Isotropic compressible ferromagnets

J. F. Nicoll
Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742
(Received 22 January 1981)

Earlier work on a model isotropic compressible ferromagnet is extended. Two different techniques are used. First, the one-particle-irreducible renormalization-group generator is applied to the n-component system to obtain the free energy and magnetic equation of state. The result, formally correct to $O(\epsilon)$, $\epsilon \equiv 4-d$, is exact in the spherical limit $n=\infty$ and for a vanishing effective rigid-system coupling constant. It includes correctly the Goldstone singularities for $n \neq 1$ and is uniformly valid in the thermodynamic space, even in the presence of first-order transitions. Second, a renormalization-group matching technique is used for $n=1$ to carry the results to $O(\epsilon^2)$.

I. INTRODUCTION

There have been several renormalization-group studies of the effects of the coupling between elastic and magnetic degrees of freedom in a ferromagnet.[1,2] The elastic modes are eliminated from the partition function to yield an effective spin Hamiltonian; in the isotropic case, the resulting Hamiltonian for a n-component spin $\vec{\phi}(x)$ in d dimensions can be written

$$H=\int d^dx \left[\tfrac{1}{2}t\vec{\phi}^2+\tfrac{1}{2}|\nabla\vec{\phi}|^2+\frac{u}{4!}(\vec{\phi}^2)^2-\vec{h}\cdot\vec{\phi}\right] + \frac{v}{4!\,\Omega}\left[\int d^dx\ \vec{\phi}^2\right]^2 . \tag{1.1}$$

In Eq. (1.1) Ω is the volume of the systems, $t\sim(T-T_c)$, $\vec{h}$ is the magnetic field, u is an effective rigid-system coupling system constant, and v is the remnant of the elastic-spin coupling. For the compressible magnet itself, v is negative, but Eq. (1.1) defines a ferromagnetic model of interest for positive v as well. If v is negative, this system may exhibit a first-order phase transition.[1,2]

The underlying mechanism of this first-order transition and, in fact, all the qualitative features of Eq. (1.1) can be exhibited by giving the exact solution of the compressible model in terms of the $v=0$ rigid case. This provides an excellent example of renormalization by hidden variables (Fisher renormalization[3]). Write

$$\exp\left[-\left(\frac{v}{4!}\int\vec{\phi}^2\right)^2\right] = \int d\psi\left(\frac{\Omega}{2\pi}\right)^{1/2}\exp\left[-\frac{\psi^2\Omega}{2}-\frac{a\psi}{2}\int(\vec{\phi})^2\right] , \tag{1.2}$$

with $a^2=-v/3$. Then the partition function for the compressible system is

$$Z_c=\int\mathfrak{D}\phi\, d\psi\left(\frac{\Omega}{2\pi}\right)^{1/2}\exp\left[-H_r-\frac{\psi^2}{2}\Omega-\frac{a}{2}\int\vec{\phi}^2\right] , \tag{1.3}$$

where H_r is the Hamiltonian of the rigid system [Eq. (1.1) with $v=0$]. This can be evaluated exactly. Introducing the Gibbs free energy of the compressible system by $Z_c=\exp(-\Omega G_c)$, the functional integral over the spin variables gives

$$\exp(-\Omega G_c)=\int d\psi\left(\frac{\Omega}{2\pi}\right)^{1/2} \times\exp\left[-\Omega\left(G_r(t+a\psi)+\frac{\psi^2}{2}\right)\right] , \tag{1.4}$$

where G_r is the Gibbs potential of the rigid system. In the thermodynamic limit, $\Omega\to\infty$, the ψ integral may be evaluated by the method of steepest descent yielding

$$G_c=G_r\left(t+\frac{v}{3}\psi\right)-\frac{v}{6}\psi^2 , \tag{1.5a}$$

$$\psi=\frac{\partial G_r}{\partial t}\left(t+\frac{v}{3}\psi\right) , \tag{1.5b}$$

where ψ has been rescaled $\psi\to-a\psi$. The compressible magnet is thus a simple example of a system with a hidden variable (ψ) subject to a constraint.[4] This form of constraint can change a transition from second order to first,[3,4] as well as (for positive α, the specific-heat exponent) changing the exponents. Equation (1.5) also shows that the compressible system involves only quantities which appear in the rigid

system and that, therefore, a solution method which applies for $v=0$ should be directly extendable to $v\neq 0$.

Global solutions of the renormalization-group equations for Eq. (1.1) were given for the disordered phase by the present author and collaborators[5–8] and the equation of state was calculated in an expansion around the disordered state as examples of the use of differential renormalization-group techniques. However, a fully exponentiated expression including the effects of the transverse or Goldstone singularities was given only for the rigid $v=0$ case. More recently, Bruno and Sak[9–11] have reexamined the model. Using the approach introduced by Nelson and Rudnick,[12] they rederived the disordered-phase susceptibility[9] given in Ref. 6; applying Rudnick's method[13] they have studied[10] the Ising ($n=1$) first-order phase transition of the compressible Hamiltonian Eq. (1.1); finally, they have given a partial description of the free energy and equation of state.[11]

