak disorder yields

$$\Gamma_+=1+\sqrt{3}\langle\epsilon^2\rangle+\frac{8\langle(\delta V)^2\rangle}{V_0^2}, \tag{31}$$

whereas in the limit of weak disorder at $E=0$ the largest eigenvalue of Eq. (15) satisfies

$$(\lambda_+)^2=\left[\left[\langle V^2\rangle\left\langle\frac{1}{V^2}\right\rangle\right]^{1/2}+\frac{\langle\epsilon^2\rangle}{2V_0^2}\right]^2$$
$$=1+\frac{\langle\epsilon^2\rangle}{V_0^2}+4\frac{\langle(\delta V)^2\rangle}{V_0^2}, \tag{32}$$

so $\Gamma_+>\lambda_+^2$ and

$$\langle\rho^2\rangle/\langle\rho\rangle^2\propto\exp\left[N\ln\left[\frac{\Gamma_+}{\lambda_+^2}\right]\right],$$

which diverges as $N\to\infty$. In addition, for the case of purely diagonal disorder, Γ is determined by the simpler equation

$$\Gamma^3-(1+\langle\epsilon^4\rangle/V_0^4)\Gamma^2$$
$$+(\langle\epsilon^4\rangle/V_0^4-1-6\langle\epsilon^2\rangle^2/V_0^4)\Gamma+1=0. \tag{33}$$

For a Gaussian distribution $P(\epsilon)$, $\langle\epsilon^4\rangle=3\langle\epsilon^2\rangle$, and this cubic factors with negative one as a root. Then one can obtain a simple expression for Γ_+ for *any* degree of disorder

$$\Gamma_+=1+\frac{\langle\epsilon^4\rangle}{2}+\left[\langle\epsilon\rangle^4+\frac{\langle\epsilon^4\rangle^2}{4}\right]^{1/2}. \quad (34)$$

One can verify from Eqs. (15) and (34) that in this case $\Gamma_+/\lambda_+^2>1$ for any degree of disorder. Because of this poor behavior of the distribution of ρ as $N\rightarrow\infty$ we shall see in Sec. V that it is really the geometric and not the arithmetic mean of ρ which gives a value of ρ representative of the distribution. In addition, it is convenient to study not $\langle\ln\rho\rangle$ but $\langle\ln(1+\rho)\rangle$, which is non-negative and will have no pathologies associated with the perfectly conducting ($\rho=0$) members of the ensemble. We study this quantity numerically in Sec. IV.

IV. NUMERICAL STUDIES

A. Off-diagonal disorder for $E\neq0$

The numerical work of the next two sections is primarily oriented toward answering the specific question of whether and for what range of parameters the behavior $\langle\ln(1+\rho)\rangle\propto\sqrt{N}$ persists if we consider off-diagonal disorder away from band center, or off-diagonal disorder mixed with diagonal disorder at $E=0$. It is not intended as an exhaustive numerical study of the statistical properties of the Anderson model with general disorder. Our calculations are again based on Eq. (9) for ρ in terms of the matrix elements of the promotion matrix. We computed numerically the promotion matrices of an ensemble of chains of varying lengths N, and then calculated the quantities $\langle\ln(1+\rho)\rangle$ and $\ln(\langle1+\rho\rangle)$, where the angle bracket denotes the ensemble average. Off-diagonal disorder was introduced into the promotion matrices by randomly generating the hopping matrix elements $V_{n,n+1}$ according to a rectangular probability distribution of width $2W_V$ centered around V_0 which we set equal to unity. Our basic result is that $\langle\ln(1+\rho)\rangle\propto N$ as $N\rightarrow\infty$ for all $E\neq0$ even down to $E=10^{-4}V_0$, so that the behavior of the geometric mean for off-diagonal disorder at $E=0$ does appear to be a very special case, at least in one dimension. In Fig. 1 we present a plot of $\langle\ln(1+\rho)\rangle$ versus N with $W_V=0.25$ for several nonzero values of E. Note that for all nonzero energies even for $E=0.01$ we get a very precise linear dependence down to $N=100$. Qualitatively, this kind of behavior for $E\neq0$ is not too surprising, as may be seen by a consideration of the recursion relations for P_n^{11}. For diagonal disorder at $E=0$ the recursion relation is $P_n^{11}=-(\epsilon_n/V_0)P_{n-1}^{11}-P_{n-2}^{11}$ which has already been shown numerically to lead to the behavior $\langle\ln(1+\rho)\rangle\propto N$.[3] For off-diagonal disorder at $E\neq0$ the recursion relation is

$$P_n^{11}=\frac{E}{V_{n,n+1}}P_{n-1}^{11}-\frac{V_{n-1,n}}{V_{n,n+1}}P_{n-2}^{11}.$$

