Exactly soluble Ising models on hierarchical lattices

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Certain approximate renormalization-group recursion relations are exact for Ising models on special hierarchical lattices, as noted by Berker and Ostlund. These lattice models provide numerous examples of phase coexistence and critical points at finite temperatures, including cases of continuously varying critical exponents and phase transitions without phase coexistence. The lattices are, typically, quite inhomogeneous and may possess several inequivalent limits as infinite lattices.

The discovery of a large class of exactly soluble lattice models exhibiting a variety of first-order phase transitions and critical phenomena (with nonclassical exponents) at finite temperatures would ordinarily give rise to considerable excitement in the world of statistical mechanics. However, the very important observation of Berker and Ostlund[1] that the approximate Migdal-Kadanoff[2] real-space recursion relations applied to Ising models on Bravais lattices provide the *exact* solution for an Ising model on a very different lattice, which in effect amounts to such a discovery, has received very little attention.

We wish to point out that these "hierarchical lattices," as we shall call them, and the corresponding Ising (and Potts, etc.) models, are interesting objects in and of themselves, and the associated phase transitions are worth studying for a variety of reasons. To begin with, they provide elementary examples of duality transformations, continuously varying critical exponents, phase transitions without true long-range order (e.g., without spontaneous magnetization), and similar phenomena whose study in exactly soluble models has usually required considerable effort and mathematical skill. Second, these lattices are much more inhomogeneous; i.e., they have a much lower symmetry, than are Bravais lattices, and thus they may provide insights into other low-symmetry problems such as random magnets, surfaces, and the like. Third, a study of models on which real-space renormalization-group methods are exact may throw some light on the situations in which such methods are probably to some degree misleading.[3] In this Communication we report several results related to the first and second of these topics.

Previous studies of Ising models on hierarchical lattices include investigations of some cases in which the phase transitions only occur at zero temperature,[4–6] and calculations on Bethe lattices with a surface.[7,8] Forgacs and Zawadowski[9] have noted another situation in which a Migdal type of recursion relation is exact for a special model, but which does not seem to be associated with a hierarchical lattice as we define it here. Our definition also excludes those self-similar lattices of Gefen *et al.*[6] which (presumably) possess phase transitions at finite temperatures. Also note that Dyson's *hierarchical model,*[10] in which each spin interacts with an infinite number of neighbors, does not involve a hierarchical *lattice* on which each (or almost every) spin interacts with a finite number of neighbors. Bernasconi *et al.*[11,12] have used the lattice of Fig. 1 in a study of random conductance networks.

One example of a hierarchical lattice is shown in Fig. 1. Five of the zero-order or *primitive* bonds in Fig. 1(a) are assembled to form a unit or "bond" of order one in Fig. 1(b). At the next stage, five of these units are assembled *in the same manner* to form a bond of order two in Fig. 1(c). Repetition of this procedure produces units of arbitrarily high order. One can, alternatively, view Fig. 1 as a prescription for "miniaturization": The bond of order N shown (schematically) in Fig. 1(a) possesses an internal structure revealed in Fig. 1(b), where the lines are bonds of order $N-1$, and in still greater detail in Fig. 1(c), where the bonds are of order $N-2$.

Note that such a lattice contains sites with different coordination numbers, 3×2^n, $n=0,1,2,\ldots$. Let $\sigma_i=\pm 1$ be the Ising spin variable associated with the ith site. The dimensionless interaction associated

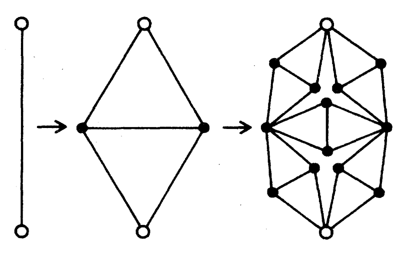

FIG. 1. (a)–(c) Construction of a particular hierarchical lattice.

with the primitive bond from i to j is

$$-\mathcal{H}_0/kT = \hat{H}_0(\sigma_i, \sigma_j) = K_0\sigma_i\sigma_j \ , \tag{1}$$

and the total (dimensionless) Hamiltonian $\hat{H}$ is the sum of $\hat{H}_0$ over all primitive bonds. (One could, of course, include a magnetic field in $\hat{H}_0$.) The partition function is obtained by successive decimation: $e^{\hat{H}}$ is first summed over spins at clusters of neighboring sites of coordination number 3, producing an effective interaction $\hat{H}_1$ between spins at pairs of sites adjacent to each cluster. The next sum is over spins at sites of coordination number 6, then 12, etc.