The simple renormalization-group matching technique of Ref. 12, while very effective for a wide variety of problems, is not well suited for treating the Goldstone singularities present in the n-component system ($n\neq 1$). In this approach, renormalization-group equations which are simplified and approximate forms of the full equations are solved and the renormalized parameters used in a presumably more rapidly convergent perturbation theory. In the ordered phase of an isotropic n-component system, however, there are two length scales, corresponding to the transverse and longitudinal susceptibilities, and these cannot easily be represented by a single matched value of the renormalized couplings. The generator methods described in Ref. 8, on the other hand, are able to incorporate the transverse modes to give the correct singularities. This may be done in such a way that the $n=\infty$, or spherical, limit is recovered exactly. An alternative method is to carefully analyze and resum those diagrams responsible for the Goldstone singularities as has been done by Schäfer and Horner.[14] Even for the single-component case, the procedures used by Bruno and Sak break down near the first-order transition (necessitating a reanalysis of the problem) and do not describe the entire phase diagram. This is again a consequence of the approximations used; as will be shown in this work, the original solution methods developed in Refs. 5–8 are valid at the first-order transition and beyond.

In Sec. II, the rigid ($v=0$) n-component results of Ref. 8 are extended to the compressible magnet ($v\neq 0$). Although formally correct only to $O(\epsilon)$, $\epsilon\equiv 4-d$, the result is exact for $u=0$ and/or $n=\infty$. Both the free energy A and the equation of state are given in compact form so that a simple study of the entire thermodynamic phase diagram is possible, including the first-order transition for all values of n.

In Sec. III, an alternate approach is applied to the special case of $n=1$. The generator methods used in Sec. II, while in principle useful to arbitrary order, are in practice limited to first order. On the other hand, Bruce and Wallace[15] have given an extension and systematization of the matching technique of Ref. 12 which can be applied to any desired order in perturbation theory. A variant of the Bruce-Wallace matching procedure has been recently used to calculate the crossover thermodynamic functions for the rigid system to $O(\epsilon^2)$.[16] The calculation is illuminated by Eq. (1.5) which permits the compressible renormalization-group equations to be determined exactly in terms of the rigid-system equations. It is then a simple task to extract the $O(\epsilon^2)$ results for $v\neq 0$.

In the Appendix, the perturbation series needed for the explicit calculations of Sec. III are listed.

Bruno and Sak have provided an extensive background and mean-field description of the compressible-magnet model[11] and the methods to be used here are described in detail in Refs. 7, 8, 15, and 16. For that reason, I will focus primarily on the new results, referring the reader to the cited works.

II. GENERAL n-COMPONENT SYSTEM

To obtain the solution to lowest order for the compressible Hamiltonian, Eq. (1.1), one begins with the expression for the magnetic equation of state in the one-particle-irreducible generator formulation. To the order needed

$$\frac{h}{M}=\lim_{l\to\infty}\left[t(l)\exp(-2l)+[u(l)+v(l)]e^{-\epsilon l}\frac{M^2}{6}\right]. \tag{2.1}$$

It is the fact that the limit of infinite l [$\exp(-l)$ gives the scale of fluctuations not yet incorporated into the renormalized couplings] is taken which allows the inclusion of both transverse and longitudinal effects. Each of these "saturates" at a different value of the parameter l. The renormalization-group flow equations are

$$\frac{\partial t}{\partial l}=t\left[2-u\left(g_1^2\frac{n-1}{3}+g_2^2\right)-\frac{v}{3}[g_1^2(n-1)+g_2^2]\right], \tag{2.2a}$$

$$\frac{\partial u}{\partial l}=\epsilon u-u^2\left(g_1^2\frac{n-1}{3}+3g_2^2\right), \tag{2.2b}$$

$$\frac{\partial v}{\partial l}=\epsilon v-\frac{v^2}{3}[(n-1)g_1^2+g_2^2]-2uv\left(g_1^2\frac{n-1}{3}+g_2^2\right). \tag{2.2c}$$

In Eqs. (2.2) g_1 and g_2 represent the transverse and longitudinal propagators

$$g_1^{-1} = 1 + \frac{h}{M}(l)\exp(2l) \ , \tag{2.3a}$$

$$g_2^{-1} = 1 + \kappa^2(l)\exp(2l) \ , \tag{2.3b}$$

where $h/M(l)$ is the partially renormalized expression for the equation of state [given inside the large parentheses in Eq. (2.1)] and $\kappa^2(l)$ is an effective partially renormalized longitudinal (mass)^2

$$\kappa^2 = t(l)\exp(-2l) + \frac{u(l)e^{\epsilon l}M^2}{2} + \frac{v(l)^{-\epsilon l}M^2}{6} \ . \tag{2.3c}$$

The solution is straightforward, and, as expected, involves only functions familiar from the $v=0$ case.

$$\frac{h}{M} = t\mathcal{T}X + \frac{u\mathfrak{U}M^2}{6} + \frac{v\mathcal{T}^2XM^2}{6} \ . \tag{2.4}$$