If one considers $E=V_0=1$ and takes a rectangular distribution of ϵ_n of width $W_\epsilon=1$ centered at zero and a similar distribution of $V_{n,n+1}$ centered at one, then trivially the random variable $E/V_{n,n+1}$ is always positive and greater than ϵ_n/V; thus one expects P_n^{11} for off-diagonal disorder to grow faster than for diagonal disorder for equal degrees of disorder and E of order unity. It does not seem obvious however, that for $E<<1$ we should get such precise linear dependence, and it would appear that more powerful mathematical techniques such as those employed by O'Connor[12] are necesary to show this, as will be discussed in the following section.

The slope of the lines in Fig. 1 is twice the inverse localization length and it is a monotonically increasing function of energy as expected, since the states should be more strongly localized nearer the band edge. The results are consistent with the energy dependence $(1/N)\langle\ln(1+\rho)\rangle\propto1/(4-E^2)$, which is the analytically derived energy dependence of $(1/N)\ln(\langle1+\rho\rangle)$ for weak disorder.

At $E=0$ we do get the result $\langle\ln(1+\rho)\rangle\propto\sqrt{N}$, in good agreement with the analytic results of Ref. 9 and the numerical results of Ref. 10. These results are shown in Fig. 2 (the solid line is the analytic result). The analytic result is only exact as $N\rightarrow\infty$, and we found that for weak disorder ($W_V=0.05$) we had to go to chains as long as 10 000 atoms before a clear square-root dependence was found. The behavior $\langle\ln(1+\rho)\rangle\propto\sqrt{N}$ is reached much more rapidly for large disorder ($W_V\geq0.5$), but then the onset of linear dependence for finite E also occurs more quickly. We note also that very large off-diagonal disorder ($W_V>1$) is somewhat unrealistic physically since in such a situation the chains have a large probability density near zero coupling which, for example, causes the arithmetic mean of ρ to diverge. It is significant that there is substantially greater scatter in the

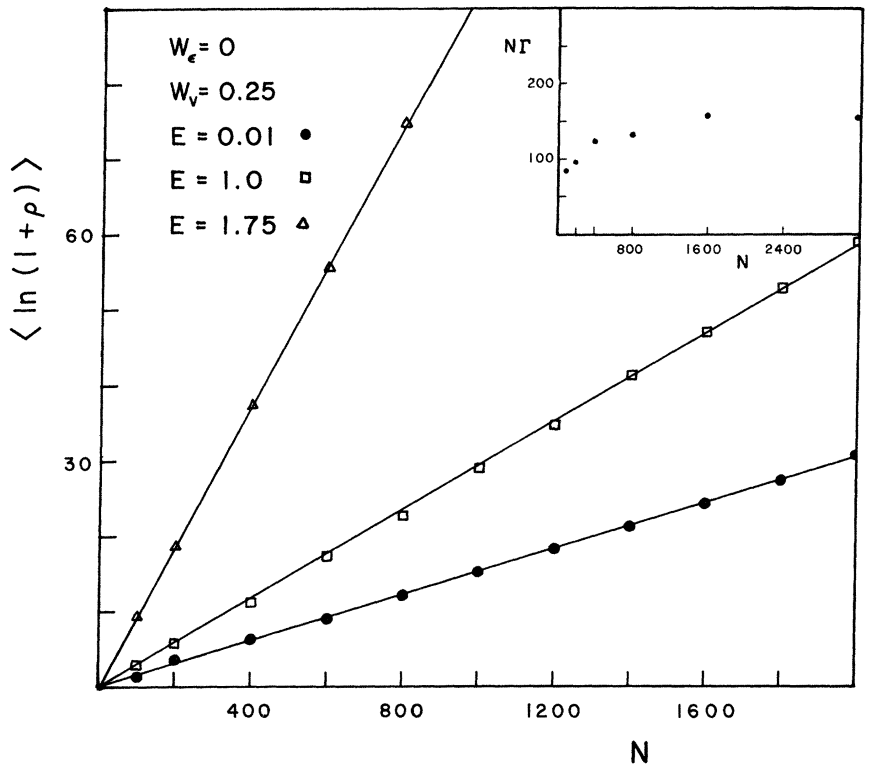

FIG. 1. Plot of $\langle \ln(1+\rho) \rangle$ vs N for a 1D Anderson model with purely off-diagonal disorder for several nonzero values of energy. The results are for $W_V=0.25$ and an ensemble of 500 chains. The inset plots N times the relative variance $\Gamma=\langle [\ln(1+\rho)-\langle \ln(1+\rho) \rangle]^2 \rangle / \langle \ln(1+\rho) \rangle^2$ vs N. Averages are over an ensemble of 500 chains.

