PHYSICAL REVIEW B VOLUME 24, NUMBER 9 1 NOVEMBER 1981

Antiferromagnetic clock models in two dimensions

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(Received 26 January 1981)

Antiferromagnetic and asymmetric p-state clock models on a square lattice are analyzed with the renormalization group, by studying the effect of p-fold symmetry-breaking fields in the planar model. For p odd and ≥ 3 ($p=3$ is the three-state Potts model) these models exhibit two phase transitions with a critical line of continuously varying exponent η ($\eta_{\min} < \eta < \frac{1}{4}$) separating them. When the couplings are antiferromagnetic in one direction only, $\eta_{\min} = 2/p^2$, while for the isotropic case $\eta_{\min} = 1/p^2$. In the latter case, the low-temperature phase is characterized by a $2p$-fold degeneracy.

I. INTRODUCTION

There has been some recent interest in the antiferromagnetic p-state Potts models. These models have a high degree of degeneracy in the ground state, resulting in an extensive entropy at zero temperature. It has been suggested by Berker and Kadanoff[1] that the degeneracy is responsible for a low-temperature phase characterized by power-law decay of correlations with a fixed exponent η. On the other hand, Monte Carlo simulations in three dimensions, and ϵ-expansion methods,[2] indicate that the three- and four-state models undergo continuous transitions to an ordered low-temperature phase.

In this paper we discuss a class of models on a square lattice which include the antiferromagnetic p state *clock* models. For p odd, these antiferromagnetic models also have highly degenerate ground states, and for $p=3$ the model coincides with the three-state Potts model.

The most general model we consider is defined with an angular variable $\theta(\vec{r})$ at each site $\vec{r} = (ma, na)$ of a square lattice of spacing a. The Hamiltonian is given by

$$-\beta\mathcal{H} = \frac{1}{T_0} \sum_{\vec{r},\vec{r}\,'} \cos\left[\theta(\vec{r}) - \theta(\vec{r}\,') - \frac{2\pi\vec{\Delta}\cdot(\vec{r}-\vec{r}\,')}{pa}\right] + \tilde{h}_p \sum_{\vec{r}} \cos p\theta(\vec{r}) \ , \quad (1.1)$$

where the first term is a sum over nearest-neighbor sites (with each pair counted once only). The true clock model is realized in the limit $\tilde{h}_p \to \infty$, when the angles $\theta(\vec{r})$ become discrete. The vector $\vec{\Delta} = (\Delta_x, \Delta_y)$ can, for reasons of periodicity, be considered to lie in or on the square $0 \leq \Delta_x, \Delta_y \leq \frac{1}{2}$. The model (1.1) for $\Delta_y = 0$ but arbitrary Δ_x has been considered recently by Ostlund.[3] Our results for this case, described below, agree with his. When Δ_x or Δ_y equals $\frac{1}{2}$, the model is antiferromagnetic in the respective directions if p is odd.

The basic observation is that by a redefinition

$$\theta(\vec{r}) \to \theta(\vec{r}) + 2\pi\vec{\Delta}\cdot\vec{r}/pa \quad (1.2)$$

the model becomes a ferromagnetic one in a staggered p-fold field

$$\tilde{h}_p \,\mathrm{Re} \sum_{\vec{r}} \exp(2\pi i\vec{\Delta}\cdot\vec{r}/a)\exp[ip\theta(\vec{r})] \ . \quad (1.3)$$

We can then, following José *et al.*,[4] consider the effect of small $\tilde{h}_p$ perturbations on the ferromagnetic planar model. This enables us to understand the phase diagram at small $\tilde{h}_p$. Of course, the extrapolation of $\tilde{h}_p \to \infty$ is questionable, but, for the case $\vec{\Delta} = 0$, rigorous statements can be made to justifiy this assumption.[5]

Our conclusions, based on this extrapolation, are as follows.[6]