In Eq. (2.4) [$\Delta \equiv (n+2)/(n+8)$]

$$\mathcal{T} = YY_2^{\Delta-1} \ , \tag{2.5a}$$

$$\mathfrak{U} = Y \ , \tag{2.5b}$$

$$X^{-1} = 1 + \frac{v}{3}\mathfrak{K} \ , \tag{2.5c}$$

$$\mathfrak{K} = \frac{3n}{(n+8)u}\frac{(Y_2^{2\Delta-1}-1)}{1-2\Delta} + \frac{3}{u}(Y_2 - Y)Y_2^{2(\Delta-1)} \ , \tag{2.5d}$$

$$Y_2^{-1} = 1 + \frac{u(n+8)}{3\epsilon}[(\kappa^2)^{-\epsilon/2} - 1] \ , \tag{2.5e}$$

$$Y^{-1} = 1 + \frac{u(n-1)}{3\epsilon}\left[\left(\frac{h}{M}\right)^{-\epsilon/2} - 1\right] + \frac{3u}{\epsilon}[(\kappa^2)^{-\epsilon/2} - 1] \ . \tag{2.5f}$$

The (h/M) in the function Y must be taken to have its full complete physical value; the definition of the effective longitudinal mass is

$$\kappa^2 = \frac{h}{M}[1 + 2(1 - Y_2Y^{-1})] + \frac{uY_2M^2}{3} = \frac{3h}{M} - 2tY_2^{\Delta} \ . \tag{2.6}$$

This differs from the choice of Ref. 8 and from the Schäfer and Horner[14] for $v=0$; it agrees for $h/M=0$ and for $n=1$. However, this choice and no other permits an exact integral of Eq. (2.4) to be made.

Following Ref. 8, the Helmholtz free energy is

$$A = \frac{t\mathcal{T}XM^2}{2!} + \frac{u\mathfrak{U}M^4}{4!} + \frac{v\mathcal{T}^2XM^4}{4!} - \frac{t^2}{2}\mathfrak{K}X \ , - \frac{n-1}{8(1-\epsilon/4)}\left(\frac{h}{M}\right)^{d/2} - \frac{1}{8(1-\epsilon/4)}\kappa^d \ . \tag{2.7}$$

In fact, only the first four terms are directly obtained by the generator methods; the last two terms are generally $O(\epsilon)$ smaller than the first and were neglected in Ref. 8. Here, the existence of the possible first-order transition requires more care. The choice of κ^2 made in Eq. (2.6) guarantees that Eq. (2.4) is the exact derivative of the Helmholtz potential of Eq. (2.7).

The choice made in Eq. (2.6) was made entirely on pragmatic grounds so that Eq. (2.4) would follow exactly from Eq. (2.7). However, it essentially coincides, as least for $u=u^*$, with the definition introduced recently by Lawrie[17] who uses a formalism designed to connect the $(v=0)\phi^4$ theory to the nonlinear σ model in order to extract the Goldstone modes. His results coincide with Eq. (2.4) with Eq. (2.6) defining an effective longitudinal scale.

Equations (2.4) and (2.7) are exact in the spherical limit, $n=\infty$, in which limit they reduce to

$$\frac{h}{M} = \tilde{Y}\left(t + \frac{u+v}{6}M^2\right) \ , \tag{2.8a}$$

$$A = \tilde{Y}\left(\frac{tM^2}{2!} + \frac{u+v}{4!}M^4\right) - \frac{t^2}{2}\frac{3}{u+v}(1-\tilde{Y}) - \frac{n}{8(1-\epsilon/4)}\left(\frac{h}{M}\right)^{d/2} \ , \tag{2.8b}$$

$$\tilde{Y}^{-1} = 1 + \frac{n}{3}(u+v)\left[\left(\frac{h}{M}\right)^{-\epsilon/2} - 1\right] \ . \tag{2.8c}$$

That is to say, the system reduces to a spherical model with coupling constant $(u+v)$. There is a second exact limit: $u=0$ (v positive!). This leads to Eq. (2.9) with u set equal to zero for all values of n. This is expected from Eq. (1.5) since the spherical model is the Fisher-renormalization counterpart to the Gaussian model.[7]

For general u, v, and n, the mechanism of the Fisher renormalization and possible first-order transition resides in the behavior of the factor X. For v positive, its asymptotic properties are governed by the sign of the rigid-system specific-heat exponent α. Note that the leading term of the specific heat C is just $\mathfrak{K}$ and that $(2\Delta-1) = -\alpha/\epsilon\nu = (n-4)/(n+8)$. If α is negative, then $\mathfrak{K}$ has a finite maximum at the critical point and the asymptotic behavior of the system is unchanged. The v-dependent terms would appear as corrections to scaling with a weak specific-heat character. On the other hand, if α is positive,

then the divergence of the specific heat drives X to zero and the well-known exponent renormalization occurs. For example, the disordered-phase inverse correlation length $\xi^{-1}=\kappa$ with

$$\kappa^2 = tY_2^{\Delta}X \quad . \tag{2.9a}$$

The singularity in $\mathfrak{K}$ is such that $X \sim Y_2^{1-2\Delta}$ and, of course, $Y_2 \sim \kappa^{\epsilon}$ yielding

$$\kappa^2 = t\kappa^{\epsilon(1-\Delta)} \quad . \tag{2.9b}$$

This shows that the new value of the thermal exponent $1/\nu$ is given by

$$1/\nu' = 2-\epsilon(1-\Delta) \quad , \tag{2.10a}$$

in contrast to the rigid system

$$1/\nu = 2-\epsilon\Delta \quad . \tag{2.10b}$$

Note that two temperature eigenvalues are related by Fisher renormalization if $1/\nu+1/\nu'=d$.

