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Quantum fluctuations in two-dimensional superconductors

S. Doniach
Department of Applied Physics, Stanford University, Stanford, California 94305
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For granular metallic films or two-dimensional arrays of Josephson junctions, fluctuations in the number of Cooper pairs are conjugate to the superconducting phase and can lead to zero-point destruction of phase coherence at zero temperature. In two dimensions the resulting critical behavior is shown to be that of the λ point of ^{4}He in three dimensions, with the charging energy playing the role of temperature. The nature of the quantum→classical crossover and its effect on the frequency dependence of precursor diamagnetism near the quantum critical point are discussed.

I. INTRODUCTION

The idea that zero-point fluctuations of the phase difference of the superconducting order parameter across a Josephson junction can become large as the effective local capacitance, C, of the junction tends to zero was proposed by Anderson in 1963.[1] Recently measurements by a number of groups have demonstrated the effects of this zero-point motion on the junction critical current.[2] For any assembly of coupled junctions, Abeles[3] pointed out that the zero-point phase fluctuations could actually destroy the phase coherence of the assembly for C below a critical value, even at zero temperature. A mean-field theory of this quantum phase transition has been worked out by Simanek[4] and Efetov.[5]

For two-dimensional arrays of junctions, or weakly coupled granular films for which the BCS coherence length $\xi_{\rm BCS}$ is smaller than the grain size, vortex-pair fluctuations will be expected to dominate the superconducting-to-paracoherent (resistive) transition at finite temperature.[6,7] The purpose of this paper is to show that space-time quantum fluctuations of the superconducting order parameter for a two-dimensional array of junctions will also strongly modify the nature of the zero-temperature paracoherent critical point away from the mean-field results.

II. EFFECTIVE GINZBURG-LANDAU FREE-ENERGY FUNCTIONAL

For grain sizes large compared to ξ, the Anderson formulation for charging fluctuations leads to the Hamiltonian

$$H = -\sum_{\langle ij\rangle} J\cos(\phi_i-\phi_j) - \alpha\sum_{\langle ij\rangle}\left(\frac{\partial}{\partial\phi_i}-\frac{\partial}{\partial\phi_j}\right)^2$$
$$= H_1 + H_0 \ , \qquad (1)$$

where $\alpha = 2e^2/C$ and we take the sum to be over nearest-neighbor pairs of grains on a two-dimensional lattice.[8] In order to discuss the phase diagram for this model it is convenient to use a Hubbard-Stratanovich[9] representation for the partition function in terms of "coarse-grained" classical local field variables $\vec{\psi}_k(\tau)$:

$$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\left[e^{-\beta H_0} T \exp\left(-\int_0^\beta d\tau\, H_1(\tau)\right)\right] \ , \qquad (2)$$

where T is the time-temperature ordering operator and $H_1(\tau) = e^{H_0\tau} H_1 e^{-H_0\tau}$. Then (2) may be rewritten as

$$Z = Z_0 D^{1/2} \int \prod_k \mathfrak{D}\psi_k(\tau) e^{-\mathfrak{F}[\psi]} \ , \qquad (3)$$

where

$$\mathfrak{F}[\psi] = \int_0^\beta d\tau \sum_k J_k^{-1} \vec{\psi}_k^{\,*}(\tau)\cdot\vec{\psi}_k(\tau) - \frac{1}{\beta}\ln\left[\left\langle T\exp\left(-2\int_0^\beta d\tau \sum_k \vec{\psi}_k(\tau)\cdot\vec{S}_{-k}(\tau)\right)\right\rangle\right] \qquad (4)$$

and $D = \prod_k J_k$ is the determinant of the coupling matrix.

Here

$$J_k = \frac{J}{N}\sum_{\langle ij\rangle}\exp(i\vec{k}\cdot\vec{R}_{ij}) \ ,$$

$\langle A\rangle_0$ denotes $\mathrm{Tr}(e^{-\beta H_0}A)/Z_0$, with

$$Z_0 = \mathrm{Tr}\, e^{-\beta H_0}, \quad \vec{S}_k = \sum_i e^{i\vec{k}\cdot\vec{R}_i}\begin{Bmatrix}\cos\phi_i\\ \sin\phi_i\end{Bmatrix} \ .$$

III. MEAN-FIELD STABILITY CONDITION

The mean-field approximation consists in replacing $\psi_k(\tau)$ by $N\delta_{k,0}\psi$ in Eq. (4) and looking for the saddle point in the free-energy functional:

$$\frac{\mathcal{F}_{\rm MF}}{N} = \frac{\beta}{\mathfrak{z}J}\psi^2 - \ln\left[\left\langle T\exp\left(-2\sum_i \vec{\psi}\cdot\int_0^\beta d\tau\,\vec{S}_i(\tau)\right)\right\rangle_0\right]$$
$$= \frac{\beta}{\mathfrak{z}J}\psi^2 - \ln\left\{\mathrm{Tr}\left\{\exp\left[-\beta\left(H_0 - 2\vec{\psi}\cdot\sum_i \vec{S}_i\right)\right]\right\}\Big/Z_0\right\}, \qquad (5)$$

where $\mathfrak{z}$ = number of nearest neighbors and $\beta = 1/T$ is the inverse temperature (units are chosen so that Boltzmann's constant k_B has the value unity).

