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Generalized diffusion coefficient in one-dimensional random walks with static disorder

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A real-space renormalization-group method is applied to one-dimensional random walks with static disorder. In agreement with previous results we find that the presence of disorder leads to a non-Markovian diffusion equation with a $t^{-3/2}$ long-time tail. The effective diffusion coefficient and the coefficient of the long-time tail are computed for several disordered random walks.

I. INTRODUCTION

This paper is concerned with the long-time and large-distance properties of one-dimensional random walks with static disorder. In addition to serving as models for conductivity in various materials,[1,2] these random walks display non-Markovian features which are similar to those found in more complicated disordered systems such as dense fluids.[3] In both cases the non-Markovian features appear in the transport equations as memory kernels which decay as a power law. The existence of these long-time tails in disordered random walks was recently discovered by several authors.[2,4–6] These authors analyzed specific models using methods tailored to their models. Here we apply a renormalization-group (RG) method to a class of disordered random walks including some of the models previously studied. This RG calculation is a real-space version of the method used by Forster, Nelson, and Stephen[7] in their study of the long-time properties of fluids.

The random walks which we consider are those in which a group of noninteracting walkers hop between nearest-neighbor sites irregularly spaced on a line. Associated with each site are arbitrarily chosen waiting time distributions for jumping to the right and to the left. For example, if we choose exponential waiting-time distributions we obtain a random walk described by a master equation with random transition rates. Many of the properties of this random walk are reviewed in Ref. 6.

In Sec. II we give a perturbative solution to the master equation with disorder in the transition rates and step lengths. This solution displays the expected $t^{-3/2}$ long-time tail in the generalized diffusion equation but gives inexact results for the value of the diffusion coefficient and the coefficient of the long-time tail.

In Sec. III we show via the renormalization group that the perturbation expansion of Sec. II can be rearranged to give exact results for the asymptotic properties of the random walk with disorder in the waiting-time distributions and step lengths, so long as the disorder is not too strong.

II. PERTURBATIVE SOLUTION TO THE MASTER EQUATION

Consider the master equation

$$\frac{\partial \hat{P}_n(t)}{\partial t} = \sum_m W_{nm} \hat{P}_m(t) \tag{2.1}$$

with nearest-neighbor hopping,

$$W_{nm} = W_n \delta_{nm-1} + W_{n-1}\delta_{nm+1} - (W_n + W_{n-1})\delta_{nm} \ . \tag{2.2}$$

The W_n are to be interpreted as transition rates and the $\hat{P}_n(t)$ as occupation probabilities for site n at time t. The lattice sites are at position x_n on a line. The separation between lattice sites is given by l_n:

$$l_n \equiv x_{n+1} - x_n \ . \tag{2.3}$$

The W_n and l_n are random variables, possibly correlated with one another at the same site but with no correlations between sites. The distributions for W_n and l_n are taken to be translationally invariant.

We will solve Eqs. (2.1)–(2.3) perturbatively by expanding around an ordered system with the same average properties as the disordered one. The ordered system has step length l and transition rate W given by

$$W \equiv \langle W_n \rangle , \tag{2.4}$$

and

$$l \equiv \langle l_n \rangle , \tag{2.5}$$

where the bracket is an average over the distribution of l_n and W_n.

For the ordered system the problem is easily solved by taking Fourier and Laplace transforms of the master equation, Eqs. (2.1)–(2.3), with a single transition rate and step length,

$$zP_q(z) - P_q(t=0) = -4\sin^2(ql/2)\, W P_q(z) , \tag{2.6}$$

where

$$P_q(z) = \int_0^\infty dt \sum_n \exp(iqnl - zt)\hat{P}_n(t) . \tag{2.7}$$

The solution to Eq. (2.6) is

$$P_q(z) = G_q(z) P_q(t=0) , \tag{2.8}$$

where the Green's function is given by

$$G_q(z) = [z + 4\sin^2(ql/2)W]^{-1} . \tag{2.9}$$

For small q or large distances, $G_q(z)$ reduces to the diffusive Green's function

$$G_q(z) \sim \frac{1}{z + q^2 D} \quad \text{as } q \to 0 , \tag{2.10}$$

with diffusion coefficient

$$D \equiv l^2 W . \tag{2.11}$$

For the irregular lattice we must first go to a continuous space variable x and then take a continuum Fourier transform. Let

$$\hat{P}(x,t) = \sum_n \delta(x - x_n)\hat{P}_n(t) , \tag{2.12}$$

and

$$P(q,z) \equiv \int_{-\infty}^{+\infty} dx \int_0^\infty dt\, e^{iqx - zt}\hat{P}(x,t) . \tag{2.13}$$