If v is negative, the possibility of a first-order transition arises. If α is positive, the factor X would inevitably diverge somewhere near the putative second-order critical point. If α is negative, then $\mathfrak{K}$ again has a finite maximum

$$\mathfrak{K}_{\max} = \frac{3n}{(n+8)u|\alpha/\epsilon\nu|} \quad , \tag{2.11a}$$

and X will diverge at or before this maximum is reached if

$$|v| \geq \frac{n+8}{n}\left|\frac{\alpha}{\epsilon\nu}\right|u \quad . \tag{2.11b}$$

Thus, the explicit forms of the solutions given here exhibit precisely those properties deducible from the fixed-point structure reviewed in Table I. The rigid-system fixed point is stable only if α is negative. In that case, its domain of attraction consists of all positive v and those negative v lying above the separatrix joining the Gaussian and Fisher-renormalized fixed points. Those lying below this separatrix, that is, those satisying Eq. (2.11b), will "run away," indicating a first-order transition. For positive α the Fisher-renormalized fixed point is the stablest and lies in the positive u, positive v quadrant. The entire range of negative v thus is outside the domain of attraction and runs away.

It will now be assumed that v lies in the region implicated in a first-order transition. The argument follows different lines for $n \neq 1$ and $n=1$. The $n \neq 1$ case will be considered first.

The Goldstone singularities suppress the v term relative to the u term in the equation of state so that an ordered state with $h=0$ is described by ($\kappa^2 = uY_2M^2/3$)

$$-tY_2^{\Delta} = \frac{\kappa^2}{2}X^{-1} = \frac{\kappa^2}{2}\left[1+\frac{v}{u}\left(\frac{n}{n+8}\frac{Y_2^{-\alpha/\epsilon\nu}-1}{\alpha/\epsilon\nu}+Y_2^{-\alpha/\epsilon\nu}\right)\right] \quad . \tag{2.12}$$

The point $t=0$ is reached for a finite value of κ at which $X^{-1}=0$. This ordered phase can persist to $t>0$ with $X^{-1}<0$. Eventually the free energy of the disordered state is lower and a first-order transition occurs.

The Goldstone modes suppress all but the fourth and sixth terms in the ordered state:

$$A_- = -\tfrac{1}{2}t^2X_-\mathfrak{K}_- - \frac{1}{8}\frac{(\kappa_-)^d}{1-\epsilon/4} \quad , \tag{2.13a}$$

while in the disordered phase

$$A_+ = -\tfrac{1}{2}t^2X_+\mathfrak{K}_+ - \frac{n}{8}\frac{(\kappa_+)^d}{1-\epsilon/4} \quad , \tag{2.13b}$$

where $(\pm)$ subscripts denote values in the disordered and ordered states, and

$$tY_2^{\Delta}(\kappa_+)X_+ \equiv \kappa_+^2 \quad . \tag{2.13c}$$

From general considerations[18] the value of t at the transition is $O(v)$, X_-^{-1} is $O(v)$ with $\mathfrak{K}_\pm \sim O(1/\epsilon)$ and κ_- still $O(1)$. Thus A_- is $O(1)$ and κ_+^2/κ_-^2

TABLE I. The four fixed points of the compressible ferromagnet Hamiltonian Eq. (1.1) to $O(\epsilon)$. The fixed-point values of u and v are u^* and v^*. The eigenvalues associated with u, v, and t are λ_u, λ_v, and $\lambda_t(=1/\nu)$, respectively.

Fixed point	u^*	v^*	λ_u	λ_v	λ_t
Gaussian	0	0	ϵ	ϵ	2
Spherical	0	$3\epsilon/n$	ϵ	$-\epsilon$	$2-\epsilon$
Rigid	$3\epsilon/(n+8)$	0	$-\epsilon$	α/ν	$2-\epsilon\Delta$
Fisher renormalized	$3\epsilon/(n+8)$	$(3/n)\alpha/\nu$	$-\epsilon$	$-\alpha/\nu$	$2-\epsilon(1-\Delta)$

$\sim O(\epsilon)$. If X_+ were also $O(1)$ then A_+ would be $O(\epsilon)$ and equating A_+ and A_- would be equivalent to setting $A_-=0+O(\epsilon)$.