The mean-field stability condition $\partial\mathcal{F}_{\rm MF}/\partial\psi = 0$ gives, on expanding to second order in $\vec{\psi}$ and orienting $\vec{\psi}$ along the x axis,

$$1 = 2\mathfrak{z}JT\int_0^\beta d\tau\,d\tau'\langle T[S_i^x(\tau)S_i^x(\tau')]\rangle_0$$
$$= 2\mathfrak{z}JT\int_0^\beta d\tau\,d\tau' g_0(\tau-\tau') \ . \qquad (6)$$

Using a quantum rotor representation $|m_i\rangle = (1/\sqrt{2\pi})\exp(im_i\phi_i)$, one has

$$g_0(\tau-\tau') = \frac{1}{Z_0}\sum_{\{m_i\}}\exp\left(-\beta\alpha\sum_{\langle ij\rangle}(m_i-m_j)^2\right)$$
$$\times\langle m_i|T[S_i^x(\tau)S_i^x(\tau')]|m_i\rangle \ , \qquad (7)$$

which is a correlation function of the two-dimensional discrete Gaussian model. At very low temperatures, $\beta\alpha >> 1$ and the main contribution comes from a state in which all m_i have the same value. Setting this equal to zero gives

$$g_0(\tau) = e^{-\alpha|\tau|} \qquad (8)$$

from which the mean-field stability condition (6) becomes

$$1 = 2\mathfrak{z}\frac{J}{\alpha}\left(1 - \frac{(1-e^{-\beta\alpha})}{\beta\alpha}\right) \quad \text{for } \beta\alpha >> 1 \ . \qquad (9)$$

This leads to a stability boundary at low temperatures given by

$$\alpha_0(T) = 2\mathfrak{z}J - T + O(T^2/J) \ . \qquad (10)$$

At high temperature ($\beta\alpha << 1$), all the rotors may be treated as independent, so that (7) becomes

$$g_0(\tau) \cong \frac{1}{Z_0}\sum_{m_i} e^{-\beta\alpha m_i^2} e^{-\alpha|\tau|}\cosh(2m_i\alpha|\tau|) \ ,$$

from which the stability condition (6) becomes

$$1 = \frac{2\mathfrak{z}J}{Z_0}\sum_m e^{-\beta\alpha m^2}\left(\frac{\beta^2}{2} - \frac{\beta^3\alpha}{6} + \cdots\right)$$
$$\cong (\mathfrak{z}J/T)(1-\alpha/3T) \qquad (11)$$

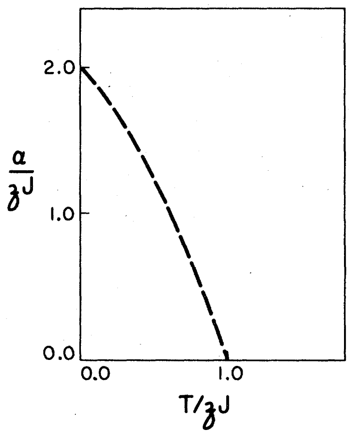

FIG. 1. Sketch of mean-field stability boundary as a function of quantum coupling α and temperature T.

leading to a high-temperature mean-field phase boundary (to order α),

$$T_0(J,\alpha) \cong \mathfrak{z}J - \alpha/3 \ .$$

The mean-field phase boundary is sketched in Fig. 1.

IV. FLUCTUATION EXPANSION AT LOW TEMPERATURE

For values of α and T near the mean-field stability line, the general free-energy functional (4) may be expanded in powers of $\psi_k(T)$ using a cumulant expansion. This leads to a Ginzburg-Landau-Wilson (GLW) form for $\mathcal{F}[\psi]$ which, as we shall see, has a critical superconducting-paracoherent phase boundary below the mean-field line.[10] In general, the coupling constants of this effective functional are only aproximately calculated using the cumulant expansion. However, we expect the universality class of the resulting critical behavior to be correctly represented using this approach even if the coupling constants are only given approximately in terms of the parameters of the original Hamiltonian (1).