We can invert the transformation from $\hat{P}_n$ to $\hat{P}(x)$ by convoluting $\hat{P}(x)$ with the characteristic function $\chi(x)$,

$$\chi(x) \equiv \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases} \tag{2.14}$$

so

$$\hat{P}_n = \lim_{a \to 0+} \int dx\, \chi(x - x_n)\hat{P}(x) . \tag{2.15}$$

The small positive number a is taken to zero only at the end of a calculation. Note that for sufficiently small a,

$$\chi(x_n - x_m) = \delta_{nm} . \tag{2.16}$$

This allows us to write the master equation in the continuum form,

$$zP(q,z) = \int dq'\, W(q,q')P(q',z) + P(q,t=0) , \tag{2.17}$$

with solution

$$P(q,z) = \int dq'\, G(q,q';z)\, P(q',t=0) , \tag{2.18}$$

where

$$W(q,q') \equiv \frac{\chi_{-q'}}{2\pi} \sum_{mn} \exp(iqx_m - iq'x_n) W_{mn} , \tag{2.19}$$

and

$$\chi_q \equiv \int dx\, e^{iqx}\chi(x) , \tag{2.19'}$$

and

$$G(q,q';z)^{-1} = z\delta(q - q') - W(q,q') . \tag{2.20}$$

We now seek a perturbation expansion for the average or effective Green's function which must take the translationally invariant form,

$$\delta(q - q')\mathcal{G}_q(z) \equiv \langle G(q,q';z) \rangle . \tag{2.21}$$

By analyzing the average Green's function instead of the Green's function for a single realization we simplify the calculation and the results at the expense of losing information about the short-distance properties of the random walk. On the other hand, the effective Green's function should accurately describe the long-time and large-distance behavior of a single random walk.

We expand $G(q,q';z)$ in powers of $\delta W(q,q')$, the deviation of $W(q,q')$ from its uniform system value,

$$\delta W(q,q') \equiv W(q,q') - \langle W(q,q') \rangle . \tag{2.22}$$

Up to second order this gives

$$G(q,q';z)=\delta(q-q')G_q(z)+\frac{\delta G(q,q';z)}{\delta W(q_1,q_2)}\delta W(q_1,q_2)$$
$$+\frac{\delta^2 G(q,q';z)}{\delta W(q_1,q_2)\delta W(q_3,q_4)}\delta W(q_1,q_2)\delta W(q_3,q_4)+O(\delta W^3)\ . \tag{2.23}$$

Here integration over wave vectors repeated in parentheses is implied and the functional derivatives are evaluated in the uniform system. Noting that

$$\frac{\delta^2 G(q,q';z)}{\delta W(q_1,q_2)\delta W(q_3,q_4)}=G_q(z)\delta(q-q_1)G_{q_3}\delta(q_2-q_3)\delta(q_4-q')G_{q'}\ , \tag{2.24}$$

and averaging gives

$$\mathscr{G}_q(z)=G_q(z)+G_q^2(z)\int dq_1\, G_{q_1}(z)\frac{\langle \delta W(q,q_1)\delta W(q_1,q')\rangle}{\delta(q-q')}\ . \tag{2.25}$$

In Appendix A the correlation function is simplified and the wave-vector integration done yielding

$$\mathscr{G}(q,z)=G_q(z)+\Delta G_q^2(z)q^2 D\left[1-\frac{z^{1/2}}{\sqrt{z+4W}}\right]\ , \tag{2.26}$$

where

$$\Delta\equiv\left\langle\left[\frac{\delta l}{l}+\frac{\delta W}{W}\right]^2\right\rangle . \tag{2.27}$$

From the Green's function the z-dependent generalized diffusion coefficient $\mathscr{D}(z)$ can be obtained and expanded in powers of $z^{1/2}$,

$$\mathscr{D}(z)\equiv-\frac{z^2}{2}\frac{\partial^2}{\partial q^2}\mathscr{G}(q,z)\bigg|_{q=0}$$
$$=D(1-\Delta)+\frac{\Delta}{2}D\sqrt{z/W}+O(z)\ . \tag{2.28}$$

Thus second-order perturbation theory leads to a generalized diffusion equation with a $z^{1/2}$ or $t^{-3/2}$ long-time tail. The coefficient of this long-time tail and the modification of the diffusion coefficient are both proportional to the mean-square fluctuation in the transition rates and the step lengths.

III. RENORMALIZATION-GROUP CALCULATION

A. RG transformation

The difficulty with naive perturbation theory is that contributions to the diffusion coefficient and the long-time tail appear at all orders of the perturbation expansion. Thus the result of Eq. (2.28) is exact only in the limit of weak disorder. The renormalization-group approach presented in this section circumvents this difficulty by rearranging the perturbation expansion so that the diffusion coefficient and long-time-tail coefficient are moved entirely to the first two terms of the series. The method also leads naturally to the analysis of disordered walks described by arbitrary waiting-time distributions.

The idea of the RG approach is to iteratively transform the original, strongly disordered process into a weakly disordered process without changing the asymptotic properties of the original process. Perturbation theory is then applied to the weakly disordered process. Each step in the transformation consists of eliminating odd-numbered lattice sites. If the walker is at an odd-numbered site in the untransformed process, its position after the elimination is assigned to be that of the previous site visited. Since the difference between the positions of the walker before and after the elimination is one lattice spacing or less, the large distance properties of both walks are the same. After the elimination of odd-numbered sites, the remaining sites are renumbered and lengths and times rescaled so that the new process looks as much as possible like the old process.