To self-consistently check that, in fact, X_+ is $O(1)$ it is useful to include the leading term in A_+ to give the first-order condition:

$$0=-\tfrac{1}{2}t^2(X_-\mathcal{K}_- - X_+\mathcal{K}_+)-\tfrac{1}{8}(\kappa_-)^d \ . \qquad (2.14a)$$

After some manipulation

$$-\frac{1}{X_-}\left[1+2\frac{\kappa_+^2}{\kappa_-^2}\frac{Y_2^{-\Delta}(\kappa_+)}{Y_2^{-\Delta}(\kappa_-)}\right]=\frac{|v|}{3}Y_2^{2\Delta}(\kappa_-)\kappa_-^{-\epsilon} \ , \qquad (2.14b)$$

$$\frac{\kappa_+^2 Y_2^{-\Delta}(\kappa_+)}{\kappa_-^2 Y_2^{-\Delta}(\kappa_-)}\left[\frac{4-n[Y_2(\kappa_+)/Y_2(\kappa_-)]^{-\alpha/\epsilon\nu}}{4-n}\right]=\frac{u}{6}Y_2\kappa_-^{-\epsilon} \ . \qquad (2.14c)$$

For $n=1$ the v term is not suppressed so that the ordered phase is characterized by

$$-tY_2^{\Delta}X=\frac{uY_2M^2}{6}+\frac{vY_2^{2\Delta}M^2X}{6} \qquad (2.15a)$$

or

$$-tY_2^{\Delta}=\frac{\kappa^2}{2}\,\mathfrak{z} \ , \qquad (2.15b)$$

$$\mathfrak{z}=X^{-1}+\frac{v}{u}Y_2^{2\Delta-1}$$

$$=1+\frac{v}{u}\left[\frac{1}{9}\frac{Y_2^{-\alpha/\epsilon\nu}-1}{\alpha/\epsilon\nu}+Y_2^{-\alpha/\epsilon\nu}\right] \ , \qquad (2.15c)$$

duplicating Eq. (2.12). Now all the terms in the free energy contribute to A_-. Combining terms, Eq. (2.14) is recovered with $n=1$ and $-\mathfrak{z}_-$ replacing $-X_-^{-1}$.

From Eq. (2.14) it appears that κ_+^2/κ_-^2 is $O(u)$. For example setting $u=u^*$

$$\frac{\kappa_+^{1/\nu}}{\kappa_-^{1/\nu}}\left[\frac{4-n(\kappa_+/\kappa_-)^{-\alpha/\nu}}{4-n}\right]=\frac{u^*}{6} \ , \qquad (2.16)$$

if the $(\kappa_+/\kappa_-)^{-\alpha/\nu}$ factor is neglected $(\kappa_+/\kappa_-)^{1/\nu}=\epsilon/2(n+8)$. One then needs to consider

$$\left[4-n\left(\frac{\epsilon}{2(n+8)}\right)^{-\alpha}\right]\Big/(4-n) \ .$$

An ϵ expansion would give

$$\left[4-n\left(\frac{\epsilon}{2(n+8)}\right)^{-\alpha}\right]\Big/(4-n)=1+\frac{n}{4-n}\alpha\ln\frac{\epsilon}{2(n+8)}$$

$$=1+\frac{n}{n+8}\frac{\epsilon}{2}\ln\frac{\epsilon}{2(n+8)} \ , \qquad (2.17)$$

The $\epsilon\ln\epsilon$ factor is of intermediate size. It seems simplest to leave Eq. (2.16) as it stands, recognizing that $(\kappa_+/\kappa_-)^{1/\nu}$ does not have an expansion in powers of ϵ. Thus, the factor $[4-n(\kappa_+/\kappa_-)^{-\alpha/\nu}]/(4-n)$ will be treated as $O(1)$ without further specification. In any event κ_+^2/κ_-^2 is $O(u)$ and in the present case, this factor can be dropped from Eq. (2.14b) that is, to this order,

$$-\frac{1}{X_-}=\frac{|v|}{3}Y_2^{2\Delta}(\kappa_-)\kappa_-^{-\epsilon} \ . \qquad (2.18)$$

Note that the factor which drives the transition in both the $n\neq 1$ and $n=1$ cases has the form of the rigid-system specific heat (at least its leading part) and that the rigid specific-heat amplitude ratio is $n/4$.

Any other thermodynamic function of interest can be obtained from A and h/M or their derivatives. A few examples will be given in Sec. III for the Ising-like $n=1$ case.

Although the use of implicitly defined functions is avoided by the seemingly more direct methods of Refs. 12 and 13, in fact, it is precisely the implicit nature of the expressions which permits them to be valid at and below the first-order transition and to include the Goldstone mode singularities properly. The approximations which appear to simplify the solution of the renormalization-group equations fail in the regions of interest.

A constrast may also be made by considering how the two approaches treat the mean-field stability line, marking the onset of the first-order transition. For the compressible magnet, it is described by $u+v=0$. In the procedure introduced by Rudnick,[13] approximate renormalization-group equations are solved until the parameters flow to the stability line; at that point the flow is stopped and a perturbative analysis of the system is applied. In the present work, on the other hand, the flows have well-defined asymptotic properties and can be integrated out to their limits. If the behavior of the "renormalized couplings" is then considered the stability boundary is *crossed* at $t=0$ and, in fact, the renormalized couplings lie on the far side for the positive t states (t small). For example, if the $n\neq 1$ case is considered one may define a "renormalized" pair of couplings by