Writing $\psi_k(\tau)$ in terms of Matsubara frequencies $\omega_m = 2\pi mT$ ($m = 0, \pm1, \pm2, \ldots$),

$$\vec{\psi}_k(\tau) = \sum_m e^{i\omega_m\tau}\vec{\psi}_{k,\omega_m}, \quad 0 \le \tau \le \beta \ , \qquad (12)$$

J may be expanded to fourth order in ψ as

$$\mathcal{F} = \sum_{k,m,m'}\vec{\psi}^{\,*}_{k,\omega_m}\cdot\vec{\psi}_{k,\omega_{m'}}[\beta J_k^{-1}\delta_{mm'} - 2g(\omega_m,\omega_{m'})]$$
$$+ \int_0^\beta d\tau\,U\sum_i[(\vec{\psi}_i)^2]^2 \ , \qquad (13)$$

where

$$g(\omega_m,\omega_{m'}) = \frac{1}{\beta^2}\int_0^\beta d\tau\,d\tau' e^{-i\omega_m\tau}e^{-i\omega_{m'}\tau'}g(\tau-\tau')$$

and the τ dependence of the fourth-order cumulant has been neglected. In the low-temperature regime ($\beta\alpha \ll 1$), we can use the limiting form (8) for $g(\tau)$ to give

$$g(\omega_m,\omega_{m'})=\delta_{mm'}\frac{\beta\alpha}{\omega_m^2+\alpha^2}-(1-e^{-\beta\alpha})\frac{\alpha^2-\omega_m\omega_{m'}}{(\omega_m^2+\alpha^2)(\omega_{m'}^2+\alpha^2)}\ , \tag{14}$$

which in general leads to damped propagation of the fluctuation modes. However, as $T\to 0$ ($\beta\to\infty$), the second term becomes $O(T/\alpha)$ with respect to the first and the modes may be treated as purely propagating.

Expanding J_k^{-1} to order k^2 and Eq. (14) to order ω_m^2 the low-temperature limiting form reduces to

$$\mathcal{F}=\beta\left\{\sum_{k,\omega_m}\vec{\psi}^{\,*}_{k,\omega_m}\cdot\vec{\psi}_{k,\omega_m}\left[(\mathfrak{z}J)^{-1}\left[1-\frac{k^2a^2}{2}\right]-\frac{2}{\alpha}\left[1-\frac{\omega_m^2}{\alpha^2}\right]\right]+U\sum_l\int_0^\beta d\tau[\vec{\psi}_l(\tau)\cdot\vec{\psi}_l(\tau)]^2\right\}\ , \tag{15}$$

with $U=(\mathfrak{z}J)^2/6\alpha^3$, which is an $n=2$ GLW model defined in a two-dimensional position-space and a finite time slice of "thickness" $\beta=1/T$.

In the limit $\beta\to\infty$, the sums over ω become integrals and the critical properties of Eq. (15) become those of a three-dimensional $n=2$ model, i.e., those of the λ point of liquid ^{4}He in three dimensions with the quantum coupling constant α playing the role of temperature.

V. CRITICAL-POINT PHASE BOUNDARY AT LOW TEMPRATURE

At $T=0$, the universal critical properties of the model may be deduced from what is known about the $d=3$, $n=2$ critical point.

The role of temperature is now played by the quantum fluctuation energy α, which varies as the sample parameters are changed. The critical value α_c will occur at some fraction of the mean-field instability coupling α_0, and for $\alpha<\alpha_c$ the superfluid density, ρ_S, will vary as

$$\rho_S(\alpha)\propto(\alpha_c-\alpha)^\beta\ , \tag{16}$$

where $\beta\cong\frac{2}{3}$.

At finite temperature, only the $\omega_m=0$ modes of the system will exhibit critical behavior so that the system will behave as a $d=2$ X-Y model, i.e., we expect it to show Kosterlitz-Thouless critical behavior at a renormalized two-dimensional critical temperature $T_{2D}(\alpha)$. In order to estimate the effect of quantum fluctuations on this transition, one needs a way to estimate the renormalization of the local superfluid density ρ_{SL} which determines the vortex-vortex coupling

$$U(\vec{r}_{ij})=-q_iq_j\ln(r_{ij}/\xi) \tag{17}$$

(where r_{ij} is the vortex spacing, $q_i=n_iq$ and $n_i=\pm1,\pm2,\ldots$ the vorticity) via the hydrodynamic relation for the vortex charge,

$$q^2=(\hbar^2/m)\rho_{SL}\ . \tag{18}$$

For α close to α_c, the zero-temperature coherence length for quantum order-parameter fluctuations,