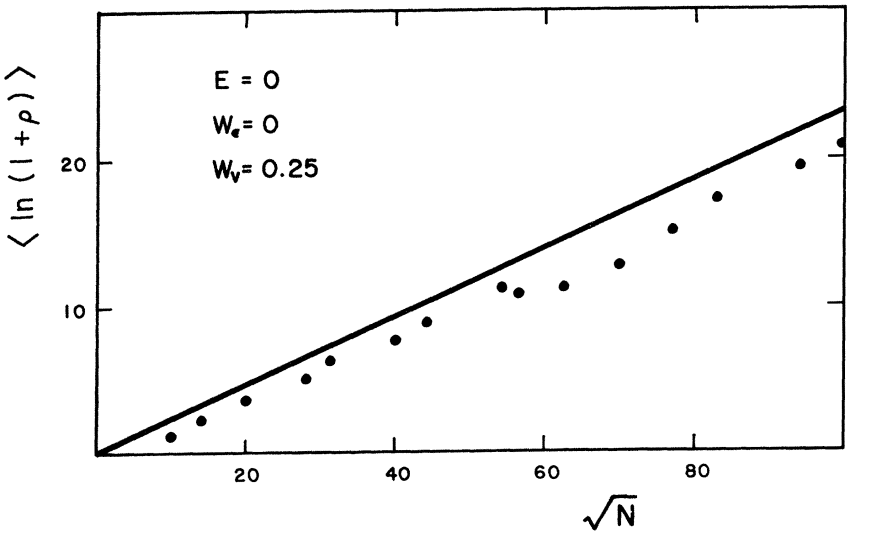

FIG. 2. Plot of $\langle \ln(1+\rho) \rangle$ vs $\sqrt{N}$ for purely off-diagonal disorder and $E=0$ for $W_V=0.25$ and an ensemble of 500 systems. The solid line is the analytic result of Ref. 9 which is only exact at $N \rightarrow \infty$.

data at $E=0$ than for $E\neq 0$. It was shown in Ref. 9 that for off-diagonal disorder at $E=0$, the relative variance of $\ln(1+\rho)$, which is the ratio of the variance to the square of the mean, was independent of N, unlike the case for a Gaussian distribution with nonzero mean, where this ratio decreases as N^{-1}. We computed this ratio, Γ, given by

$$\Gamma=\frac{\langle[\ln(1+\rho)-\langle\ln(1+\rho)\rangle]^2\rangle}{\langle\ln(1+\rho)\rangle^2}$$

for $E\neq 0$ and found that it did in fact decrease as N^{-1} as $N\to\infty$ (see inset to Fig. 1), whereas for $E=0$ it converged to the analytic result $(\Pi/2-1)$ and remained constant as N increased. This behavior was reflected in the variation of $\langle\ln(1+\rho)\rangle$ from ensemble to ensemble. We found that for ensembles of 500 chains of length 3200, the value of $\langle\ln(1+\rho)\rangle$ typically varied by only $\pm 0.5\%$ from ensemble to ensemble for $E=0.5$, whereas for $E=0$ it varied by $\pm 5\%$.

The behavior of the $\ln(\langle 1+\rho\rangle)$ was much worse, as expected. For an ensemble of 1000 chains of length 400 at $E=0$ and $W_V=0.25$ the value of $\ln(\langle 1+\rho\rangle)$ typically varied by $\pm 15\%$ as compared to $\pm 2\%$ for $\langle\ln(1+\rho)\rangle$. The growth of $\ln(\langle 1+\rho\rangle)$ was approximately linear with N and in fair agreement with the analytic result Eq. (21) up until $N=400$, at which point it begins to grow more slowly, roughly as $\sqrt{N}$ with large fluctuations. The explanation for this behavior is that as N increases the numerical averaging procedure is seeing less of the long tail of the distribution, as will be discussed in detail in Sec. V.

