PHYSICAL REVIEW B VOLUME 24, NUMBER 9 1 NOVEMBER 1981

Griffiths singularity in ferromagnetic alloys

Ronald Fisch
Department of Physics, Washington University, St. Louis, Missouri 63130
(Received 10 August 1981)

Griffiths's result for percolating Ising ferromagnets is extended to the case of binary ferromagnetic alloys. The results indicate that the Griffiths singularity is a general and characteristic phenomenon in random Ising systems. This lends strong support to the contention of McCoy and Wu that a single length-scaling picture is not adequate to describe phase transitions in quenched random magnets.

I. INTRODUCTION

It is now over ten years since Griffiths[1] proved that the magnetization of a bond-diluted Ising ferromagnet is a nonanalytic function of the magnetic field at $H=0$, for all temperatures less than or equal to $T_c(1)$, the critical temperature of the undiluted ferromagnet. Subsequent work[2,3] revealed that the nonanalytic behavior was due to an essential singularity at $H=0$. On this basis, it was concluded[3,4] that the Griffiths singularity is too weak to be experimentally observable. This was used as an argument[4] to support the contention that renormalization-group calculations[5–7] for the phase transition in random ferromagnets were essentially correct, even though they did not contain a Griffiths singularity.

On the other hand, it is known that the Griffiths singularity is closely related to the Lifschitz tail[8] in the density of states of an electron in a random potential, as both phenomena arise from essentially the same mechanism.[3,9] Few people would argue that the Lifschitz tail is an inessential and unimportant detail of the problem of the electron in a random potential. Furthermore, the work of McCoy and Wu,[10] although it is based on a special and perhaps somewhat anomalous model, gives rigorous results which are not consistent with the starting assumptions of a naive renormalization-group approach. In this paper we extend Griffiths's arguments to show that the Griffiths singularity occurs in a wide class of random Ising ferromagnets. From this we conclude that the Griffiths singularity is a characteristic phenomenon of random magnetic systems. Any theory which ignores this phenomenon is seriously deficient. A renormalization-group calculation which does account for the Griffiths singularity in the one-dimensional diluted Ising chain has been given by Grinstein *et al.*[11] Chalupa[12] has presented nonrigorous arguments which indicate that the effect is also present in one-dimensional ferromagnetic alloys.

II. EXISTENCE OF THE GRIFFITHS SINGULARITY

We will explicitly consider Hamiltonians of the form

$$H=-\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j \ , \tag{2.1}$$

where $\langle ij\rangle$ means a sum over nearest neighbors on a lattice, the $\{\sigma_i\}$ are Ising variables, and the $\{J_{ij}\}$ are uncorrelated quenched random bond variables with the probability distribution

$$P(J)=x\delta(J-J_1)+(1-x)\delta(J-J_2) \ . \tag{2.2}$$

We will assume $0 \leq J_2 < J_1$, and, of course, $0 \leq x \leq 1$.

The case $J_2=0$ is the randomly diluted model considered by Griffiths.[1] Using the Lee-Yang circle theorem,[13] he showed that $H=0$ is an accumulation point of the zeros of the magnetization function $M(H)$ for any nonzero value of x, if the temperature satisfies $T \leq T_c(1)$, the critical temperature of the $x=1$ model. Now, because of the GKS inequalities,[14] the zeros of $M(H)$ can only move toward $H=0$ if T is decreased or if J_2 is increased. Therefore $H=0$ is an accumulation point of the zeros of $M(H)$ for *any* non-negative value of J_2, if $T \leq T_c(1)$.

Bergstresser[15] has proven that if $J_2 < J_1$ and $x<1$, then $T_c(x) < T_c(1)$. [The critical temperature, T_c, is the temperature at which the spontaneous magnetization, $M(0)$, disappears.] Therefore if $x<1$ there is a finite range of temperature below $T_c(1)$ in which $M(H)$ is nonanalytic at $H=0$, but $M(0)=0$, for any J_2 which satisfies $0 \leq J_2 < J_1$.