$$\xi_0(\alpha)\propto(\alpha_c-\alpha)^{-\nu}\ , \tag{19}$$

will determine the length scale beyond which the superfluid density $\rho_S(\alpha)$ will describe the superfluid hydrodynamics. At finite temperature, leading to finite "film thickness" in the time dimension, $d_{\text{eff}}(T)=a(\alpha/T)$ [using Eq. (15) to relate the ω scale to the k scale] the system will behave three-dimensionally until a crossover temperature T_x is reached at which $d_{\text{eff}}\lesssim\xi_0(\alpha)$, giving

$$T_x(\alpha)\approx\alpha a/\xi_0(\alpha)\propto(\alpha_c-\alpha)^\nu\ . \tag{20}$$

At this temperature, T will still be very small compared to α_c, and so $\xi(\alpha)$ will not have changed very much from its zero-temperature value. Hence for $T\sim T_x(\alpha)$, ρ_{SL} will still be determined mainly by the quantum fluctuations and may be equated with $\rho_S(\alpha, T=0)$.

Using this relation and Eq. (18), the vortex charge at temperatures T such that the system is starting to behave two-dimensionally is given by

$$q^2\approx q_{\text{class}}^2(1-\alpha/\alpha_c)^\beta\ , \tag{21}$$

where q_{class}^2 is the $\alpha=0$ limit (classical) vortex charge. This leads to a predicted two-dimensional vortex unbinding temperature given via $T_{2D}\approx 2q^2$ as

$$T_{2D}(\alpha)\approx T_{2D}(0)(1-\alpha/\alpha_c)^\beta\ . \tag{22}$$

At this temperature, the ratio of the effective time thickness to the zero-temperature coherence length is given by

$$d_{\text{eff}}(T_{2D})/\xi_0(\alpha)\propto(1-\alpha/\alpha_c)^{\nu-\beta}\ . \tag{23}$$

Since ν and β are of the same order of magnitude for

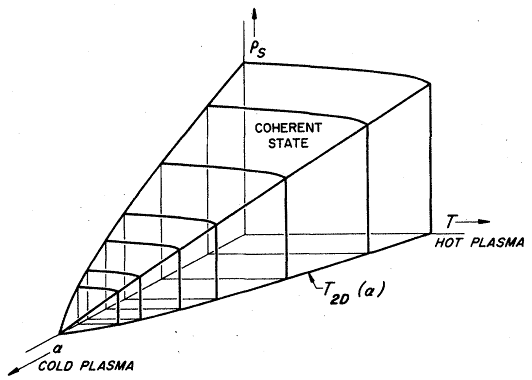

FIG. 2. Sketch of the effective superconducting density $\rho_S(\alpha,T)$, as measured in a diamagnetic response experiment, as a function of quantum fluctuation coupling constant α and temperature T expressed in units of the Josephson coupling energy J. The universal jump at T_{2D}, rescaled as α varies, is exhibited for several arbitrary choices of α.

$d=3$ and $n=2$,[10a] it may be seen that finite "time-size" effects will not play a very strong role in modifying Eq. (16) as T is raised to $T_{2D}(\alpha)$, so that, pending more detailed numerical investigation, Eq. (22) should provide a reasonable estimate of the universal part of the dependence of $T_{2D}(\alpha)$ on $\alpha_c-\alpha$.

Finally, following Nelson and Kosterlitz,[11] the global superfluid density $\rho_S(T,\alpha)$ measured, say, by the diamagnetic response will then exhibit a "universal jump" at $T_{2D}(\alpha)$ whose height is proportional to $T_{2D}(\alpha)$. So a plot of the effect of quantum zero-point fluctuations on the global superfluid density will exhibit a "cliff face" of universal jump as α is varied, dropping to zero at $\alpha\rightarrow\alpha_c$. (See Fig. 2.)

VI. EFFECT OF QUANTUM FLUCTUATIONS ON THE PRECURSOR DIAMAGNETISM

In order to test out the above ideas experimentally, it is necessary to have some way of characterizing α/J for a given sample. This may be done by measuring the precursor diamagnetism for temperatures above the two-dimensional critical temperature $T_{2D}(\alpha)$. In order for the unrenormalized coupling constants to be seen, $T-T_{2D}(\alpha)$ must be outside the critical region. In the presence of a magnetic field with vector potential $\vec{A}(x)$ the Josephson part of the Hamiltonian may be written