Indicating the parameters of the walk after the elimination of odd sites with a bar, the new step lengths are

$$\bar{l}_{2n}=l_{2n}+l_{2n+1}\ . \tag{3.1}$$

There is no analogous transformation on the set of W's because, after the elimination of odd numbered sites, there is no longer an exponential waiting-time distribution for jumping between even-numbered sites. Thus we are forced to consider transformations on arbitrary sets of waiting-time distributions. A set of waiting-time distributions is equivalent to a non-Markovian master equation. The connection between the waiting-time picture and the generalized master-equation picture can be found in Ref. (8) and is reviewed in Appendix B.

Let $\hat{p}_n(t)$ $[\hat{q}_n(t)]$ be the probability density for jumping from site n to site $n+1$ $[n-1]$ after waiting at site n for a time t. Let $p_n(z)$ and $q_n(z)$ be their Laplace transforms. After the elimination of odd-numbered sites the new waiting-time distributions for jumping to the right are given by

$$\bar{p}_{2n}(z)=p_{2n}(z)p_{2n+1}(z)\sum_{s=0}^{\infty}[p_{2n}(z)q_{2n+1}(z)+q_{2n}(z)p_{2n-1}(z)]^s . \qquad (3.2)$$

The prefactor is the probability density for jumping directly from site $2n$ to site $2n+2$. $p_{2n}(z)q_{2n+1}(z)$ and $q_{2n}(z)p_{2n-1}(z)$ represent jumps from site n to $n+1$ or $n-1$ and then back to n. These jumps can occur any number of times in any order and in the transformed process are counted as waiting at site n. Summing the geometric series gives

$$\bar{p}_{2n}(z)=p_{2n}(z)p_{2n+1}(z)\times[1-p_{2n}(z)q_{2n+1}(z)-q_{2n}(z)p_{2n-1}(z)]^{-1} , \qquad (3.3)$$

and, by the same reasoning,

$$\bar{q}_{2n}(z)=q_{2n}(z)q_{2n-1}(z)\times[1-p_{2n}(z)q_{2n+1}(z)-q_{2n}(z)p_{2n-1}(z)]^{-1} . \qquad (3.4)$$

$\bar{l}_{2n}$, $\bar{p}_{2n}(z)$, and $\bar{q}_{2n}(z)$ fully describe the random walk after the elimination of the odd-numbered sites. To complete the RG transformation the remaining lattice sites are renumbered and time and space are rescaled so that some average parameters of the random walk are unaffected by the RG transformation. If the average lattice spacing is to be unaffected by the RG transformation then lengths must be rescaled by two. Indicating the transformed walk by a prime we have,

$$l_n'=\bar{l}_{2n}/2=(l_{2n}+l_{2n+1})/2 . \qquad (3.5)$$

Leaving the time rescaling factor arbitrary for the moment and denoting it by λ we have

$$p_n'(z)=\bar{p}_{2n}(z/\lambda) \qquad (3.6)$$

and

$$q_n'(z)=\bar{q}_{2n}(z/\lambda) . \qquad (3.7)$$

B. Fixed points of the RG transformation

The fixed-point step length l^* is a constant which must equal the average initial step length,

$$l^*=l . \qquad (3.8)$$

There are various fixed-point waiting-time distributions depending on the choice of λ and the symmetry imposed on the fixed point. Supposing that $p_n(z)=q_n(z)$, the fixed-point equation is

$$p^*(z)=p^*(z/\lambda)^2[1-2p^*(z/\lambda)^2]^{-1} . \qquad (3.9)$$

For small z, the solution to Eq. (3.9) is given by

$$p^*(z)=\tfrac{1}{2}(1-Tz^{\alpha}) , \qquad (3.10)$$

where T is an arbitrary constant and α is related to λ by

$$\lambda^{\alpha}=4 . \qquad (3.11)$$

$\alpha=1$ and $\lambda=4$ correspond to an ordinary diffusion process with diffusion constant $D=l^2/2T$. $\alpha<1$ gives the nonanalytic waiting-time distributions discussed by Scher and Montroll[9] which lead to processes in which the mean-square displacement grows more slowly than t.

For the diffusive case the fixed-point equation is solved by

$$p^*(z)=\tfrac{1}{2}\operatorname{sech}(2\sqrt{zT}) . \qquad (3.12)$$

The inverse Laplace transform of this function is related to a theta function and $\hat{p}^*(t)$ is the waiting-time distribution that would be obtained by replacing a continuous one-dimensional diffusion process by a discrete one according to the following rule. The diffusing walker is counted as waiting at the last lattice site crossed until the next lattice site is crossed.

Starting from a disordered random walk and applying the RG transformation many times with the appropriate rescaling factor we expect to approach one of the above fixed points; which fixed point and how it is approached determines the long-time and large-distance properties of the disordered random walk. The approach to the fixed point is analyzed by considering the effect of the RG transformation on moments of the random parameters defining the process.