$$(u+v)_R=X^{-1}u+vYY_2^{2(\Delta-1)} \ . \qquad (2.19)$$

One factor of Y has been removed to prevent this expression from vanishing identically in the ordered state. The factor of X^{-1} has been added to show the relationship to Eq. (2.12). The remaining factor of Y drives the "renormalized" v to zero in the ordered state, leaving only the term uX^{-1}. This goes to zero at $t=0$, thereby reaching the stability line and, in fact, is negative in the positive t, ordered-phase domain. For $n=1$ there are no Goldstone effects and one may choose to consider the renormalized

couplings as

$$(u+v)_R = u + vXY_2^{-1/3} \quad . \tag{2.20}$$

Again, at $t=0$, the stability boundary is reached and then for positive t, it is crossed. In fact, even after the first-order transition has occurred, Eq. (2.20) is still negative and, in fact, is more negative on the disordered side than it is on the ordered. This follows from the fact that $XY_2^{-1/3}$ is an increasing function of κ^{-1} and that $\kappa_+^2 < \kappa_-^2$.

This crossing of the mean-field stability line, while initially surprising, in fact, is a perfectly natural, sensible occurrence. It is easy to check, for example, that the curvature of the free energy is positive at either of the two minima, and remains positive for the ordered phase even at temperatures above the first-order transition, if the free-energy expression is used to describe that metastable state. For temperatures and/or magnetization far removed from the first-order transition point the renormalized couplings return to $u+v$; that is, away from the transition the Y functions and X tend to unity and the mean-field picture prevails. Thus, if the original Hamiltonian parameters satisfy $u+v>0$, then the crossing of the stability line by the "renormalized" parameters has no disruptive significance.

As utilized by Bruno and Sak, the methods of Refs. 12 and 13 cannot describe the entire $h=0$ axis; the solutions break down above some $t_{\min}$ ($t_{\min}<0$) and below $t_{\max}$ ($t_{\max}>0$). As shown here, this may be attributed entirely to the methods employed. Furthermore, the methods used to probe the first-order transition itself by integrating to the stability line and applying perturbation theory[13] must inevitably give rise to a sequence of logarithms which are not easily recognized as the expansion of power-law singular terms such as given in Eq. (2.14).

The present approach, in sharp contrast, is capable of describing with a single expression the entire range of thermodynamic variables, even in the presence of Goldstone singularities. It achieves this by considering the asymptotic behavior of the renormalization-group trajectories and by being implicit in nature. The implicit character, which uses physically meaningful quantities in the formulation, will remain self-consistently valid even at first-order transitions. Unfortunately both the Nelson-Rudnick approach[12,13] and the differential generator methods are hard to extend to higher order in perturbation theory. In Sec. III, a modification[15] of the matching method which is more systematic is used to carry the Ising-like case to $O(\epsilon^2)$.

III. $n=1$ CASE: $O(\epsilon^2)$

A powerful method for studying crossover equations of state has recently been described by Bruce and Wallace[15] (for applications, cf. Ref. 16 and Theumann[19]). Renormalization-group equations of a field-theoretic nature (as distinguished from those obtained from a differential generator or equivalent approach) are exploited in a systematization of the matching technique.[12] These renormalization-group flow equations differ from generator equations in that the propagator factors are absent in the former. For example, the lowest-order field-theoretic flow equations for the compressible magnet are identical to Eq. (2.2) with $g_1=g_2=1$. The absence of the propagator factors has several immediate consequences. First, there is now no distinction between transverse and longitudinal contributions. It, therefore, seems difficult to separate the corresponding singularities. At present, the method is directly applicable for Ising-like systems or the disordered phase of non-Ising systems. Second, the lack of propagators prevents any saturation of fluctuation effects. This *necessitates* a matching approach. Third, and in compensation, the nonlinear equations are far more easily solved without propagator factors. This is what allows $O(\epsilon^2)$ calculations to be simply made. As will be shown below, the full thermodynamic phase diagram, including the first-order transition, may be obtained showing that the failure of the methods employed in Refs. 9–11 is not due to matching, per se, but rather has to do with the nature of the approximations employed.

For the purposes of this section the Hamiltonian will be defined as

$$H = \int \left[\tfrac{1}{2}t\,\vec{\phi}^2 + \tfrac{1}{2}|\vec{\nabla}\vec{\phi}|_\Lambda^2 + \frac{u\Lambda^\epsilon}{4!}(\vec{\phi}^2)^2\right] d^dx + \frac{v\Lambda^\epsilon}{4!\,\Omega}\left[\int \vec{\phi}^2\, d^dx\right]^2 - \int \vec{h}\cdot\vec{\phi}\, d^dx \quad . \tag{3.1}$$

In Eq. (3.1), Λ is a lattice cutoff effectively limiting the momentum integrals to $|k| \lesssim \Lambda$. There are, of course, many ways of imposing such a cutoff which lead to differences in the calculated quantities which can be, to the order needed, removed by simple changes of scale.[16] This adjustable scale will be chosen to agree with the sharp cutoff implicitly used in the generator approach of Sec. II. The renormalization-group equations follow[15] from the existence of a renormalized theory in the $\Lambda \to \infty$ limit. This implies a specific connection between the dependence of the free energy on t, u, v, and Λ. Equation (1.5) can be written as (by Legendre transform)

$$A_c = A_r\left(t + \frac{v\Lambda^\epsilon}{3}\psi\right) - \frac{v\Lambda^\epsilon}{6}\psi^2 \quad , \tag{3.2a}$$

$$\psi = \frac{\partial A_r}{\partial t}\left(t + \frac{v\Lambda^\epsilon}{3}\psi\right) \quad . \tag{3.2b}$$