$$H_1=-J\sum_{\langle ij\rangle}\cos(\phi_i-\phi_j-A_{ij}) \ , \tag{24}$$

where

$$A_{ij}=\int_{x_i}^{x_j} d\vec{l}\cdot\vec{A}(x) \ .$$

Using a complex notation for the phase variables defined in real space (not to be confused with the complex representation of the k-space transform) this becomes

$$H_1=-\tfrac{1}{2}J\sum_{\langle ij\rangle}(e^{iA_i}S_i^*S_je^{-iA_j}+\text{c.c.}) \ , \tag{25}$$

where

$$A_i=\int_{-\infty}^{x_i} d\vec{l}\cdot\vec{A}(x)$$

and the Hubbard-Stratanovich representation becomes

$$\mathfrak{F}[\psi]=\int_0^\beta d\tau\sum_{ij}\psi_i^*e^{-iA_i}J_{ij}^{-1}e^{iA_j}\psi_j -\frac{1}{\beta}\ln\left[\left\langle T\exp\left[-\int_0^\beta d\tau\sum_i\psi_i^*S_i(\tau)\right]+\text{H.c.}\right\rangle_0\right] \ , \tag{26}$$

where the ψ variables have been redefined $\psi_i\rightarrow\psi_i\exp(-iA_i)$ to take up the gauge factors in Eq. (25).

Expanding the first term of Eq. (26) in powers of the lattice constant a, and inserting the cumulant expansion (15) for the second term, the continuum limit becomes

$$\mathfrak{F}[\psi]=\int_0^\beta\int \alpha^2x\left[-(\mathfrak{z}J)^{-1}\left(a^2\psi^*(x,\tau)[\vec{\nabla}-i\vec{A}(x)]^2\psi(x,\tau)+\psi^*(x,\tau)\psi(x,\tau)r(T,\alpha)-\frac{2}{\alpha^3}\left|\frac{\partial\psi}{\partial\tau}\right|^2+U[|\vec{\psi}(x,\tau)|^2]^2\right)\right] \ , \tag{27}$$

where

$$r(T,\alpha)=[(\mathfrak{z}J)^{-1}-2/\alpha+2T/\alpha^2] \ . \tag{28}$$

The diamagnetic susceptibility in the paracoherent region may now be calculated by expanding in powers of $\vec{A}(x)$ to give, in the Gaussian limit,

$$\chi(\alpha,T)=\frac{\partial^2(-T\ln Z)}{\partial(\vec{\nabla}\times\vec{A})^2}=\chi_0\lim_{q\rightarrow 0}\frac{\partial}{\partial q^2}\sum_{k,\omega_m}\frac{\vec{k}\cdot\vec{q}}{\xi^{-2}+(\vec{k}+\vec{q})^2+c^{-2}\omega_m^2}\,\frac{a^{-2}}{\xi^{-2}+k^2+c^{-2}\omega_m^2} \ . \tag{29}$$

Here $\chi_0 = \frac{4}{3}\pi T(a/\phi_0^2)$ is a susceptibility per unit volume, with ϕ_0 the quantum of flux (hc/e), $\xi^{-2}(\alpha,T) = a^{-2}\mathfrak{z}Jr(T,\alpha)$ is the Ginzburg-Landau coherence length in the paracoherent region and c is the group velocity of the Josephson plasmon modes given by ($\omega_m = 2\pi m T$ has dimensions of energy),

$$c = \alpha a \ . \qquad (30)$$

On performing the k integration, Eq. (29) becomes[12]

$$\frac{\chi(\alpha,T)}{\chi_0} = \sum_m \frac{2a^{-2}}{(\xi^{-2}+c^{-2}\omega_m^2)} = \frac{\xi c}{2Ta^2}\coth\left[\frac{\xi^{-1}c}{2T}\right] \ . \qquad (31)$$

Note that $\chi(\alpha,T)$ remains finite as $T \to 0$ for $\alpha > \alpha_c$ since χ_0 is proportional to T. Since Eq. (31) has been obtained in a Gaussian approximation, it will not be valid as the critical phase boundary is approached. Nonetheless, we use it to get an indication of the classical-to-quantum crossover as (T,α) are increased away from the critical boundary. To lowest order in $(2J\mathfrak{z} - \alpha)$, ξ^{-2} may be written as

$$\xi^{-2}(T,\alpha) = \xi_0^{-2}\left[\frac{T - T_0(\alpha)}{T_0(\alpha)}\right] = \xi_0^{-2}t \ , \qquad (32)$$

where $T_0(\alpha)$ is the mean-field estimate of the critical boundary

$$T_0(\alpha) = (2\alpha/\alpha_0)(\alpha_0 - \alpha) \ , \qquad (33)$$

with $\alpha_0 = 2\mathfrak{z}J$, and $\xi_0^{-2} = a^{-2}T_0/\alpha = a^{-2}(\alpha_0 - \alpha)/\alpha$, and Eq. (31) may be reexpressed as

$$\frac{\chi(\alpha,T)}{\chi_0} = \frac{(\xi_0 c/a^2)}{2(1+t)\sqrt{t}}\coth\left[\frac{\xi_0^{-1}c}{2T_0}\frac{\sqrt{t}}{(1+t)}\right] \ . \qquad (34)$$