III. TWO-DIMENSIONAL CASE

It is widely believed although not rigorously proven[1,16] that when the Griffiths singularity occurs in the magnetization, the free energy, F, will also be nonanalytic. This expectation has been confirmed in the one-dimensional randomly diluted Ising model by the calculation of Wortis.[2] In this section, we will assume that the presence of the Griffiths singularity will cause the free energy to be nonanalytic, and we will analyze the implications of this for the quenched random-bond binary-alloy Ising ferromagnet on a square lattice. This is an interesting thing to do because of the duality relation which is satisfied by this system[17] for J_1 and J_2 positive.

The duality relation works as follows. Let $K_1 = J_1/kT$, and $K_2 = J_2/kT$. Then the free energy is a function of the variables K_1, K_2, and x: $F = F(K_1, K_2, x)$. [The F we are talking about is the ensemble average free energy, which is a probability one object. This means that all Hamiltonians of the form (2.1) for a given lattice structure which satisfy the same bond probability distribution function (2.2) have the same free energy per site in the infinite volume limit, except for a set of measure zero.] Define the dual variables K_1^* and K_2^* by the Kramer-Wannier-Onsager relations[18]

$$\tanh(K_1^*) = \exp(-2K_1) \quad , \tag{3.1a}$$

$$\tanh(K_2^*) = \exp(-2K_2) \quad . \tag{3.1b}$$

These relations are symmetric: $K_2 = K_1^*$ if and only if $K_1 = K_2^*$. The result of the duality transformation is

$$F(K_1^*, K_2^*, x) = F(K_1, K_2, x) + \tau \tag{3.2}$$

where τ is a trivial (analytic) term. Inspection of the probability distribution function (2.2) gives us the additional relation

$$F(K_1, K_2, x) = F(K_2, K_1, 1-x) \quad . \tag{3.3}$$

These two relations allow us to relate the high-temperature behavior of a system with a concentration x of strong bonds to the low-temperature behavior of a system with a concentration $1-x$ of strong bonds. This is because relations (3.1) imply that if $K_1 > K_2$, then $K_1^* < K_2^*$, since $\tanh(K)$ is a monotonically increasing function and $\exp(-2K)$ is a monotonically decreasing function.

Because of this relation, if the free energy is nonanalytic for a finite range of temperatures below $T_c(1)$ for $0 < x < 1$, as we have proven the magnetization is, then the free energy must also be nonanalytic for a finite range of temperatures above $T_c(0)$. It seems likely that the free energy is nonanalytic at all temperatures between $T_c(0)$ and $T_c(1)$. We will show that this is actually the case for the special value $x = \frac{1}{2}$.

The case $x = \frac{1}{2}$ is self-dual.[17] This is because for $x = \frac{1}{2}$, relation (3.3) maps the free energy of the system back onto the energy of the *same* system at a different temperature. The self-duality temperature, T^*, is the temperature for which the relations $K_2 = K_1^*$ and $K_1 = K_2^*$ are satisfied. It obeys the equation

$$\sinh(2J_1/kT^*)\sinh(2J_2/kT^*) = 1 \quad . \tag{3.4}$$

It is likely that $T_c(\frac{1}{2}) = T^*$. While we cannot prove this, we can prove the inequality

$$T_c(\tfrac{1}{2}) \leq T^* \quad . \tag{3.5}$$

Therefore a Griffiths singularity exists for all temperatures between T^* and $T_c(1)$. Using the assumption that this implies a nonanalyticity in the free energy, and the duality relation (3.2), we find that, for $x = \frac{1}{2}$, F is nonanalytic for all temperatures between $T_c(0)$ and $T_c(1)$. [Note that $T_c(0)$ and $T_c(1)$ are dual temperatures; i.e., $J_2/T_c(0) = J_1/T_c(1)$.]

The reason why $T_c(\frac{1}{2}) \leq T^*$ is because it is not possible for the expectation value of a spin variable $\langle\sigma\rangle$ and the expectation value of a dual variable $\langle\mu\rangle$ (often called a disorder variable[19]) to both be nonzero at the same temperature. Let r_1 and r_2 be any two points on the lattice. Then

$$\langle\sigma(r_1)\mu(r_2)\rangle = 0 \quad , \tag{3.6}$$

because $\sigma\mu$ is a spinor operator[19] (i.e., it goes into minus itself under a 360° rotation). But if we let the distance between r_1 and r_2 become large, the operators must decouple:

$$\lim_{|r_1 - r_2| \to \infty} \langle\sigma(r_1)\mu(r_2)\rangle - \langle\sigma(r_1)\rangle\langle\mu(r_2)\rangle = 0 \quad . \tag{3.7}$$

Putting (3.6) and (3.7) together, and using the fact that our choices of r_1 and r_2 were essentially arbitrary, we obtain

$$\langle\sigma(r_1)\rangle\langle\mu(r_2)\rangle = 0 \tag{3.8}$$

for any r_1 and r_2. This can only be true if either $\langle\sigma(r_1)\rangle = 0$ for any choice of r_1, or $\langle\mu(r_2)\rangle = 0$ for any choice of r_2, or both.