Consider the moments of δl, the deviation of the step length from its average. To simplify the notation we will henceforth suppress the site subscript n by using $+$ or $-$ to indicate $n+1$ or $n-1$ and by leaving implicit the renumbering of lattice sites by the RG transformation. Thus Eq. (3.5) becomes, for the fluctuations of the step lengths

$$\delta l' = \tfrac{1}{2}(\delta l + \delta l_+) \ . \tag{3.13}$$

The transformation on the nth moment of δl is thus

$$\langle \delta l'^n \rangle = \frac{1}{2^n} \langle (\delta l + \delta l_+)^n \rangle \ . \tag{3.14}$$

Since δl are independent variables in the original random walk and are not coupled by the RG transformation, Eq. (3.14) becomes

$$\langle \delta l'^n \rangle = \frac{1}{2^{n-1}} \langle \delta l^n \rangle \ . \tag{3.15}$$

Thus the mean-squared fluctuations in the step length are reduced by a factor of two at each iteration and the higher moments of δl diminish more rapidly. This is essentially a statement of the central limit theorem for sums of original step lengths.

In the case of the waiting-time distributions the situation is less simple because the RG transformation is nonlinear and acts upon a set of functions. Instead of considering the full waiting-time distribution we will derive recursion relations for two parameters associated with $p_n(z)$ and $q_n(z)$. The first is the probability of jumping to the right or to the left and the second is the mean time for waiting at a site before jumping. Let the probability of jumping to the right [left] from site n be p_n [q_n] and observe that it is just the $z=0$ value of $p_n(z)$ [$q_n(z)$] so that it follows immediately from Eqs. (3.3), (3.4), (3.6), and (3.7) that

$$p' = pp_+(1 - pq_+ - qp_-)^{-1} \tag{3.16}$$

and

$$q' = qq_-(1 - pq_+ - qp_-)^{-1} \ . \tag{3.17}$$

Notice that if $p+q=1$ then $p'+q'=1$ so that if trapping is not present in the original process it will not arise through RG transformations. Since Eqs. (3.16) and (3.17) are also nonlinear it is difficult to show directly that p_n and q_n converge to their fixed-point values of $\frac{1}{2}$. However, from the lattice functions p_n and q_n we can construct a new lattice function τ_n which obeys linear recursion relations.

Let

$$s_n = \begin{cases} \prod_{m=1}^{n} \dfrac{q_m}{p_m} \ , & n > 0 \\ 1 \ , & n = 0 \\ \prod_{m=0}^{n+1} \dfrac{p_m}{q_m} \ , & n < 0 \end{cases} \tag{3.18}$$

and let

$$\tau_n = s_n \Big/ \lim_{N\to\infty} \frac{1}{2N} \sum_{j=-N}^{N-1} s_j \ . \tag{3.19}$$

From Eqs. (3.16) and (3.17) it follows that:

$$\frac{p'}{q'} = \frac{pp_+}{qq_-} \tag{3.20}$$

and

$$s' = p_1(s + s_+) \ , \tag{3.21}$$

so the recursion relations for τ_n are

$$\tau' = \frac{\tau + \tau_+}{2} \ . \tag{3.22}$$

This is identical to the recursion relation for l_n so the same results apply if τ_n is uncorrelated from site to site; τ_n converges to its average value of 1, its mean-square fluctuation diminishes by a factor of two at each iteration, and its higher moments diminish by higher powers of two. Of course, if the disorder in the right/left jump probabilities is sufficiently strong, τ_n may be ill defined. The subsequent analysis is thus restricted to problems where τ_n is well defined and uncorrelated with τ_m for $n \neq m$.

The definition of τ_n can be inverted to yield

$$p = \frac{\tau_-}{\tau + \tau_-} \tag{3.23}$$

so that, as the disorder in τ_n diminishes, so also does the disorder in p_n.

Next consider the effect of the RG transformation on the mean time for waiting at a site, T_n. Defining $T_n(z)$ by

$$T(z) \equiv \frac{1}{z}[1 - p(z) - q(z)] \ , \tag{3.24}$$

then the mean waiting time is given by

$$T=\lim_{z\to 0} T(z) . \tag{3.25}$$

Henceforth we will assume that this quantity exists at each site. In Appendix C we derive the following recursion relation for T_n from Eqs. (3.3), (3.4), (3.6), and (3.7)

$$T'=\frac{1}{\lambda}\left[\frac{T+pT_+ +qT_-}{pp_+ +qq_-}\right] . \tag{3.26}$$