(A) $\Delta_x \neq 0$, $\Delta_y \neq 0$. The results in this case are typified by those for the pure antiferromagnet $\vec{\Delta} = (\frac{1}{2}, \frac{1}{2})$. For p odd and ≥ 3 the models exhibit the two infinite-order phase transitions characteristic of a ferromagnetic clock model with $2p$ states.[4] The intermediate "floating" phase has a continuously varying exponent η in the range $1/p^2 \leq \eta \leq \frac{1}{4}$. The low, but nonzero, temperature phase is characterized by an effective $2p$-fold degeneracy. The order parameter, for example for the three-state Potts model, with a spin $s(\vec{r})$ capable of being in states A,B,C at each site, is a complex number

$$M = \langle \delta_{s,A} + \omega\delta_{s,B} + \omega^2\delta_{s,C}\rangle_{\mathrm{I}} - \langle \delta_{s,A} + \omega\delta_{s,B} + \omega^2\delta_{s,C}\rangle_{\mathrm{II}} \ , \quad (1.4)$$

where $\omega = e^{2\pi i/3}$, and I and II refer to the two sublattices. In the presence of a weak uniform field, there should be a single, Ising-like, transition, with an effective two-fold degeneracy of the low-temperature phase.

(B) $\Delta_x \neq 0$, $\Delta_y = 0$. Once again, for $p \geqslant 3$ there are two infinite-order transitions separated by a "floating" phase with $2/p^2 \leqslant \eta \leqslant \frac{1}{4}$. However, in this case, for $\Delta_x < \frac{1}{2}$, the low-temperature phase is ferromagnetic with a p-fold degeneracy. For the antiferromagnetic case, $\Delta_x = \frac{1}{2}$, the low-temperature phase cannot exist and the floating phase extends down to zero temperature. This feature is not apparent in our small $\tilde{h}_p$ analysis. A uniform field now acts to destroy the transition.

The layout of this paper is as follows. In the next section we analyze the case (A), treating the pure antiferromagnetic model in detail. In Sec. III the case (B) of a one-dimensional modulation is considered, and we close in Sec. IV with some further remarks on extensions to other models and dimensionalities.

II. PURE ANTIFERROMAGNET: $\vec{\Delta} = (\frac{1}{2}, \frac{1}{2})$

After the transformation (1.1), the Hamiltonian becomes

$$-\beta \mathcal{H} = \frac{1}{T_0} \sum_{\langle \vec{r}, \vec{r}' \rangle} \cos[\theta(\vec{r}) - \theta(\vec{r}')] + \tilde{h}_p \sum_{\vec{r}} (-1)^{m+n} \cos p\theta(\vec{r}) \ . \tag{2.1}$$

When $\tilde{h}_p = 0$, for T_0 sufficiently small, Eq. (2.1) will renormalize onto the Gaussian fixed line with an effective temperature T.[4,7] When $T > \pi/2$, the fixed line is unstable to the formation of vortices, and in this case we assume, as usual, that the flows will end in a high-temperature fixed point. Since vortex and h_p perturbations decouple to lowest orders, we are therefore led to consider a perturbed Gaussian model

$$-\beta \mathcal{H}_G = -\frac{1}{2T} \sum_{\langle \vec{r}, \vec{r}' \rangle} [\theta(\vec{r}) - \theta(\vec{r}')]^2 + \tilde{h}_p \sum_{\vec{r}} (-1)^{m+n} \cos p\theta(\vec{r}) \ . \tag{2.2}$$

In the case of a uniform field h_p, José *et al.*[4] showed that the Gaussian fixed line is stable to h_p perturbations if $T > 8\pi/p^2$. Thus a segment of the line is completely stable if $p > p_c = 4$. Models with $p > p_c$ and h_p initially small enough will exhibit two phase transitions, with $4/p^2 \leqslant \eta \leqslant \frac{1}{4}$ on the critical line separating them.

In our case we have a staggered field $\tilde{h}_p$. First we show that this is always irrelevant. To $O(\tilde{h}_p^2)$ the partition function has the form

$$Z = Z_G \left(1 + \tilde{h}_p^2 \sum_{m,n} \sum_{m',n'} (-1)^{m-m'+n-n'} G_p(m-m', n-n') + O(\tilde{h}_p^4) \right) , \tag{2.3}$$

where

$$G_p \sim \text{const}[(m-m')^2 + (n-n')^2]^{-p^2 T/4\pi} \tag{2.4}$$

as $|\vec{r} - \vec{r}'| \to \infty$. We now average over 2×2 cells containing sites (m,n), $(m+1,n)$, $(m,n+1)$, and $(m+1,n+1)$ where m and n are even. The $O(\tilde{h}_p^2)$ term is now proportional to

$$\sum_{\substack{m,n \\ \text{even}}} \sum_{\substack{m',n' \\ \text{even}}} \delta_m^2 \delta_n^2 G_p(m-m', n-n') \ , \tag{2.5}$$