Hence, for $\xi_0^{-1}c/2T_0 \gtrsim 1$ the quantum limit (where the argument of the hyperbolic cotangent is $>>1$) will extend to small values of t, i.e., inside the critical region. Using Eqs. (32) and (33) this condition reads

$$1 - \alpha/\alpha_0 \lesssim \tfrac{1}{4} \ .$$

Hence there will be a range of coupling constants, α, above which quantum fluctuations will be expected to dominate the diamagnetic fluctuations in the region of the critical phase boundary even at finite temperatures. The crossover temperature is given in this approximation by

$$\frac{\xi_0^{-1}c}{2T_0}\frac{\sqrt{t_x}}{(1+t_x)} = 1 \ ,$$

giving

$$t_x \cong 4(1 - \alpha/\alpha_0) \ . \qquad (35)$$

Since the phase boundary becomes renormalized to lower temperatures by the non-Gaussian ψ^4 interactions, Eq. (35) may be expected to provide an overestimate of the classical-to-quantum crossover regime (see hatched region in Fig. 3). So observation of the quantum exponent $1/\sqrt{t}$ temperature dependence of fluctuation diamagnetism as opposed to the $1/t$ Curie-like behavior of the classical precursor fluctuations[12] will provide a signal for samples which have entered the quantum regime. The magnitude of the coefficient of the $1/\sqrt{t}$ term in expression (34) for χ, proportional to $1/\xi_0^{-1}a = [\alpha/(\alpha_0 - \alpha)]^{1/2}$, will give a measure of closeness to the critical value of the quantum coupling. The above Gaussian estimate relates this to the mean-field value α_0, though renormalization of the Gaussian propagator will reduce this to α_c. In practice this may be hard to determine from a static measurement owing to uncertainties about other material parameters, and a frequency-dependent measurement will give a better way to estimate this parameter. This is discussed in the remainder of the paper.

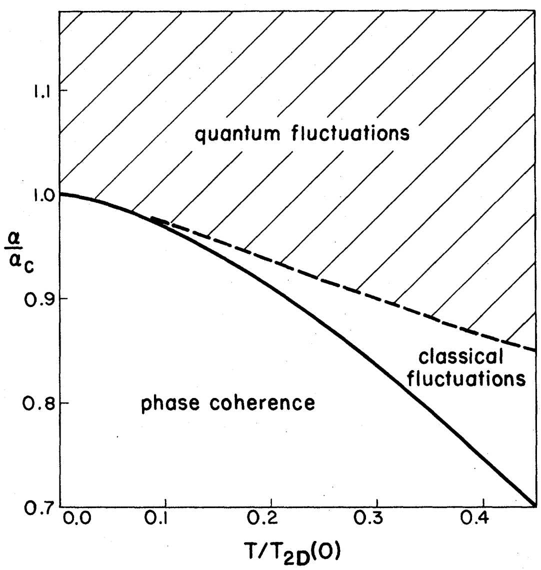

FIG. 3. Sketch of the classical-to-quantum crossover line in the vicinity of α_c. The Gaussian estimate [Eq. (35)] for $t_x = [T_x/T_{2D}(\alpha) - 1]$ has been tacked on to the renormalized curve for $T_{2D}(\alpha)$ [Eq. (22)].

VII. FREQUENCY DEPENDENCE OF THE DIAMAGNETIC FLUCTUATIONS–LOW-TEMPERATURE LIMIT

As may be seen from Eq. (29), the measurement of a diamagnetic response corresponds to the excitation of pairs of order-parameter propagators $\langle \psi(x,\tau)\psi^*(x',\tau')\rangle$ summed over the Brillouin zone of wave vectors. The effect of the quantum term in the Hamiltonian is now to produce propagating Josephson plasmon modes of energy