Due to the GKS inequalities,[14] if $\langle\sigma\rangle = 0$ at some temperature, it must also be zero at all higher temperatures. Since the magnetization is precisely the configuration average of $\langle\sigma\rangle$, and since for $x = \frac{1}{2}$ the $\langle\sigma\rangle$ and the $\langle\mu\rangle$ are related to each other by the duality transformation, the result, (3.5), follows.

IV. CONCLUSION

In this work we have shown that the Griffiths singularity is present not only in the diluted random-bond Ising ferromagnet, but also in the random-bond binary-alloy Ising ferromagnet. It seems straightforward to extend our results to any bounded probability distribution of ferromagnetic bonds. Further, it is not apparent why mixing in some antiferromagnetic bonds should cause the result to fail. (Of course, the presence of antiferromagnetic bonds makes it essentially impossible to prove anything, since then one no longer has either the Lee-Yang circle theorem[13] or the GKS inequalities[14] to work with.) In the two-dimensional case we have been able to show (although not entirely rigorously) that the Griffiths singularity is present for a range of temperatures on both sides of the critical temperature.

These results demonstrate that the Griffiths singularity is a characteristic phenomenon of random Ising ferromagnets, and perhaps of an even larger class of magnetic systems. Any theory which does not take this phenomenon into account cannot be considered to be an accurate description of these systems. Thus we are supporting the contention of McCoy and Wu[20] that random ferromagnets are too complicated to be properly described by a single length scaling picture.

In closing, we would like to make a comment about the relationship, to which we have already alluded, between the problem we have studied here and the problem of the electron in a random potential. It is often remarked that the Griffiths singularity is analogous to the Lifschitz tail. It is also widely believed that the dc conductivity is an order parameter for the electronic problem[21] analogous to the magnetization in the ferromagnet. Therefore it is reasonable to expect that an understanding of the nature of the ferromagnetic transition in the presence of the Griffiths singularity will lead to a better understanding of the nature of electronic states close to the mobility edge.

[1]R. B. Griffiths, Phys. Rev. Lett. 23, 17 (1969).
[2]M. Wortis, Phys. Rev. B 10, 4665 (1974).
[3]A. B. Harris, Phys. Rev. B 12, 203 (1975).
[4]Y. Imry, Phys. Rev. B 15, 4448 (1977).
[5]T. C. Lubensky, Phys. Rev. B 11, 3573 (1975).
[6]G. Grinstein and A. Luther, Phys. Rev. B 13, 1329 (1976).
[7]D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. 68, 1960 (1975) [Sov. Phys. JETP 41, 981 (1975)].
[8]I. M. Lifschitz, Adv. Phys. 13, 483 (1964).
[9]W. F. Wreszinski, Phys. Lett. 62A, 15 (1977).
[10]B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard University, Cambridge, Mass., 1973).
[11]G. Grinstein, A. N. Berker, J. Chalupa, and M. Wortis, Phys. Rev. Lett. 36, 1508 (1976).
[12]J. Chalupa, J. Phys. A 12, 1257 (1979).
[13]T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952).
[14]R. B. Griffiths, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, London, 1972), Vol. 1, pp. 72–84.
[15]T. K. Bergstresser, J. Phys. C 10, 3831 (1977).
[16]M. A. Mikulinskii, Zh. Eksp. Teor. Fiz. 53, 1071 (1967) [Sov. Phys. JETP 26, 637 (1968)].
[17]R. Fisch, J. Stat. Phys. 18, 111 (1978).
[18]L. Onsager, Phys. Rev. 65, 117 (1944).
[19]L. P. Kadanoff and H. Ceva, Phys. Rev. B 3, 3918 (1971).
[20]Reference 10, p. 377.
[21]E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).