B. General disorder at $E=0$

We next consider $\langle\ln(1+\rho)\rangle$ numerically for the Anderson model with both off-diagonal and diagonal disorder at $E=0$. Again, previous work on the Anderson model with only diagonal disorder suggested that we should find $\langle\ln(1+\rho)\rangle\propto N$ for large diagonal disorder, and similar to our previous results we find that even a small amount of diagonal disorder gives a linear dependence on N. In Fig. 3 we plot $\langle\ln(1+\rho)\rangle$ versus N for $W_V=0.25$ and various values of W_ϵ; note the linear dependence down to $W_\epsilon=0.01$. We found that as we increased W_V the linear behavior could be seen clearly for even smaller values of W_ϵ. The inverse localization length $(1/2N)\langle\ln(1+\rho)\rangle$ grew more rapidly with W_ϵ than it did with energy in the case of purely off-diagonal disorder, as one might expect (until very near the band edge). We also calculated

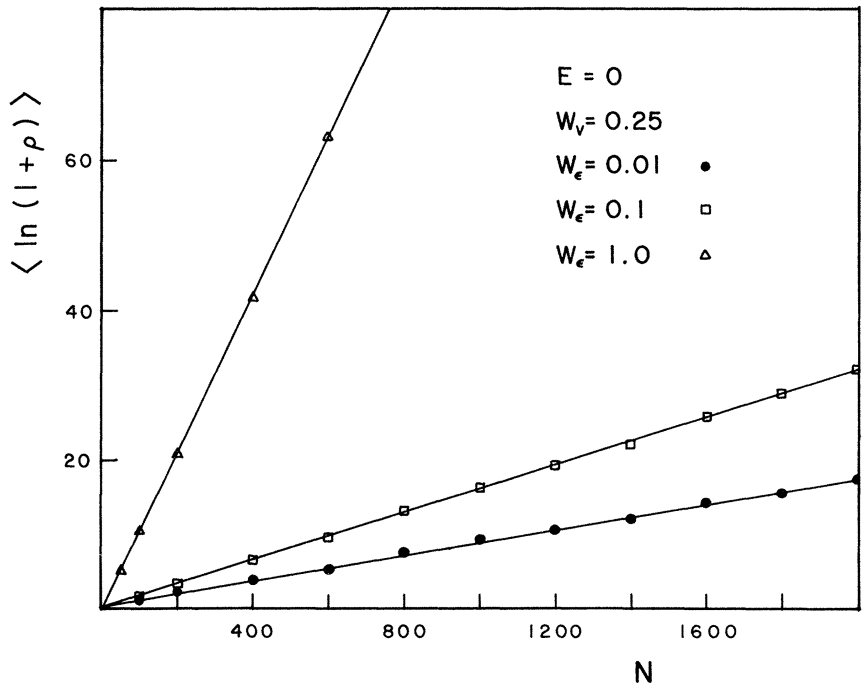

FIG. 3. Plot of $\langle\ln(1+\rho)\rangle$ versus N for both diagonal and off-diagonal disorder at $E=0$. Results are for an ensemble of 400 chains with $W_V=0.25$ and several values of W_ϵ.