The renormalization-group operator $\mathfrak{R}$ is of the form

$$\mathfrak{R} = \Lambda\partial_\Lambda + \beta_u(u)\partial_u + \beta_v\partial_v + \bar{\gamma}_2(u,v)t\partial_t - \frac{\eta(u)}{2}M\partial_M \ . \tag{3.3}$$

Applying this operator to Eq. (3.2) and requiring that $\mathfrak{R}$ annihilate all singular terms shows that $\beta_u(u)$ and $\eta(u)$ are, as indicated, independent of v and that

$$\bar{\gamma}_2 = \gamma_2(u) + B(u)v/3 \ , \tag{3.4a}$$

$$\beta_v = -\epsilon v + B(u)v^2/3 + 2\gamma_2(u)v \ , \tag{3.4b}$$

where $\gamma_2(u) = 2 - 1/\nu$ at the fixed point $\beta(u^*) = 0$ and $B(u)$ appears in the kernel of the renormalization-group equation

$$\mathfrak{R}A = -\frac{B(u)t^2}{2}\Lambda^{-\epsilon} \ .$$

These relationships hold to all orders in ϵ and u and for all n. [The expressions for these quantities to $O(\epsilon^2)$ are given in Ref. 15 and the Appendix.] They simply embody the Fisher-renormalization relation Eq. (3.2) in differential form.

The flow equations relate systems with different values of Λ; setting $\Lambda = \exp(-l)$ and (and specializing to $n = 1$)

$$\frac{\partial t}{\partial l} = t\bar{\gamma}_2 \ , \tag{3.5a}$$

$$\frac{\partial u}{\partial l} = -\beta_u(u) \ , \tag{3.5b}$$

$$\frac{\partial v}{\partial l} = -\beta_v \ . \tag{3.5c}$$

The solutions are simple

$$t(l) = t\mathcal{T}X \ , \tag{3.6a}$$

$$u(l)\exp(-\epsilon l) \equiv u\mathfrak{U} \ , \tag{3.6b}$$

$$v(l)\exp(-\epsilon l) = v\mathcal{T}^2X \ , \tag{3.6c}$$

$$X^{-1} = 1 + \frac{v}{3}\mathcal{K} \ , \tag{3.6d}$$

$$\mathcal{T} = \exp\left[-\int_0^l \gamma_2[u(l)]\,dl\right] \ , \tag{3.6e}$$

$$\mathcal{K} = \int_0^l B(u(l))\mathcal{T}^2\exp(\epsilon l)\,dl \ . \tag{3.6f}$$

Defining ΔA and Δh as the fluctuation contributions to the Helmholtz free energy and equation of state, then[15,16]

$$A = \frac{t\mathcal{T}X}{2!}\mathfrak{D}M^2 + \frac{u\mathfrak{U}}{4!}(\mathfrak{D}M^2)^2 + \frac{v}{4!}\mathcal{T}^2X(\mathfrak{D}M^2)^2 - \frac{t^2\mathcal{K}X}{2} + \Delta A(l) \ , \tag{3.7a}$$

$$\frac{h}{M} = \mathfrak{D}\left[t\mathcal{T}X + u\frac{\mathfrak{U}}{6}(\mathfrak{D}M^2) + v\frac{\mathcal{T}^2X}{6}\mathfrak{D}M^2 + \frac{\Delta h}{M}(l)\right] \ , \tag{3.7b}$$

with $\mathfrak{D} \equiv \exp\int_0^l \eta[u(l)]\,dl$. These expressions hold to all orders in perturbation theory and for all values of l. The most convenient choice of a match point is $l = l^*$ such that $\Delta h(l^*) = 0$. This provides a parallel expression to that of Sec. II and simplifies the description of the coexistence surface.

To $O(\epsilon^2)$ the renormalization factors are[15,16]

$$\mathfrak{D} = Y^{-\eta/\omega}\exp[-(p-\bar{u})\eta/\omega] \ , \tag{3.8a}$$

$$\mathcal{T} = Y^{(2-1/\nu)}\exp[D_1(p-\bar{u})] \ , \tag{3.8b}$$

$$\mathfrak{U} = Y^{\epsilon/\omega} \ , \tag{3.8c}$$

$$\mathcal{K} = \frac{\exp 2D_1(1-\bar{u})}{\epsilon\bar{u}} \times \left[\frac{Y^{-\alpha/\omega\nu}-1}{\alpha/\nu\epsilon} + \frac{Y^{1-\alpha/\omega\nu}-1}{1-\alpha/\omega\nu}\right] \times \left[\frac{\epsilon}{\omega} - 1 + 2D_1\right](1-\bar{u}) \ , \tag{3.8d}$$

where $Y = (1-p)/(1-\bar{u})$, $\bar{u} = u/u^*$, $u(l) = u^*p$, η, ν, α, ω are the rigid-system critical-point exponents, and D_1 is given by