From the lattice function T_n we can construct a new lattice function, U_n which obeys a simpler recursion relation. Let

$$U\equiv T\left[\frac{1}{\tau}+\frac{1}{\tau_-}\right]. \tag{3.27}$$

From Eqs. (3.26) and (3.23) it follows that:

$$U'=\frac{2}{\lambda}U+\frac{1}{\lambda}(U_+ +U_-)+\frac{2}{\lambda}(\epsilon_- U_- -\epsilon_+ U_+) , \tag{3.28}$$

where

$$\epsilon=p-\tfrac{1}{2} . \tag{3.29}$$

Suppose the time rescaling factor, λ, is set to be four. From translational invariance it follows that the average value of U_n is unchanged by the RG transformation,

$$\langle U'\rangle=\langle U\rangle\equiv U^* . \tag{3.30}$$

After many RG iterations ϵ_n becomes small and the recursion relation for U_n is approximately given by

$$U'=\tfrac{1}{2}U+\tfrac{1}{4}(U_+ +U_-) . \tag{3.31}$$

This recursion relation induces recursion relations on the moments of δU. Even if δU is initially uncorrelated from site to site, Eq. (3.31) couples δU at neighboring sites so that we obtain coupled equations for $\langle \delta U\delta U\rangle$ and $\langle \delta U\delta U_+\rangle$. Using translational invariance these are easily seen to be

$$\begin{bmatrix}\langle \delta U'\delta U'\rangle\\ \langle \delta U'\delta U'_+\rangle\end{bmatrix}=\begin{bmatrix}\frac{3}{8} & \frac{1}{2}\\ \frac{1}{16} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}\langle \delta U\delta U\rangle\\ \langle \delta U\delta U_+\rangle\end{bmatrix} . \tag{3.32}$$

The eigenvalues of this transformation are $\frac{1}{2}$ and $\frac{1}{8}$. By adding the ϵ terms from Eqs. (3.28)–(3.32) and then diagonalizing the linear part one can show that only the eigenvector of $\frac{1}{8}$ has ϵ contributions.

Thus far we have shown that the zeroth and first moments with respect to time of the waiting-time distributions converge to their fixed-point value. We now sketch an argument that the fixed point is stable. This will be the case if all the remaining moments with respect to time have recursion relations whose linear part have eigenvalues all less than one. From Eqs. (3.3), (3.4), (3.6), and (3.7) it is easy to show that these moments satisfy

$$\delta p^{(n)\prime}=\frac{1}{\lambda^n}[\delta p^{(n)}+\delta p_+^{(n)}+\tfrac{1}{2}(\delta p^{(n)}+\delta q^{(n)}+\delta q_+^{(n)}+\delta p_-^{(n)})+H] , \tag{3.33}$$

where

$$\delta p^{(n)}\equiv(-)^n\int_0^\infty t^n\hat{p}(t)\,dt = \left.\frac{\partial^n p(z)}{\partial z^n}\right|_{z=0} , \tag{3.34}$$

and similarly for $\delta q^{(n)}$. H contains nonlinear terms and linear contributions from $\delta p^{(m)}$ and $\delta q^{(m)}$ for $m<n$. Consider the linearized recursion relations for the second moments of $\delta p^{(n)}$. Since these relations are block triangular in the index n it follows from the Schwartz inequality that for all eigenvectors involving $\delta p^{(n)}$ the associated eigenvalue must be less than $4^{(1-n)}$. Thus for $n>1$ the second (and higher moments) of $\delta p^{(n)}$ and $\delta q^{(n)}$ decay by at least a factor of 16 with each RG iteration. For $n=0$ we have seen, by analyzing τ that, if τ exists, $\langle \delta p^{(0)}\delta p^{(0)}\rangle$ ultimately decays by a factor of 2 with each RG iteration. For $n=1$ the analysis of δU shows that second moments of the combination $\delta p^{(1)}+\delta q^{(1)}$ ultimately decay at least by a factor 2. It is easy to deduce from Eq. (3.33) that the second moments of the other combination, $\delta p^{(1)}-\delta q^{(1)}$, decay at least by a factor of 8. Thus the fixed point is stable to small analytic perturbations so long as all the moments of $\delta p^{(n)}$ and $\delta q^{(n)}$ exist. We assume, without proof, that the fixed point is also stable to arbitrary analytic perturbations so long as all the moments of $\delta p^{(n)}$ and $\delta q^{(n)}$ exist.

If the original random walk is described by the master equation of Eqs. (2.1) and (2.2) then the quantities τ_n and U_n are simply related to the transition rates W_n. Using Eq. (3.19) and the results of Appendix B we have that

$$\tau_n = \left\langle \frac{1}{W_n} \right\rangle^{-1} \frac{1}{W_n} , \tag{3.35}$$

and that U is a constant for all n

$$U_n = \left\langle \frac{1}{W_n} \right\rangle . \tag{3.36}$$

Thus, for the master equation, a necessary condition for convergence to the diffusive fixed point is the existence of the average $\langle 1/W_n \rangle$. For a discussion of the situation where $\langle 1/W_n \rangle$ diverges, see Ref. 6.

C. Generalized diffusion coefficient

After any number of RG recursions we can return to the master equation picture (cf. Appendix B) and construct a perturbation series for the generalized diffusion coefficient. The formalism of Sec. II can be taken over as it stands except that the transition rates must be replaced by z-dependent transition rates. These transition rates can then be further expanded in terms of the moments of the waiting time distribution $\delta\tau$, δU, $\delta p^{(2)}, \delta q^{(2)}, \ldots$ yielding a perturbation series in terms of these parameters. As the fixed point is approached the structure of the perturbation series simplifies and the diffusion coefficient and long-time tail coefficient can be read off exactly.