The hierarchical arrangement of sites means that at each stage the spins that are summed out or "decimated" belong to *finite* clusters interacting with a *finite* number of neighbors outside the cluster (but not with spins in other clusters). We shall use the term "hierarchical" to denote lattices having the property that the two finite numbers just mentioned are bounded, independent of the cluster and the stage of decimation. In the case under consideration they are, of course, constant, which is what makes this Ising model "soluble" by elementary methods.

The aggregation instructions for a number of other hierarchical lattices are shown in Fig. 2. In each case two stages of aggregation are shown, starting with the primitive bond in Fig. 2(a). Figure 2(b) is that discussed by Berker and Ostlund.[1] In Fig. 2(c) the dashed line is a *noniterated* bond representing an energy as in (1) but with an interaction constant K' independent of K_0 and also independent of the stage of aggregation. [Thus the five dashed lines in the second part of Fig. 2(c) all correspond to the same interaction.]

Hierarchical lattices can also be constructed by aggregation of units involving 3, 4, etc., sites. In the example in Fig. 3, four squares of order $N-1$ are

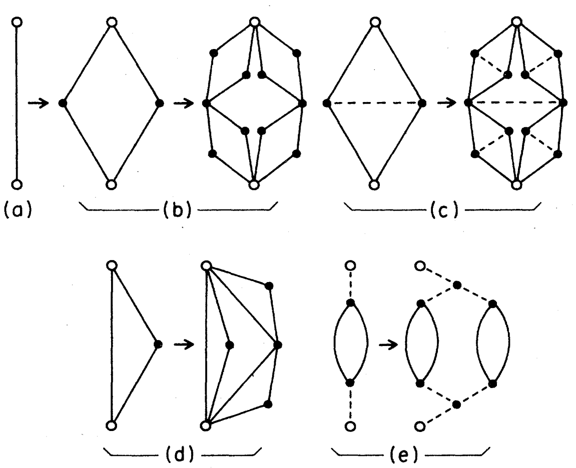

FIG. 2. (a)–(e) Construction of additional hierarchical lattices.

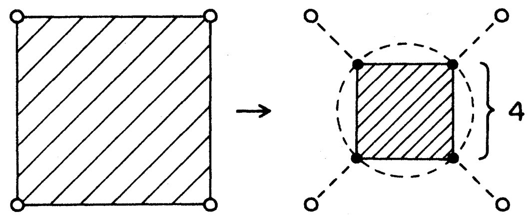

FIG. 3. Hierarchical lattice using aggregation of squares.

combined by placing them on top of each other and identifying the corner vertices. Noniterated bonds then connect these vertices to those of the square of order N, and the dashed circle stands for various noniterated two- and four-spin interactions. Kadanoff's "lower bound" bond-shifting approximation[13] to a square lattice is the exact solution[14] to the Ising model on the lattice in Fig. 3 if the straight dashed bonds are equal to p (in his notation) and the other noniterated interactions are chosen appropriately.

In all of these cases the *aggregation number B* is defined as the number of *iterated* units which are assembled at each step to form the unit of next high order. It is 4, 4, 3, 2, and 4 in Figs. 2(b), 2(c), 2(d), 2(e), and Fig. 3, respectively. Note that noniterated units (dashed bonds) are *not* counted in *B*.

More than one aggregation procedure can be used in constructing a hierarchical lattice. An example is shown in Fig. 4, whose significance will be clear if it is regarded as a prescription for miniaturization. If one chooses the two primitive bonds to be ferromagnetic but of different strengths, the result is a hierarchical model which mimics certain features of the "rectangular" Ising model (i.e., an Ising model on a square lattice with unequal horizontal and vertical exchange interactions).