where the finite difference operator δ_m^2 is defined by

$$\delta_m^2 f(m) = f(m+1) + f(m-1) - 2f(m) \ . \tag{2.6}$$

Note that Eq. (2.5) does not sum to zero because only even (m,n) are counted. This expression can, in the large distance limit, be replaced by an integral which, after angular terms have been averaged out, has the form

$$\tilde{h}_p^2 \int_{|\vec{r} - \vec{r}'| >> a} \frac{d^2r \, d^2r'}{a^4} \left(\frac{a}{|\vec{r} - \vec{r}'|} \right)^{p^2 T/2\pi + 4} \ . \tag{2.7}$$

From Eq. (2.7) the renormalization-group equation for $\tilde{h}_p$ under an infinitesimal change of scale $a \to ae^l$ may be read off. It is

$$\frac{d\tilde{h}_p}{dl} = -\frac{p^2 T}{4\pi} \tilde{h}_p \ . \tag{2.8}$$

Thus $\tilde{h}_p$ is always irrelevant except at $T = 0$. However, this is not the whole story, since, as was pointed out for the case $p = 2$ by Knops,[8] under renormalization a *uniform* $\cos 2p\theta$ field will be generated. The easiest way to see this is to add a term $h_{2p} \sum_{\vec{r}} \cos 2p\theta(\vec{r})$ to the Hamiltonian, and consider the $O(h_{2p}\tilde{h}_p^2)$ term in Z which is

$$h_{2p}\tilde{h}_p^2 \sum_{\vec{r}, \vec{r}_1, \vec{r}_2} (-1)^{m_1+n_1+m_2+n_2} \langle \cos 2p\theta(\vec{r}) \cos p\theta(\vec{r}_1) \cos \theta(\vec{r}_2) \rangle_G \ . \tag{2.9}$$

At large distances the correlation function in Eq. (2.9) is proportional to

$$\left[\frac{a}{|\vec{r}-\vec{r}_1|}\right]^{2p^2T/2\pi}\left[\frac{a}{|\vec{r}-\vec{r}_2|}\right]^{2p^2T/2\pi}\left[\frac{|\vec{r}_1-\vec{r}_2|}{a}\right]^{p^2T/2\pi} . \tag{2.10}$$

We now average $\vec{r}_1$ and $\vec{r}_2$ over 2×2 cells, as before, by applying the finite difference operator $\delta_{m_1}\delta_{m_2}\delta_{n_1}\delta_{n_2}$. The dominant term in the limit $|\vec{r}-\vec{r}_1|\to\infty$, $|\vec{r}_1-\vec{r}_2|$ fixed, which we shall need, is generated when this operator acts only on the third factor in Eq. (2.10). This piece has the form, on replacing sums over even (m_1,n_1) and (m_2,n_2) by integrals, and averaging over angular terms

$$h_{2p}\tilde{h}_p^2 AT\int\frac{d^2r d^2r_1 d^2r_2}{a^6}\left[\frac{a}{|\vec{r}-\vec{r}_1|}\right]^{2p^2T/2\pi}\left[\frac{a}{|\vec{r}-\vec{r}_2|}\right]^{2p^2T/2\pi}\left[\frac{\vec{r}_1-\vec{r}_2}{a}\right]^{p^2T/2\pi-4} , \tag{2.11}$$

where A is positive for $T<4\pi/p^2$. On changing $a\to ae^l$ in the cutoff $|\vec{r}_1-\vec{r}_2|>a$, a term proportional to

$$lh_{2p}\tilde{h}_p^2 AT\int\frac{d^2r d^2r_1}{a^4}\left[\frac{a}{|\vec{r}-\vec{r}_1|}\right]^{4p^2T/2\pi} \tag{2.12}$$

will appear which must be interpreted as a renormalization of the $O(h_{2p}^2)$ term. Thus, to lowest order, the renormalization-group equation for h_{2p} takes the form

$$\frac{dh_{2p}}{dl}=\left[2-\frac{p^2T}{\pi}\right]h_{2p}+AT\tilde{h}_p^2 , \tag{2.13}$$

h_{2p} is thus irrelevant only for $T>2\pi/p^2$, and there will be a stable segment of the Gaussian line if $p>p_c=2$. Although $\tilde{h}_p$ is irrelevant, for sufficiently low (but nonzero) temperatures h_{2p} will be generated and renormalize to large values, resulting in a $2p$-fold degeneracy. Note that at $T=0$ no h_{2p} term is generated, and so the character of the low-temperature phase is quite different from that of the zero-temperature state.[9] The order parameter, in terms of the original variables $\theta(\vec{r})$ in Eq. (1.1) is

$$M=\langle e^{i\theta(\vec{r})}\rangle_{\mathrm{I}}-\langle e^{i\theta(\vec{r}')}\rangle_{\mathrm{II}} , \tag{2.14}$$

where I and II refer to the two sublattices. For the case $p=3$ we obtain the order parameter quoted in the Introduction.

