Universality and the power spectrum at the onset of chaos

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Two one-dimensional maps are iterated to evaluate the average height $\phi(k)$ of the peaks in the power spectrum corresponding to frequencies $\omega_{k,l}=(2l-1)\pi/2^k$, where $l=1,2,\ldots,2^{k-1}$ and $k=1,2,\ldots$ at the onset of chaos. It is shown that the ratio $\phi(k)/\phi(k+1)$ is nearly constant and for large k approaches a universal limit $2\beta^{(2)}=20.963\ldots$

Recently there has been considerable excitement over the possibility that iterated maps of the interval might provide a mathematical model for certain physical systems that undergo a transition from periodic to chaotic behavior. One reason for this excitement is the fact that there are characteristics of the maps that behave in a universal fashion at and near the transition.[1] In fact, it has proven possible to construct an analogy with critical phenomena, derive critical exponents, and, in one case, obtain a universal scaling function.[2–5]

From a physical point of view it is clear that one should consider variables exhibiting universal behavior that are directly accessible experimentally. In this Communication we discuss just such a quantity—the autocorrelation function of points under the iterated map and the Fourier transform of this function which ought to be directly related to the power spectrum of the physical system for which the iterated map is a model.[6] This allows for direct experimental tests of the physical relevance of the recently developed models of the transition to chaos.[7,8]

Consider a one-dimensional map defined by the following recursion relation:

$$x_{k+1}=f(x_k;r) \quad . \tag{1}$$

The function $f(x;r)$ is defined on the interval $a<x<b$ which it maps into itself. It is assumed that $f(x;r)$ is a smooth function of x on the interval with a single quadratic maximum and no other extremum. The variable r controls the height of the maximum. A simple example of such a function is $rx(1-x)$ in the interval $0\leq x\leq 1$ with $0\leq r\leq 4$.

The autocorrelation function of the map is defined by

$$c(j)=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}x_k x_{k+j}\equiv\langle x_0 x_j\rangle \quad , \tag{2}$$

and the Fourier transform of this autocorrelation function $C(\omega)=|x(\omega)|^2$ is

$$C(\omega)=c(0)+2\sum_{j=1}^{\infty}c(j)\cos j\omega \quad . \tag{3}$$

The transition to chaos in the iterated map is heralded by a cascade of period-doubling bifurcations. On the periodic side of the transition at a value of r such that the stable orbit has a period 2^n (i.e., $x_{k+2^n}=x_k$), $C(\omega)$ will consist of a set of δ functions at $\omega=\pi m/2^n$, with m an integer less than 2^n. As the transition is approached so that a bifurcation takes place and the orbit has a period 2^{n+1}, new contributions to $C(\omega)$ will appear in the form of δ functions at $\omega=\pi(2m-1)/2^{n+1}$ with m an integer and $2m-1<2^n$. On the chaotic side of the transition a point acted upon by the map follows a trajectory that takes it between a set of bands that merge pairwise as r is adjusted to take the map ever further into the chaotic region. This merging occurs in a sequence that is the mirror image of the bifurcation sequence on the periodic side.[9] In the chaotic regime, $C(\omega)$ will consist of the same kind of δ functions as on the periodic side with the addition of a broad-band component representing the noisy, or chaotic, aspect of trajectories under the map. In what follows we will concentrate on the δ-function contributions to $C(\omega)$. Aspects of the broad-band behavior have already been considered.[10,11]

One quantity of interest is the ratio between the coefficients of the δ functions in $C(\omega)$ which are related to the peaks in the power spectrum.

We write

$$C(\omega)=C_{0,0}\delta(\omega)+C_{0,1}\delta(\omega-\pi)+2\sum_{k=1}^{\infty}\sum_{l=1}^{2^{k-1}}C_{k,l}\delta\left(\omega-\frac{(2l-1)\pi}{2^k}\right) \tag{4}$$

and define the average $\phi(k)$

$$\phi(k)\equiv\frac{1}{2^{k-1}}\sum_{l=1}^{2^{k-1}}C_{k,l} \quad . \tag{5}$$

We find that the ratio $\phi(k)/\phi(k+1)$ is nearly constant (see Table I), and we show that it approaches a universal ratio for large k, provided