We have obtained a number of results on the properties of Ising and similar models on various hierarchical lattices, among them the following (details will be published elsewhere):

(i) For $B \geqslant 2$ the free energy per primitive unit, $B^{-N}\ln Z_N$, where Z_N is the partition function of a unit of order N, is well defined as $N \to \infty$ under fairly general conditions, including those in which the noni-

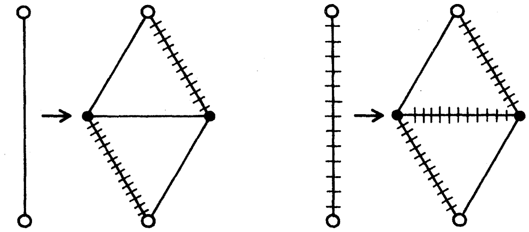

FIG. 4. Hierarchical lattice with two kinds of iterated bonds.

terated interactions are allowed to vary (within certain limits) with the aggregation step.

(ii) There is *not* a *single* infinite lattice (regarded as a connected graph) associated with the aggregation procedures of Figs. 1 and 2(b), 2(c), and 2(d). Instead there is a large number of *inequivalent* infinite lattices. The thermodynamic properties of the inequivalent lattices are the same, but local statistical properties—e.g., the average $\langle \sigma_i \rangle$—can show substantial variation within one infinite lattice, and between different inequivalent lattices.

(iii) The lattices just mentioned are extremely inhomogeneous in the sense that each group of sites equivalent to one another under the symmetries of the lattice contains a vanishingly small fraction of the total number of sites as $N \rightarrow \infty$. This is, of course, very different from the usual Bravais lattice, in which all sites are identical. The examples in Figs. 2(e) and 3 are intermediate in the sense that while not all sites are equivalent, the fraction in each equivalence class is finite in an infinite lattice.

(iv) When noniterated interactions are present, as in Figs. 2(c), 2(e), and 3, and the model has a critical point, the critical fixed point and the associated exponents depend on the value(s) of the noniterated interactions, giving rise to a continuous variation of critical exponents. This phenomenon has been noted previously in special cases[8,13] of hierarchical lattices.

(v) We can prove that phase coexistence at finite temperatures in hierarchical lattices of the types shown in Figs. 1–3 is only possible if either the effective Hamiltonian H_N or the noniterated interactions tend to infinity with the step of aggregation. In particular, if the noniterated interactions are independent of the aggregation step in Figs. 2(e) and 3, there is no phase *coexistence* at any finite temperatures, even though there are phase *transitions* (nonsmooth behavior of the free energy). Thus phase transitions without a finite order parameter, noted previously in a particular case,[8] are easily produced in hierarchical lattices.

(vi) By contrast, the lattice of Fig. 2(d), along with certain other cases in which the bond of order N contains at least one bond of order $N-1$ extending directly between its two vertices, has phase coexistence at *all* finite temperatures.

(vii) The lattice in Fig. 1 is self-dual,[11] and it is possible to work out the phase diagram of the Ashkin-Teller model[15] in complete detail, and compare it with exact and conjectural results for a square lattice.[16]

We have also studied the q-state Potts model[17] on the self-dual lattice of Fig. 4 and compared the answers with exact results on a rectangular lattice.[18,19]

In closing we wish to mention an important *unresolved* problem which we find rather perplexing: that of defining a correlation length and/or a dimensionality d for a hierarchical lattice. The two problems are not unrelated, since the aggregation number B is denoted by b^d in renormalization-group language,[1,4] with b the factor by which the correlation length (on a Bravais lattice) is reduced at each iteration. Since B is known, there is no ambiguity in defining the *thermodynamic* critical exponents using fixed-point eigenvalues. However, correlation-length exponents are not well defined unless b (equivalently, d) is known.

It may be the case that d (or b) has no well-defined meaning for hierarchical lattices in general. Certainly none of the proposals of which we are aware[4–6] seems very compelling. In any case the matter deserves further attention.

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