The critical behavior should be similar to that of the $2p$-state ferromagnetic clock model. At the upper transition T_{c2}, the susceptibility and correlation length will diverge with an essential singularity

$$\xi\propto\exp[-\mathrm{const}(T-T_c)^{-1/2}] \tag{2.15}$$

while the specific heat will have an unobservable essential singularity like ξ^{-2}. At the lower transition, T_{c1}, the order parameter M will vanish with the same type of essential singularity as $T\to T_{c1}$ from below. The correlation length will also diverge as $T\to T_{c1}$, like $|M|^{-2p^2}$.

Next, we consider the effect of adding a weak uniform field $\tilde{h}_1\sum_{\vec{r}}\cos\theta(\vec{r})$ to the Hamiltonian (1.1). After the transformation (1.2) this will give a staggered field, which will always be irrelevant, but, once again, this will generate a two-fold uniform field. The Gaussian line is always unstable to such perturbations for $T\leqslant\pi/2$,[4] and we expect the model to fall into the same universality class as the Ising model. This is reasonable on the basis of the symmetry of the ground state. For example, in the three-state Potts model, if the field favors the A state, the ground states will have a basic two-fold degeneracy, with either A on sublattice I and B or C on sublattice II, or vice versa. The remaining degeneracy between B and C is unimportant.

By the same argument, the addition of a $p=2$ field to the original model should have the same effect as cubic ($p=4$) symmetry breaking in the ferromagnetic model, which is known to correspond to another line of fixed points.[10] The antiferromagnetic clock model with $p\geqslant 3$ and a $p=2$ uniform field may then have a critical line with continuously varying exponents, like the Ashkin-Teller model.

Finally, we discuss how the results extend to the case of arbitrary nonzero Δ_x, Δ_y. Let us assume that Δ_x, Δ_y are both rational, with $\Delta_x=u_x/v_x$, $\Delta_y=u_y/v_y$ in irreducible form. The staggered field has the form

$$\tilde{h}_p\,\mathrm{Re}\left[\sum_{m,n}\exp[2\pi i(\Delta_x m+\Delta_y n)]\exp[ip\theta(\vec{r})]\right] . \tag{2.16}$$

In considering the analog of Eq. (2.3) we now average over $v_x\times v_y$ cells. The expression (2.7) is replaced by

$$\frac{\tilde{h}_p^2}{(1-\cos 2\pi\Delta_x)(1-\cos 2\pi\Delta_y)}\times\int\frac{d^2r d^2r'}{a^4}\left[\frac{a}{|\vec{r}-\vec{r}'|}\right]^{p^2T/2\pi+4} , \tag{2.17}$$

where we have displayed the full Δ_x, Δ_y dependence. Once again, the staggered field $\tilde{h}_p$ is irrelevant, but a uniform field h_{2p} is generated. The other main difference is that the coefficient A in Eq. (2.12) is now proportional to $(1-\cos 2\pi\Delta_x)^{-1}$ $(1-\cos 2\pi\Delta_y)^{-1}$. This shows that our analysis (assuming h_{2p} is small) breaks down for sufficiently small Δ_x, Δ_y. This is satisfying, since in the next section we find quite different results when one of them vanishes.

The main conclusions for the purely antiferromagnetic case extend to this case also. A subtle point arises if v_x or v_y is large, because then our analysis indicates that $\tilde{h}_p$ will only become irrelevant on scales larger than $v_x a$ or $v_y a$. If either Δ_x or Δ_y are irrational, our analysis fails completely. The fact that our final equations like Eq. (2.17) depend only on (Δ_x, Δ_y) and not (v_x, v_y) suggests that this is an artifact of our method. When $\Delta_y = 0$, we obtain, by the same method, results which agree with those obtained by methods[3] which make no reference to the rationality of Δ_x.