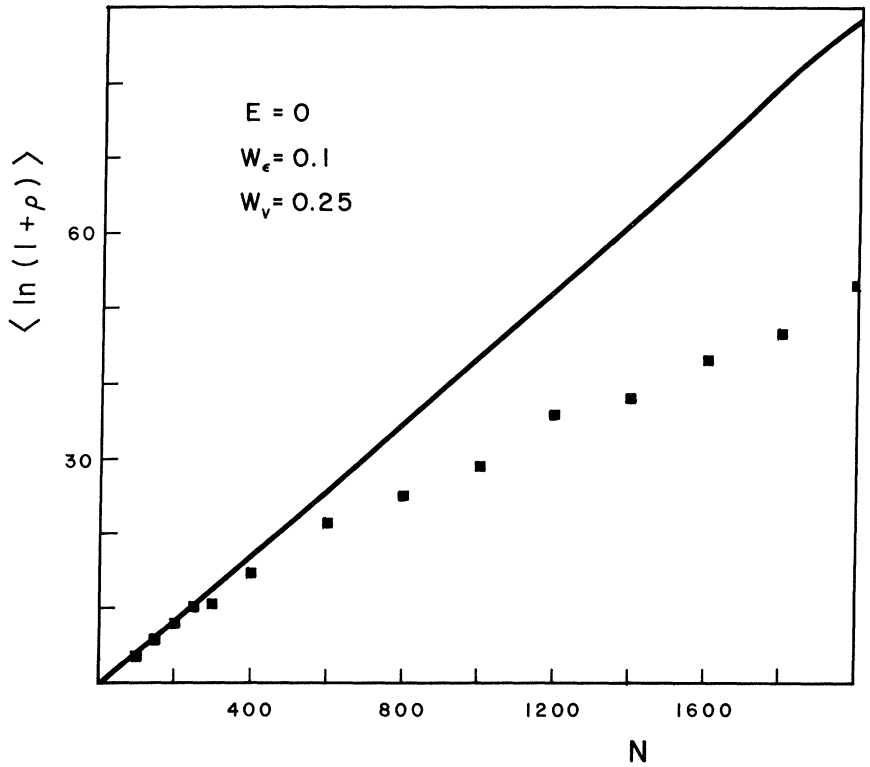

FIG. 4. Plot of $\ln(\langle 1+\rho\rangle)$ vs N for both diagonal and off-diagonal disorder at $E=0$. The solid curve is the analytic result of Eq. (41). The squares are the numerically computed average. Averages were taken over an ensemble of 1000 chains.

$\ln(\langle 1+\rho\rangle)$ with rather interesting results as is shown in Fig. 4. We found good agreement with the analytic result of Eq. (18) (the solid line) for N up to 250, after which we find that the numerical average is consistently less than the analytic result with a percentage deviation which increases with N. This is exactly the behavior one would expect based on the assumption that $\ln\rho$ is normally distributed so that a numerical averaging procedure underweights the long tail of the distribution in ρ. This kind of behavior is characteristic of one-dimensional disordered systems for reasons which we discuss in detail in Sec. V, and has been found, e.g., by Andereck and Abrahams[6] working on a different model. We discuss in our concluding remarks, however, that it may in fact be *physically* significant that an ill-converged average like the numerical average in Fig. 4 does not lead to a clear linear dependence on N.

V. STATISTICAL BEHAVIOR OF ρ

Let us summarize the results which have been obtained so far for the Anderson model with general disorder. It appears that $\langle\ln(1+\rho)\rangle \propto N$ as $N\rightarrow\infty$ in all cases except for purely off-diagonal disorder at $E=0$ where $\langle\ln(1+\rho)\rangle \propto \sqrt{N}$. It is also known analytically that the arithmetic mean satisifes $\ln\langle\rho\rangle \propto N$ in all cases despite numerical results indicating $\ln\langle\rho\rangle \propto \sqrt{N}$ for purely off-diagonal disorder at $E=0$. Finally, it appears that $\langle\rho^2\rangle/\langle\rho\rangle^2$ always diverges as $N\rightarrow\infty$, but that the relative variance of $\ln(1+\rho)$ goes to zero as N^{-1} in all cases except for purely off-diagonal disorder at $E=0$, where it is constant. All of these results may be understood by the hypothesis that in all cases $\ln\rho$ is approximately normally distributed for $\rho >> 1$, so that ρ has a lognormal distribution

$$P(\rho)=\frac{\rho^{-1}}{(2\pi N\sigma^2)^{1/2}}\exp\frac{-(\ln\rho-N\gamma)^2}{2N\sigma^2}\quad \rho >> 1, \tag{35}$$

and off-diagonal disorder at $E=0$ corresponds to the special case where $\gamma=0$.

There is a great deal of mathematical work on products of random matrices which suggests that $(1/N)\ln\rho$ should be a statistically well-behaved quantity[13]; however, there exists no general analytic proof that $\ln\rho$ is normally distributed for the Anderson model with arbitrary disorder, although this

has been shown in some special cases. Abrahams and Stephen have shown this for very large diagonal disorder[5] and their proof can be trivially generalized to both types of disorder at $E=0$. In this limit, the dominant contribution to ρ is obtained by dropping the second term in the recursion relations for $(P_N^{11})^2$ in (10):

$$(P_N^{11})^2 \simeq \frac{(\epsilon_N)^2}{(V_{N,N+1})^2}(P_{N-1}^{11})^2 \tag{36}$$

which implies

$$\ln(P_N^{11})^2 = \sum_{n=1}^{N} \ln\frac{(\epsilon_n)^2}{V_{n,n+1}} \tag{37}$$

and, since $\ln\rho \approx \ln(P_N^{11})^2$ in this approximation, by the central-limit theorem $\ln\rho$ is normally distributed and $P(\rho)$ is given by (35) with $\gamma = \langle \ln\epsilon^2 \rangle - \langle \ln V^2 \rangle$ and

$$\sigma^2 = \langle [\ln(\epsilon^2/V^2)]^2 - (\langle \ln\epsilon^2/V^2 \rangle]^2 \rangle \ .$$

Also, Stone and Joannopoulos[9] have shown that for purely off-diagonal disorder at $\epsilon=0$, and $\rho >> 1$

$$P(\rho) = \frac{2}{\rho}(8\pi N\sigma^2)^{-1/2} \exp\left[-\frac{(\ln 4\rho)^2}{8N\sigma^2} \right] \tag{38}$$

with $\sigma^2 = \langle (\ln V)^2 \rangle$.