The relationship between the generalized diffusion coefficient after N RG recursions, $\mathscr{D}^{(N)}(z)$ and the original function, $\mathscr{D}(z)$ is given by the simple scaling equation

$$\mathscr{D}(z) = \mathscr{D}^{(N)}(4^N z) . \tag{3.37}$$

From Appendix B and the definitions of τ and U we have that

$$W_{m+1,m}(0) = \frac{p_m}{T_m} = \frac{1}{\tau_m U_m} , \tag{3.38}$$

and

$$W_{m,m+1}(0) = \frac{q_{m+1}}{T_{m+1}} = \frac{1}{\tau_m U_{m+1}} . \tag{3.39}$$

At the fixed point, where the fluctuations have vanished, we have that

$$W^*_{mn} = \frac{1}{U^*}(\delta_{m,n+1} + \delta_{m,n-1} - 2\delta_{mn}) , \tag{3.40}$$

and therefore that the effective diffusion coefficient is given by

$$\mathscr{D} = l^2/U^* . \tag{3.41}$$

More care must be taken to compute the z-dependent terms in $\mathscr{D}(z)$ since with each RG recursion z is scaled up as the fluctuations in the waiting-time parameters are scaled down. Consider all the terms in the formal perturbation series which behave like $\sqrt{z}$ for small z. With each RG recursion $\sqrt{z}$ grows by a factor of 2; thus, to remain nonzero and finite, the coefficient of $\sqrt{z}$ must diminish by a factor of two. The second moments of $\delta\tau_n$, δl_n, and δU_n are the only quantities appearing in the perturbation expansion which diminish by a factor of two; thus, sufficiently near the fixed point, second order perturbation theory gives exact results for the leading, $\sqrt{z}$, correction to the diffusion coefficient.

Consider the situation where the eigenmode of $\langle \delta U \delta U \rangle$ and $\langle \delta U \delta U_+ \rangle$ with eigenvalue $\frac{1}{2}$ is not initially excited. This is true [cf. Eq. (3.36)] for the master equation with symmetric hopping described by Eqs. (2.1) and (2.2). In this case only the disorder in $\delta\tau$ and δl is relevant. Expanding $\delta W_{mn}(z)$ to lowest order in $\delta\tau$ we obtain

$$\delta W_{m,m+1}(z) \cong -\frac{\delta\tau_m}{U^*} \cong \delta W_{m+1,m}(z) . \tag{3.42}$$

Using the scaling relation, Eq. (3.37), the second-order result, Eq. (2.28), and taking the limit $N \to \infty$ we obtain an expression for the effective generalized diffusion coefficient

$$\mathscr{D}(z) = \lim_{N\to\infty} \mathscr{D}^{(N)}(4^N z) = \mathscr{D} + \frac{\Delta^*}{2} \mathscr{D}\sqrt{zU^*} , \tag{3.43}$$

with

$$\Delta^* = \left\langle \left[\frac{\delta l}{l} - \delta\tau \right]^2 \right\rangle . \tag{3.44}$$

If the process is described by a symmetric master equation then

$$\delta\tau_n = \delta \left[\frac{1}{W_{n+1,n}(0)} \right] \tag{3.45}$$

and

$$U^* = \left\langle \frac{1}{W_{n+1,n}(0)} \right\rangle . \tag{3.46}$$

Suppose, on the other hand, that τ_n and l_n are constants but that $\langle \delta U \delta U \rangle$ is nonzero and finite. For example, this occurs for the master equation

$$U_n \frac{\partial \hat{P}_n(t)}{\partial t} = [\hat{P}_{n+1}(t) + \hat{P}_{n-1}(t) - 2\hat{P}_n(t)] , \tag{3.47}$$

where U_n are uncorrelated from site to site. The perturbation expansion near the fixed point is complicated slightly because the RG recursions induce nearest-neighbor correlations in δU. However, the result is formally the same as in the previous case and given by Eqs. (3.41) and (3.43) where now

$$\Delta^* = \left\langle \left[\frac{\delta U}{U^*} \right]^2 \right\rangle . \tag{3.48}$$

For both cases we find, using a Tauberian theorem, that the memory kernel in the diffusion equation, $\hat{\mathscr{D}}(t)$, has the long-time form

$$\hat{\mathscr{D}}(t) \underset{t\to\infty}{\sim} -\Delta^* (\pi U^*)^{1/2} \mathscr{D} t^{-3/2} , \tag{3.49}$$

and that the growth of the mean-squared displacement has a $t^{1/2}$ correction

$$\frac{\langle x^2(t) \rangle}{2\mathscr{D}} - t \underset{t\to\infty}{\sim} \Delta^* \left[\frac{U^* t}{\pi} \right]^{1/2} . \tag{3.50}$$

These results agree with those obtained exactly by Zwanzig[10] and van Beijeren,[4] and, using an effective-medium approximation, by Alexander *et al.*[2]

Equation (3.50) is also in agreement with computer simulations of Eqs. (2.1) and (2.2) carried out by Richards and Renken.[11] It should be noted, however, that for their choice of parameters the predicted $t^{1/2}$ behavior was not observed until the time, measured in units of $l^2/\mathscr{D}$ was greater than 100. In their computer simulation $\Delta^* = 3.6$.