III. ONE-DIMENSIONAL MODULATION: $\vec{\Delta} = (\Delta_x, 0)$

In this case, the staggered field is

$$\tilde{h}_p \,\mathrm{Re}\left[\sum_m e^{2\pi i \Delta_x m} e^{ip\theta(\vec{r})}\right] \tag{3.1}$$

and the $O(\tilde{h}_p^2)$ contribution to the partition function is proportional to

$$\tilde{h}_p^2 \sum_{m,n} \sum_{m',n'} \exp[2\pi i \Delta_x (m-m')]\, G_p(m-m', n-n') \ . \tag{3.2}$$

As before, we let $\Delta_x = u_x / v_x$, but this time average over $v_x \times 1$ cells. This has the effect of introducing only two extra powers of $|\vec{r} - \vec{r}'|$ in the denominator, rather than four, and, in the large-distance limit, we obtain

$$\frac{\tilde{h}_p^2}{1 - \cos 2\pi \Delta_x} \int \frac{d^2 r d^2 r'}{a^4} \left[\frac{a}{|\vec{r} - \vec{r}'|}\right]^{p^2 T/2\pi + 2} . \tag{3.3}$$

The renormalization-group equation for $\tilde{h}_p$ is now

$$\frac{d\tilde{h}_p}{dl} = \left[1 - \frac{p^2 T}{4\pi}\right] \tilde{h}_p \ , \tag{3.4}$$

so that $\tilde{h}_p$ is irrelevant for $T > 4\pi/p^2$. Once again, a uniform field h_{2p} will be generated, with a renormalization-group equation

$$\frac{dh_{2p}}{dl} = \left[2 - \frac{p^2 T}{\pi}\right] h_{2p} + \left[\frac{A' T^2}{1 - \cos 2\pi \Delta_x}\right] \tilde{h}_p^2 \ , \tag{3.5}$$

where $A' > 0$ and is independent of T. Note the factor T^2, which appears after appropriate angular averages have been made. We see that h_{2p} is irrelevant for $T > 2\pi/p^2$. Thus the Gaussian fixed line is stable for $4\pi/p^2 < T < \pi/2$, and $p_c = 2\sqrt{2}$. Once again there are two transitions, the lower one now characterized by $\eta = 2/p^2$. The low-temperature phase is characterized by the staggered field $\tilde{h}_p$ becoming relevant, which means that the phase is ferromagnetic in terms of the original variables. However, when $\Delta_x = \frac{1}{2}$, the ferromagnetic state is clearly unstable, so the low-temperature phase does not in fact exist, and the floating phase extends down to zero temperature. When Δ_x is small our analysis fails, and we expect the behavior to be that of the ferromagnetic model.

All these results are in agreement with those of Ostlund,[3] who used a method based on the low-temperature domain-wall analysis of Villain and Bak.[11] This is an important check on our extrapolation to $\tilde{h}_p \to \infty$, since domain walls emerge only in this limit.

Finally we note that a uniform field $\tilde{h}_1$ in the original model will now always be relevant for $T < \pi/2$. We expect it to destroy the transition in this case. This is in agreement with the fact that the ground state now has no sublattice degeneracy.

IV. CONCLUSIONS AND FURTHER REMARKS

We have shown that the antiferromagnetic clock models (in general, the asymmetric models) have an interesting phase structure in two dimensions for $p \geqslant 3$. Although the ground state has a finite entropy per site, the conclusions of Berker and Kadanoff,[1] based on single-parameter recursion relations, appear to be wrong. Our analysis suggests that at least three parameters are required to describe the essential physics. There is an ordered low-temperature phase, whose degeneracy is, however, $2p$ fold and not that of the zero-temperature state.

Our analysis may be simply extended to $2+\epsilon$ dimensions,[12,13] when the fixed line disappears and all models considered have a single transition in the same universality class as the XY model. This is in agreement with the Monte Carlo data for the three-state model in three dimensions.[2] Our conclusions with regard to the low-temperature phase remain the same.

We have shown that models with antiferromagnetic couplings in only one direction have a quite different low-temperature phase, although they still show an intermediate floating phase in general. Finally, we mention that these methods may be applied to the general antiferromagnetic p-state Potts models near two dimensions, and work is in progress on this problem.

ACKNOWLEDGMENTS

I would like to thank S. Ostlund and S. Shenker for educational discussions. This work was supported by NSF Grant No. 78-08439.

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