$$\Omega_k = c(\xi^{-2} + k^2)^{1/2} \ . \quad (36)$$

So at finite frequencies, the diamagnetic response will contain an absorptive part even at a temperature so low that the usual normal-electron damping mechanism for quantum phase fluctuations has become very small (since the normal-electron component becomes exponentially small as $T \rightarrow 0$). This absorption corresponds to the excitation of propagating Josephson plasmon modes above a threshold frequency $\Omega_0 = c\xi^{-1}$, which will tend to zero as the critical phase boundary is approached. In order to calculate the strength of this absorption within the framework of the Hubbard-Stratanovich effective free-energy functional (15), we use the fact that the linear diamagnetic response to a time varying vector potential $\vec{A}(x,t)$ may be expressed in terms of the analytic continuation of the current-current correlation function defined for imaginary time $t \rightarrow i\tau$:

$$C_{\alpha\beta}(x,\tau) = \langle j_\alpha(x,\tau) j_\beta(0,0) \rangle_{A\rightarrow 0} = \left[\frac{\delta}{\delta A_\alpha(x,\tau)} \frac{\delta}{\delta A_\beta(0,0)} [-T \ln Z(A)] \right]_{A\rightarrow 0} , \quad (37)$$

where $Z(\vec{A})$ is determined in terms of a τ-dependent density matrix satisfying

$$\partial\rho/\partial\tau = [H(\tau), \rho], \quad \rho(0) = e^{-\beta H(0)} , \quad (38)$$

and $\vec{A}(\tau)$ is switched on at $\tau = 0$. This may be calculated in terms of the ψ variables via the explicit τ dependence of Eq. (27) introduced through $\vec{A}(\tau)$. In the Gaussian approximation, the Matsubara transform of Eq. (37) becomes

$$C_{\alpha\beta}(q, i\omega_m) = \sum_{k,\epsilon_n} \left[\frac{(k_\alpha)(k+q)_\beta}{\Omega_{k+q}^2 + (\epsilon_n - \omega_m)^2} \frac{1}{(\Omega_k^2 + \epsilon_n^2)} - \delta_{\alpha\beta} \frac{k^2}{(\Omega_k^2 + \epsilon_n^2)^2} \right] , \quad (39)$$

leading to a diamagnetic susceptibility whose absorptive part is given by

$$\chi''_{\alpha\beta}(\omega) = \frac{1}{2} \lim_{q\rightarrow 0} \frac{\partial}{\partial q^2} \{\mathrm{disc}[C_{\alpha\beta}(q,\omega)]\} , \quad (40)$$

where the discontinuity is across the real cut of expression (39) in the complex ω plane. The real part may be obtained by a Kramers-Kronig transform of Eq. (40). Using a contour integral representation, the analytic continuation of Eq. (39) becomes, as $q \rightarrow 0$, in units of $\chi_0 a^{-2}$,

$$\left[\frac{\partial}{\partial q^2} C_{\alpha\beta}(q,z) \right]_{q\rightarrow 0} = \frac{\delta_{\alpha\beta}}{2} \int d\zeta \coth\left[\frac{\beta\zeta}{2}\right] \sum_k \frac{k^2}{\Omega_k^2 - (\zeta - z)^2} \frac{1}{\Omega_k^2 - \zeta^2} , \quad (41)$$

leading to

$$\chi''(\omega) = \frac{1}{2\omega^5} \sum_k \coth\left[\frac{\beta\Omega_k}{2}\right] k^2 [\delta(\omega - 2\Omega_k) + \delta(\omega + 2\Omega_k)]$$
$$= \coth\left[\frac{\beta\omega}{4}\right] \frac{1}{2\omega^4} \left[\frac{\omega^2}{4} - \xi^{-2}c^2\right] [\Theta(\omega - 2c\xi^{-1}) + \Theta(-\omega - 2c\xi^{-1})] \ . \quad (42)$$

So at low temperatures for which damping can be neglected, the diamagnetic absorption will have a frequency gap with threshold $\omega = 2\Omega_0 = 2c\xi^{-1}(T,\alpha)$ and strength which grows quadratically as $\omega^2 - 4\Omega_0^2$ due to the current matrix elements k^2 in Eq. (42) with a maximum at $4\Omega_0$ (see Fig. 4). As the phase boundary is approached the Josephson plasma frequency Ω_0 will go to zero and so will the absorption gap.