In addition, a general proof of a central limit for a disordered harmonic chain which obeys the classical equation of motion has been given by O'Connor.[12] His proof applies with a little reinterpretation to the Anderson model with diagonal disorder and $E \neq 0$. In our notation what he shows is that $\ln[(P_N^{11})^2 + (P_N^{21})^2]$ is normally distributed as $N \to \infty$ with mean growing as $N\gamma$ and $\gamma > 0$. It seems quite possible that with some modifications these techniques may be applied to prove an analogous theorem for the logarithm of the more general bilinear forms appearing in Eq. (9), but this has not been accomplished as yet.[14] Note that even if it is true that for *any* bilinear form $t_N^2 = \sum^{ijkL} C_{ijkl} P_N^{ij} P_N^{kl}$ that $\ln(t_n^2)$ is normally distributed, the constant term in Eq. (9) means that ρ will *not* be normally distributed for $\rho \leq 2/(4 - E^2/V_0^2)$ (which is of order unity except very near the band edge). However, this will only make a difference if $\gamma = 0$ in Eq. (35), since otherwise almost all the support of $P(\rho)$ is far from unity. Finally, O'Connor's proof that $\gamma > 0$ does not apply to the case of off-diagonal disorder at $E=0$, where the random matrix products involved are always either diagonal or purely off-diagonal. In this case one can show that γ as defined by O'Connor is in fact zero, in agreement with Eq. (38).

If we assume (35) holds and the contribution to the moments of ρ from values of ρ near the origin is negligible, then the rth moment

$$\langle \rho^r \rangle = \exp(rN\gamma + \tfrac{1}{2}r^2 N\sigma^2) \ . \tag{39}$$

We see that $\ln\langle\rho\rangle \propto N$ even if $\gamma = 0$, and that $\langle\rho^2\rangle/\langle\rho\rangle^2 \to \infty$ as $N \to \infty$ in all cases. Moreover, if (35) holds, then $\langle \ln\rho \rangle \simeq \langle \ln(1+\rho) \rangle = \gamma N$ *except* when $\gamma = 0$. In this case the fact that $\ln\rho$ is not normally distributed as $\rho \to 0$ is crucial, since otherwise one would get $\langle \ln\rho \rangle = 0$. We have seen that $\gamma = 0$ for the Anderson model with purely off-diagonal disorder at $E=0$. For this case, it has been shown that $\rho = \frac{1}{4}(x + 1/x) - \frac{1}{2}$ where $\ln x$ is normally distributed with mean zero. Thus

$$\langle \ln(1+\rho) \rangle = \frac{1}{(8\pi N\sigma^2)^{1/2}} \int_{-\infty}^{\infty} \ln\left[\frac{(1-x)^2}{4x} \right] \times \frac{e^{-(\ln x)^2/8N\sigma^2}}{x} dx \tag{40}$$

letting $y = \ln x$, and keeping the dominant terms as $N \to \infty$ gives

$$\begin{aligned} \langle \ln(1+\rho) \rangle &= \frac{1}{(8\pi N\sigma^2)^{1/2}} \int_{-\infty}^{\infty} \ln(1+e^y) e^{-y^2/8N\sigma^2} dy \\ &= \frac{1}{(8\pi N\sigma^2)^{1/2}} \int_{0}^{\infty} \ln(1+e^y) e^{-y^2/8N\sigma^2} dy \\ &= \left[\frac{8\sigma^2}{\pi} \right]^{1/2} N^{1/2} \ . \end{aligned} \tag{41}$$

Note, we can drop the integral from $-\infty$ to 0 since the very small values of e^y which occur as a $y \to -\infty$ simply keep $\ln(1+e^y) \approx 0$. The integral from zero to infinity has been obtained exactly in the limit as $N \to \infty$ by squeezing it between an upper and lower bound.[9] Although it is not obvious from (40), one gets the same result for $\langle \ln\rho \rangle$ as $N \to \infty$. Thus, we see that the unusual behavior $\langle \ln(1+\rho) \rangle \propto \sqrt{N}$ for off-diagonal disorder at $E=0$ is perfectly consistent with the assumption that $\ln\rho$ is normally distributed for $\rho >> 1$, but corresponds to the special case where $\gamma = 0$. The results of Sec. IV suggest that this is the only case where $\gamma = 0$ in the Anderson model. This is consistent with the mathematical fact that off-diagonal disorder at $E=0$ is the *only* case where the promotion matrices generated by the Anderson model are always diagonal or off-diagonal.