TABLE I. The values for $\phi(k)/\phi(k+1)$ were obtained by two methods: (a) evaluating $\phi(k)$ directly from Eq. (5), and (b) evaluating $\bar{D}(2^k)$ from Eqs. (9) and (10) and substituting into Eq. (8). Both methods gave the same result, verifying numerically our analysis.

k	$\phi(k)/\phi(k+1)$	
	Feigenbaum	Parabolic
0	21.1876	21.8911
1	20.8684	20.7707
2	20.9924	21.0453
3	20.9532	20.9383
4	20.9670	20.9744

$k \ll n$, independent of the map,

$$\phi(k-1)/\phi(k) = 2\beta^{(2)} \quad , \tag{6}$$

where $\beta^{(2)}$ is a constant whose value[12] is 10.4817. . . . Furthermore, if we consider $\phi(n-1)$ for a value of the parameter r such that the mapping has a period 2^n just before the bifurcation to the period 2^{n+1}, and correspondingly $\phi'(n)$ for r' with period 2^{n+1} before the bifurcation to the period 2^{n+2}, we find that for large n, $\phi(n-1)/\phi'(n) = 2\beta^{(2)}$.

To arrive at the scaling relationship (6) let us see how $\phi(k)$ is calculated. Since the coefficient of the δ function $\delta(\omega = (2l-1)\pi/2^k)$ is given by

$$C_{k,l} = \lim_{N\to\infty} \frac{1}{N}\left[c(0) + 2\sum_{j=1}^{N-1} c(j)\cos\left(\frac{(2l-1)j\pi}{2^k}\right)\right] \tag{7}$$

we can obtain an expression for $\phi(k)$ by substituting Eq. (7) in Eq. (5). After some algebra we find

$$\phi(k) = \frac{1}{2^{k+2}}\bar{D}(2^k)\left(1 - \sum_{s=1}^{\infty}\frac{1}{2^s\beta_{s,k}^{(2)}}\right) \quad , \tag{8}$$

where

$$\bar{D}(2^k) \equiv \lim_{M\to\infty}\frac{1}{M}\sum_{s=1}^{M} D(2^k(2s-1)) \quad , \tag{9}$$

$$D(j) \equiv \lim_{N\to\infty}\frac{1}{N}\sum_{j'=0}^{N-1}[x(j+j') - x(j')]^2 = 2[c(0) - c(j)] \quad , \tag{10}$$

and

$$\beta_{s,k}^{(2)} = \left(\frac{\bar{D}(2^k)}{\bar{D}(2^{k+s})}\right) \quad . \tag{11}$$

From the nature of the universal map it will be shown that

$$\lim_{k\to\infty}\frac{\bar{D}(2^k)}{\bar{D}(2^{k+1})} = \beta^{(2)} = 10.4817\ldots \quad . \tag{12}$$

Hence for large enough k

$$\beta_{s,k}^{(2)} = (\beta^{(2)})^s \tag{13}$$

and we obtain

$$\phi(k) = \frac{1}{2^{k+1}}\left(\frac{\beta^{(2)}-1}{2\beta^{(2)}-1}\right)\bar{D}(2^k) = \frac{0.4750}{2^k}\bar{D}(2^k) \quad . \tag{14}$$

In particular

$$\frac{\phi(k)}{\phi(k+1)} = \frac{2\bar{D}(2^k)}{\bar{D}(2^{k+1})} \quad ,$$

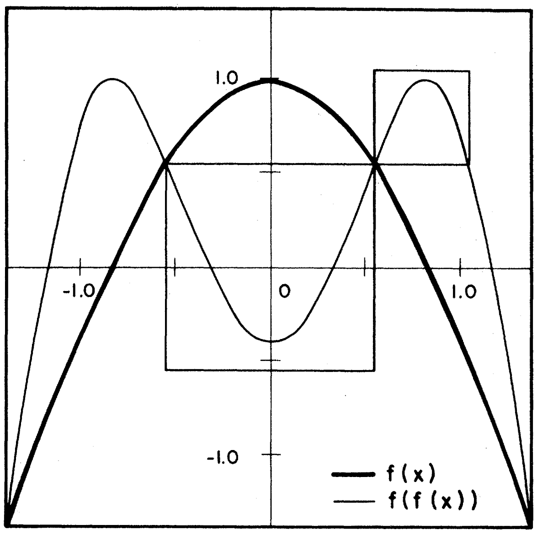
and Eq. (6) follows immediately from Eq. (12).