VIII. EFFECT OF NORMAL-ELECTRON DAMPING ON THE FREQUENCY DEPENDENCE OF THE PRECURSOR DIAMAGNETISM

The finite-frequency diamagnetic absorption derived in Sec. VII is valid only in the limit that classical damping effects which usually dominate the frequency-depenent linear response of a Josephson junction are neglected. This phenomenological damping of form $-\gamma\, d\phi/dt$ in the equation for the phase, which arises at low temperatures from voltage fluctuations induced by normal-current noise, cannot be incorporated directly into the Lagrangian form for the free-energy functional used in the above derivation. We here simulate its effects by incorporating a self-energy part in the propagator for order-parameter fluctuations used in Eq. (39). This would correspond to a Lagrangian in which coupling to heat-bath modes have been included.[13]

Writing

$$G(k, \epsilon_n) = \frac{1}{\Omega_k^2 + \epsilon_n^2 + \Pi(i\epsilon_n)} , \quad (43)$$

with

$$\Pi(i\epsilon_n)=\frac{1}{\pi}\int d\epsilon \frac{\gamma(\epsilon)\epsilon}{\epsilon_n^2+\epsilon^2} , \tag{44}$$

where $\gamma(\epsilon)$ represents a spectral function for coupling to a heat bath, the current correlation function (39) may be generalized to

$$C_{\alpha\beta}(q,i\omega_m)=\sum_{k,\epsilon_n}\{k_\alpha(k+q)_\beta G(k+q,\epsilon_n+\omega_m) \times G(k,\epsilon_n)-\delta_{\alpha\beta}k_\alpha^2[G(k,\epsilon_n)]^2\} . \tag{45}$$

Provided the coupling to the heat bath spreads over a wide range of frequencies compared to that of the Josephson modes, the self-energy (44) may be replaced by a purely imaginary damping

$$\Pi(\omega\pm i\epsilon)\to\pm i\gamma \tag{46}$$

for ω small compared to the range of ϵ over which $\gamma(\epsilon)$ varies rapidly. We then find

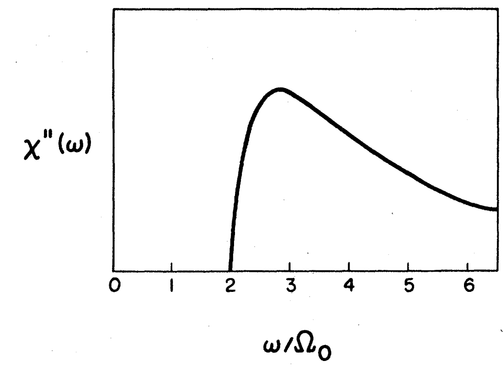

FIG. 4. Plot of the absorptive part, χ'', of the precursor diamagnetism in the low-temperature regime close to α_c. Damping effects will tend to broaden the threshold for excitation of Josephson plasmons. $\chi''(\omega)$ is given in arbitrary units.

$$\chi''(\omega)=\sum_k\frac{k^2}{\pi^2}\int dx\coth\left[\frac{\beta x}{2}\right]\frac{\gamma^2}{(\Omega_k-x)^2+\gamma^2}\frac{4\gamma[\Omega_k^2-(x-\omega)^2]}{\{[\Omega_k^2-(x-\omega)^2]^2+\gamma^2\}^2} \tag{47}$$

reducing to Eq. (45) in the limit $\gamma\to 0$. So damping effects will become important for $\gamma/\Omega_0\gtrsim 1$ and the absorption features of Fig. 4 will become washed out as the Josephson plasmon threshold approaches the critical phase boundary.

IX. CONCLUSIONS

Observation of the reduction of the superconducting transition temperature in granular-metal-film—two-dimensional arrays of Jospehson junctions as the grain-size—junction capacitance is reduced will in practice be due both to a reduction in average Josephson coupling energy J, and to an increase in the quantum fluctuation parameter α. The above analysis suggests that these may be sorted out by observation of the frequency dependence of the precursor diamagnetic fluctuations in the paracoherent phase. As shown in Fig. 4, a Gaussian approximation predicts this should display a maximum absorption at frequency proportional to $\Omega_0=c\xi^{-1}(\alpha,T)$ whose temperature dependence (for fixed α) will vary as $[T/T_{2D}(\alpha)-1]^\nu$, with $\nu=\frac{1}{2}$ in the Gaussian limit. The coefficient of this temperature-dependent factor will then give a direct measure of the scale of the quantum fluctuations in a given sample, as discussed in Sec. VI. As seen in Sec. VII, damping effects will tend to wash out the threshold of the Josephson plasmon-mode absorption spectrum, but should not change the above general relation of the frequency of maximum absorption to the inverse coherence length $\xi^{-1}(\alpha,T)$ for the paracoherent fluctuations, provided the system is close to the quantum critical point α_c, so that the low-temperature limits taken in the above discussion of fluctuation effects [Eq. (15)] are applicable. As the classical critical regime is approached, vortex fluctuations (not discussed here) will start to affect the frequency-dependent response.[14]

As the critical phase boundary is approached, the Gaussian approximation will be renormalized by mode-mode coupling effects and critical slowing down may be expected to take place. However, the effects of disorder in real granular systems will probably affect the quantitative details of this renormalization.

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