IV. DISCUSSION

The long-time properties of one-dimensional random walks with static disorder must be described by a non-Markovian hydrodynamic equation. If the disorder is not too strong, this equation takes the form of a generalized diffusion equation with a $t^{-3/2}$ decay in the memory kernel. The renormalization-group procedure separates the components of the disorder which contribute to the $t^{-3/2}$ long-time tail from those that do not. The relevant quantities are the second cumulants of the variables l_n, τ_n, and U_n. τ_n and U_n are constructed from the probabilities of jumping to the left or the right and the mean time for waiting at a site in such a way that their ensemble averages are invariant to the recursion relations. Other moments with respect to time of the waiting-time distributions decay to their fixed-point values sufficiently rapidly that they do not contribute to the $t^{-3/2}$ long-time tail, although their cumulant averages must exist if the perturbation expansion is to make sense.

The averages of τ_n and U_n exist only when the long-time behavior of the walk is diffusive. Presumably the recursion relations lead to one of the nonanalytic fixed points when the disorder is sufficiently strong that these averages diverge. Showing this to be the case remains an open problem. Another open problem is the application of the RG method to higher dimensional random walks.

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APPENDIX A

$\delta W(q,q_1)$ is defined by

$$\begin{aligned}\delta W(q,q_1) &= \frac{\chi_{-q_1}}{2\pi} \sum_{mn} \delta[\exp(iqx_m - iq_1 x_n) W_{mn}] \\ &= \frac{\chi_{-q_1}}{2\pi} \sum_{mn} \exp[i(qml - q_1 nl)][(iq\delta x_m - iq_1 \delta x_n)(\delta_{m+1,n} + \delta_{m-1,n} - 2\delta_{mn})W \\ &\quad + \delta W_m \delta_{m+1,n} + \delta W_{m-1} \delta_{m-1,n} - (\delta W_m + \delta W_{m-1})\delta_{mn}] . \end{aligned} \tag{A1}$$

After summing over m and n we have

$$\delta W(q,q_1)=4W\delta\tilde{x}_k[iq_1\sin^2(ql/2) -iq\sin^2(q_1l/2)] -4\delta\tilde{W}_k\exp(iql/2-iq_1l/2) \times\sin(q_1l/2)\sin(ql/2)\ , \quad (A2)$$

where

$$k\equiv q-q_1\ , \quad (A3)$$

$$\delta\tilde{x}_k\equiv\sum_m e^{ikml}\delta x_m\ , \quad (A4)$$

and similarly for $\delta\tilde{W}_k$. $\delta\tilde{x}_k$ can be related to $\delta\tilde{l}_k$ using Eq. (2.3)

$$\delta\tilde{l}_k=\sum_m e^{ikml}(\delta x_{m+1}-\delta x_m) =-2ie^{-ikl/2}\sin(kl/2)\delta\tilde{x}_k\ . \quad (A5)$$

Using Eqs. (A2) and (A5) and expanding to the lowest order in q we have

$$\langle\delta W(q,q_1)\delta W(q_1,q')\rangle=4q^2l^2W^2\sin^2(q_1l/2)\frac{\chi_{-q_1}}{2\pi}\frac{\chi_{-q'}}{2\pi} \times\langle(\delta\tilde{l}_k/l+\delta\tilde{W}_k/W)(\delta\tilde{l}_{k'}/l+\delta\tilde{W}_{k'}/W)\rangle\ , \quad (A6)$$

where

$$k'\equiv q_1-q'\ . \quad (A7)$$

Invoking the translational invariance and site-to-site independence of the distribution for l_n and W_n, the second-order expression for $\mathcal{G}_q(z)$ becomes

$$\mathcal{G}_q(z)=G_q(z)+\Delta G_q^2(z)\left[\frac{l}{2\pi}\right]\int_{-\pi/l}^{+\pi/l}dq_1\,[4q^2l^2W^2\sin^2(q_1l/2)G_{q_1}(z)]\ , \quad (A8)$$

where

$$\Delta\equiv\langle(\delta l/l+\delta W/W)^2\rangle\ . \quad (A9)$$

Here we have made use of the fact that, for any lattice function, f_n,

$$\int_{-\infty}^{+\infty}dq_1\,\chi_{-q_1}\tilde{f}_{q_1}=f_0=l\int_{-\pi/l}^{+\pi/l}dq_1\,\tilde{f}_{q_1}\ . \quad (A10)$$

Using Eq. (2.9), the integral of Eq. (A8) can be done yielding

$$\mathcal{G}_q(z)=G_q(z)+\Delta G_q^2(z)q^2l^2W \times\left[1-\frac{z^{1/2}}{\sqrt{z+4W}}\right] \quad (A11)$$