Some of the results of our numerical investigations are summarized in Table I. The ratio $\phi(k)/\phi(k+1)$ was calculated for both the Feigenbaum invariant map $f(x)$ and the parabolic map $rx(1-x)$. In the case of the parabolic map, r was adjusted so that there was a stable orbit of period 2^n. Our results were independent of n to the accuracy quoted here when $n \gtrsim 10$. For such an orbit we can replace N in Eq. (7) and in similar formulas by 2^n and not take the limit of an infinite number of terms. For the Feigenbaum map we adjusted the initial point so that it was mapped back into itself after 2^n iterations of the map with $n \gtrsim 10$. Again, to the accuracy quoted here, our results were independent of the precise value of n. For the initial point which mapped into itself after 2^n iterations of the map, we took $(-1/\alpha)^n x_f$, where x_f is the unstable fixed point of $f(x)$ that is closest to the origin. The period 2^n orbit executed by this point is unstable, in contrast to the orbits in the periodic regime. The results in Table I show that for both cases the universal limiting ratio $\beta^{(2)}$ is approached for sufficiently high $\phi(k)$'s.

Finally, we would like to comment briefly on our calculation of $\beta^{(2)}$ directly from the Feigenbaum invariant map $f(x)$, which is displayed in Fig. 1. This function has a single maximum at $x=0$, in the region $-1/\alpha \leq x \leq 1$, a region which is mapped into itself by $f(x)$. As shown in Fig. 1, the function $f(f(x))$, the iterate of $f(x)$, has three extrema in this region. There are two subregions, centered about two of the extrema, that are mapped into themselves by $f(f(x))$. In the subregion centered about $x=0$ the function $f(f(x))$ is exactly a scaled-down, inverted version of $f(x)$, the scale factor being α. In the other region $f(f(x))$ is approximately a scaled-down, shifted $f(x)$, that is,

$$f(f(x)) \cong (1/\alpha')f(\alpha'(x-x_0)) + x_0 \quad ,$$

where $\alpha' = (1+1/\alpha)/[1-f(1/\alpha)]$. The mean-square width of these two regions is smaller than the



Feigenbaum invariant map

FIG. 1. Feigenbaum invariant map $f(x)$ (dark curve) and the iterated map $f(f(x))$ (light curve) are displayed. The subregions about the extrema which are mapped into themselves by $f(f(x))$ are indicated by the squares.

width of the original region, by the factor $\frac{1}{2}(1/\alpha^2 + 1/\alpha'^2)$. Likewise iterating $f(f(x))$ we find four regions mapped into themselves by

$$f^{(4)}(x) \equiv f(f(f(f(x)))) ,$$

with a mean-square width reduced by a factor of approximately $[\frac{1}{2}(1/\alpha^2 + 1/\alpha'^2)]^2$. The ratio of the mean-square width of the 2^n regions mapped into themselves by $f^{(2^n)}(x)$ to that of the $2^{(n+1)}$ regions mapped into themselves by $f^{[2^{(n+1)}]}(x)$ is just $\beta^{(2)}$ where in our approximation $\beta^{(2)} = [\frac{1}{2}(1/\alpha^2 + 1/\alpha'^2)]^{-1} = 10.31$. It can be readily seen that in this approximation the widths of the allowed regions are distributed according to a binomial distribution. This estimate for $\beta^{(2)}$ can be systematically improved, and $\beta^{(2)}$ can be calculated to arbitrary accuracy by considering the ratios of the mean-square widths of regions mapped into themselves by $f^{(2^n)}(x)$. The value of $\beta^{(2)}$ quoted in this paper was obtained by this procedure.

In conclusion, we note that, although it will be necessary for the experimentalists to look at orbits deep in the bifurcation scheme in order to test the limiting behavior, we predict that, with precision, it ought to be possible to observe ratios in reasonably good accord with the limit predicted by Eq. (6) early in the sequence.

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[12]The ratio $\phi(k)/\phi(k+1) = 20.963$ is to be compared with the ratio $\mu^2 = 43.2$ of Feigenbaum (Ref. 6). Feigenbaum's ratio describes the way in which interpolations between subharmonic peaks in the power spectrum behave. For more details the reader is referred to Refs. 6. It is noteworthy that the derivation of Feigenbaum's ratio parallels ours up to the final step, at which point he replaces a sum with an integral.