Returning to the general case described by (35) with $\gamma \neq 0$, it is obvious that the geometric mean of ρ, $\exp(\langle \ln\rho \rangle) = e^{N\gamma}$, will be the median value of the distribution of ρ and will be representative of the distribution. To be precise, it is easy to show from (35) that for any positive ϵ

$$\text{Prob}(e^{N(\gamma+\epsilon)} > \rho > e^{N(\gamma-\epsilon)}) = 2\,\text{erf}\left[\frac{N^{1/2}}{\sigma^2}\epsilon\right] - 1 , \tag{42}$$

where $\text{erf}(x)$ is the error function defined by $\text{erf}(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-x^2/2} dx$. $\text{Erf}(x)$ converges to unity for positive arguments very rapidly, e.g., $[1 - erf(5)] < 5 \times 10^{-6}$. For off-diagonal disorder at $E = 0$ where $\langle \ln(1+\rho) \rangle \propto N^{1/2}$ the results of Ref. 9 imply the weaker result

$$\text{Prob}[\exp(2\sigma N^{1/2-\epsilon}) < \rho < \exp(2\sigma N^{1/2+\epsilon})] = 2[\text{erf}(N^{\epsilon}) - \text{erf}(N^{-\epsilon})] , \tag{43}$$

which again goes to unity as $N \to \infty$. Thus, it appears the geometric mean will always give a value of ρ representative of $P(\rho)$ despite the long tail characteristic of lognormal distributions which skews the arithmetic mean to be much larger than a typical value of ρ.

In fact the probability of finding a value of ρ greater than or equal to $\langle \rho \rangle$ decreases quite rapidly with N which means that numerical averaging over finite ensembles may miss these large but improbable values entirely. A simple calculation based on Eq. (38) shows that for off-diagonal disorder at $E = 0$ if one assumes a rectangular distribution of V with $2W = V_0 = 1$,

$$\text{Prob}(\rho \geq \langle \rho \rangle) = 2\{1 - \text{erf}[(0.45)\sqrt{N}\,]\} . \tag{44}$$

This is less than 10^{-6} for $N = 200$. So even for a chain as short as 200 atoms if one considers an ensemble of 1000 systems the probability of ever seeing a value as large as the arithmetic mean as found analytically is quite small; less than 1%. We believe this is the explanation for the numerical calculations which indicate that $\ln\langle \rho \rangle \propto \sqrt{N}$ for off-diagonal disorder. Exactly the same difficulty has been encountered in numerical work by Andereck and Abrahams[6] on a system of uniformly spaced delta-function potentials of random strength. In this case $\gamma \neq 0$, and the theory predicts $\ln(\langle 1+\rho \rangle) = 2\langle \ln(1+\rho) \rangle$, but numerical averaging finds that $\ln(\langle 1+\rho \rangle)$ is consistently less than $2\langle \ln(1+\rho) \rangle$ (see their Fig. 3). They correctly attribute this discrepancy to the fact that averaging finite ensembles will miss the long tail of $P(\rho)$ and show that an analytic calculation at $E = 0$ does give $\ln(\langle 1+\rho \rangle)$ equal to twice $\langle \ln(1+\rho) \rangle$ which they compute numerically. It is also worth noting that their results show that the discrepancy between the arithmetic mean as computed numerically and as calculated analytically *increases* with the length of the chains being averaged as we found in our numerical work and as our argument leading to (42) suggests it should.

Finally, we must say a brief word about averages of the conductance, $g = 1/\rho$, which have been performed numerically on the Anderson model using the Kubo formula with controversial results.[15,16] From (35) one sees that $\langle g \rangle = \exp[-N(\gamma - \frac{1}{2}\sigma^2)]$ so that $\langle g \rangle$ always decays more slowly than $1/\langle \rho \rangle$ with N and if $\gamma < \frac{1}{2}\sigma^2$, $\langle g \rangle$ will even grow which would mistakenly suggest extended states. For the independent scattering model studied by Landauer,[1] Anderson *et al.*,[4] and Andereck and Abrahams,[6] $\gamma = \frac{1}{2}\sigma^2$ and $\langle g \rangle$ is independent of N. In the Anderson model for off-diagonal disorder at $E = 0$ an analytic expression for $\langle g \rangle$ diverges for all N,[9] whereas the numerical calculations show only a weak tendency towards localization.[10] Clearly the arithmetic mean of the conductance is even a less representative quantity to study than $\langle \rho \rangle$. However, if one studies the geometric mean then trivially

$$\exp(\langle \ln g \rangle) = \exp(-\langle \ln\rho \rangle) = e^{-N\gamma} ,$$

which will be a value representative of the distribution of g by arguments similar to those given above.