APPENDIX B: RELATION BETWEEN WAITING-TIME DISTRIBUTIONS AND THE GENERALIZED MASTER EQUATION

Let $\hat{\psi}_{mn}(t)$ for $m\neq n$ be the probability density for jumping from n to m after waiting at site n for a time t. Let $\hat{Q}_{mn}(t)$ be the probability density for arriving at m at time t after starting at n. $\psi_{mn}(z)$ and $Q_{mn}(z)$, their Laplace transforms, are related by

$$Q(z)=\sum_{p=0}^{\infty}[\psi(z)]^p=[1-\psi(z)]^{-1}\ , \quad (B1)$$

where the matrix indices have been omitted. Let $\hat{T}_m(t)$ be the probability density of remaining at m for a time t after arriving there. Its Laplace transform $T_m(z)$ is given by

$$T_m(z)=\frac{1}{z}\left[1-\sum_n\psi_{nm}(z)\right]\ . \quad (B2)$$

The Green's function $\hat{G}_{mn}(t)$, is thus a product of T and Q

$$G_{mn}(z)=T_m(z)Q_{mn}(z)\ . \quad (B3)$$

Suppose we start with the generalized master equation

$$\frac{\partial\hat{P}_m(t)}{\partial t}=\sum_n\int_0^t\hat{W}_{mn}(\tau)\hat{P}_n(t-\tau)\,d\tau\ . \quad (B4)$$

The Green's function for this process is

$$G(z)=[z-W(z)]^{-1}\ . \quad (B5)$$

Comparing Eq. (B5) with Eq. (B3) we have

$$W_{mn}(z)=\frac{\psi_{mn}(z)}{T_n(z)}-\sum_l \psi_{ln}(z)\Big/T_n(z)\ . \tag{B6}$$

Consider the case of nearest-neighbor hopping where

$$p_n(z)=\psi_{n+1,n}(z)\ , \tag{B7}$$

and

$$q_n(z)=\psi_{n-1,n}(z)\ . \tag{B8}$$

Here

$$T_n(z)=\frac{1}{z}[1-p_n(z)-q_n(z)]\ , \tag{B9}$$

$$W_{n+1,n}(z)=\frac{p_n(z)}{T_n(z)}\ , \tag{B10}$$

$$W_{n-1,n}(z)=\frac{q_n(z)}{T_n(z)}\ , \tag{B11}$$

and

$$W_{nn}(z)=-\frac{p_n(z)+q_n(z)}{T_n}\ . \tag{B12}$$

APPENDIX C: RECURSION RELATION FOR THE MEAN WAITING TIME

Let T_j be defined by

$$T_j\equiv\lim_{z\to 0} T_j(z)\ . \tag{C1}$$

T_j can be interpreted as the mean time for waiting at site j. From Eqs. (3.3), (3.4), (3.6), and (3.7) we have

$$T'(\lambda z)=\frac{1}{\lambda z}\left[1-\frac{p(z)p_+(z)}{D(z)}-\frac{q(z)q_-(z)}{D(z)}\right], \tag{C2}$$

where

$$D(z)\equiv 1-p(z)q_+(z)-q(z)p_-(z)\ , \tag{C3}$$

or

$$T'(\lambda z)=\frac{1}{\lambda z D(z)}\{\,1-p(z)[p_+(z)+q_+(z)]-q(z)[p_-(z)+q_-(z)]\,\} \tag{C4}$$

$$=\frac{1}{\lambda D(z)}[T(z)+p(z)T_+(z)+q(z)T_-(z)]\ . \tag{C5}$$

Taking the limit $z\to 0$ and using $p+q=1$ we obtain

$$T'=\frac{1}{\lambda}\left[\frac{T+pT_++qT_-}{pp_++qq_-}\right]. \tag{C6}$$

[1]S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, in *Springer Series in Solid-State Sciences*, edited by J. Bernasconi and T. Schneider (Springer, Berlin, 1981), Vol. 23, p. 277.

[2]S. Alexander, J. Bernasconi, W. R. Schneider, R. Biller and R. Orbach, Proceedings of the International Conference on Low-Dimensional Conductors, Boulder, Colorado, 1981.

[3]B. J. Alder and T. E. Wainwright, Phys. Rev. A 1, 18 (1970); J. R. Dorfman and H. van Beijeren, in *Modern Theoretical Chemistry*, edited by B. J. Berne (Plenum, New York, 1977), Vol. 6, p. 65.

[4]H. van Beijeren, Rev. Mod. Phys. (in press).

[5]P. Grassberger, Physica (Utrecht) A 103, 558 (1980).

[6]S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. 53, 175 (1981).

[7]D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A 16, 732 (1977).

[8]J. Klafter and R. Silbey, J. Chem. Phys. 72, 843 (1980).

[9]H. Scher and E. W. Montroll, Phys. Rev. B 12, 2455 (1975).

[10]R. Zwanzig (unpublished).

[11]P. M. Richards and R. L. Renken, Phys. Rev. B 21, 3740 (1980); P. M. Richards (private communication).