VI. CONCLUSIONS

Thus, to summarize, all the known analytic and numerical results concerning the one-dimensional Anderson model with arbitrary disorder are consistent with and suggest the hypothesis that in all cases $P(\rho)$ is lognormally distributed for $\rho >> 1$ with off-diagonal disorder at $E = 0$ corresponding to the special case where $\gamma = 0$ in Eq. (35). We conclude that the behavior of $\ln\tilde{\rho}$ in this special case is not relevant to experiment. The results of this work shed some light on the question of whether it is the scale resistance, $\tilde{\rho}$, or the average resistance, $\langle \rho \rangle$, which will be measured for example in an experiment on the low-temperature resistance in a thin metallic wire with impurities. It is generally believed that such disordered systems are

better described as many-channel scattering systems than as the rigorously one-channel scatterer that we have studied above.[4] Setting aside this issue for the moment, let us suppose that the one-channel result was directly applicable to experiment. A zero-temperature scaling law may be related to experiment in the weakly localized regime, according to an argument due to Thouless,[17] by replacing the sample length by an "effective" sample length $(D\tau_{in})^{1/2}$, where D is the diffusion constant and τ_{in} is the inelastic scattering time. Then substituting τ_{in} as a function of temperature allows one to predict the temperature dependence of the resistance at low temperatures. The distance $(D\tau_{in})^{1/2}$ is the distance an electron diffuses elastically before it scatters inelastically, thus it is, roughly speaking, the distance over which standard localization theory which treats disordered systems of noninteracting electrons at zero temperature, is valid. Therefore, we imagine the sample as partitioned into segments of length $(D\tau_{in})^{1/2}$. In a rigorously one-dimensional system, since the current is uniform in steady state, the resistance of the sample should be obtained simply as the sum of the resistances of each of these segments and thus in principle should be proportional to the average resistance. However, if the temperature is such that the number of segments of the sample of length $(D\tau_{in})^{1/2}$ is not very large, then measuring the resistance of a single sample will be like doing a numerical average over a finite ensemble.

Thouless originally estimated that the length $(D\tau_{in})^{1/2}$ should be about 10 μm at 2 K,[18] but experimental results have indicated that it is in fact on the order of 1 μm.[19,20] Since the thin wires studied experimentally, for example, by Giordano *et al.*[21] were typically about 200 μm in length, such a wire would be roughly equivalent to 200 "samples" of lengths $(D\tau_{in})^{1/2}$, each sample several thousand atoms in length. However, since the localization length in these samples is greater than $(D\tau_{in})^{1/2}$, which corresponds in our notation to $\gamma N < 1$, we are not in the regime of exponentially large statistical fluctuations in the resistance, and the role of statistical fluctuations in this weakly localized regime is still unclear. If experiments become capable of probing the strongly localized regime, where the resistance depends exponentially on sample length, then our results suggest that such experiments will not find the clear linear dependence of the logarithm of the resistance on L, but instead will find a slower than linear dependence of the sort depicted in Fig. 4, which converges towards $\ln\tilde{\rho}$ as $L \to \infty$. This general argument should remain valid in the many-channel case, since in a system where there are several conducting channels in parallel one would expect the presence of a small number of highly resistive segments to have even less effect on the total resistance. The preceding arguments do not, of course, consider the influence of many-body effects on the low-temperature resistance, which appear to be quite significant.[22]

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[11]$\tilde{\rho}$ is defined according to the prescription of Anderson *et al.* (Ref. 4) by finding a variable $f(\rho)$ which is statistically well behaved and defining $\tilde{\rho} \equiv f^{-1}(\langle f(\rho) \rangle)$. Although $f(\rho)$ is not unique it appears that all ap-

propriate functions will be closely related to $\ln\rho$ in one dimension so that $\tilde{\rho}$ will be closely related to the geometric mean, and we will often refer to $\tilde{\rho}$ as the geometric mean.

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