$$D_1 = \frac{2-1/\nu}{\epsilon} - \frac{u^*}{\epsilon} = \tfrac{19}{54}u^* \ , \tag{3.8e}$$

$$u^* = \tfrac{\epsilon}{3}(1 + \tfrac{17}{27}\epsilon) \ . \tag{3.8f}$$

All the l dependence is embodied in

$$Y^{\epsilon/\omega}\frac{\bar{u}}{p} = \exp(-\epsilon l) \ . \tag{3.8g}$$

At the $\Delta h(l^*) = 0$ match point

$$Y^{\epsilon/\omega}\frac{\bar{u}}{p} = \exp(-\epsilon l^*) = \kappa^\epsilon\exp-\left(\frac{\epsilon}{2}L\right) \ , \tag{3.9a}$$

where

$$\kappa^2 = t\mathcal{T}X + \frac{u\mathfrak{U}(\mathfrak{D}M^2)}{2} + \frac{v\mathcal{T}^2X\mathfrak{D}M^2}{6} \ , \tag{3.9b}$$

$$L = \frac{u^*p}{w}\frac{f-1}{2} + \frac{u^*pq}{4}(f+1) \ , \tag{3.9c}$$

with $f = 4 + \pi^2 - 8\lambda$ ($\lambda \sim 1.17$) or $f \sim 4.5$.[20] In Eq. (3.9c)

$$q = \frac{t\mathcal{T}\mathfrak{U}\mathfrak{D}M^2}{\kappa^2} \ , \tag{3.10a}$$

$$w = 1 + \frac{v}{3u}\frac{\mathcal{T}^2X}{\mathfrak{U}} \ . \tag{3.10b}$$

There are four basic variables in this problem: t, M, u, and v. These are thus four global nonlinear scaling fields. Alternately one can consider a single scaling field, κ, and three global renormalization-group invariants p, q, and w. Although these are invariants of the renormalization-group flow, they do depend upon the values of t and M. For example, $q=3$ on the coexistence surface and $w=2/3$ on the "renormalized" stability line, reached at $t=0$.

At the $\Delta h=0$ match point the free energy A is

$$A=\frac{t\mathcal{T}X}{2!}\mathfrak{D}M^2+\frac{u\mathfrak{U}}{4!}(\mathfrak{D}M^2)^2 +\frac{v}{4!}\mathcal{T}^2X(\mathfrak{D}M^2)^2-\frac{t^2\mathcal{K}X}{2} -\frac{\kappa^d}{8(1-\epsilon/4)}\left[1+\frac{u^*p}{w}(1-f)-\frac{u^*pq}{2}(3-f)\right] . \tag{3.11}$$

Note that this reduces to the result of Sec. II if the two-loop contributions are dropped. It is also instructive to observe that if the specific heat at constant field, C_h, match point were used instead of the Δh match point, then for $v=h=0$, $C_h=\mathcal{K}\,(t>0)$, and $C_h=\mathcal{K}+(3/u)\mathcal{T}^2/\mathfrak{U}\,(t<0)$. With the present match point, they represent the *leading* behavior of the rigid-system specific heat as do the corresponding expressions for general n given in Sec. II.

The description of the first-order transition (for $v<0$) proceeds as in Sec. II. At $h/M=0$, $\kappa^2=u\mathfrak{U}(\mathfrak{D}M^2)/3$

$$-t\mathcal{T}=\frac{\kappa^2\mathfrak{z}}{2} , \tag{3.12a}$$

$$\mathfrak{z}=1+\frac{v}{3}\left[\mathcal{K}+\frac{3\mathcal{T}^2}{u\mathfrak{U}}\right] . \tag{3.12b}$$

As before, $t=0$ marks the crossing of stability line ($\mathfrak{z}=0$) and the ordered phase persists with $\mathfrak{z}$ negative. The free energy in the ordered phase is

$$A_-=-\frac{3}{8}\frac{(\kappa_-^2)^2X_-\mathfrak{z}_-}{u\mathfrak{U}_-}-\frac{t^2}{2}X_-\mathfrak{z}_- -\frac{(\kappa_-)^d}{8(1-\frac{1}{4}\epsilon)}\left[1+\frac{u^*p_-}{w_-}(1-f)-u^*\tfrac{3}{2}p_-(3-f)\right] \tag{3.13a}$$

while in the disordered phase

$$A_+=-\frac{t^2}{2}X_+\mathcal{K}_+-\frac{(\kappa_+)^d}{8(1-\epsilon/4)}\left[1+\frac{u^*p_+}{w_+}(1-f)\right] . \tag{3.13b}$$

The location of the first-order transition and the κ_+^2/κ_-^2 ratio are determined by

$$-\mathfrak{z}_-\left[1+2\,\frac{\kappa_+^2\mathcal{T}_-}{\kappa_-^2\mathcal{T}_+}\right]=\frac{|v|}{3}\mathcal{T}_-^2(\kappa_-)^{-\epsilon}\frac{(1-3u^*p_-)}{(1-\epsilon/4)} , \tag{3.14a}$$

$$\frac{\kappa_+^2}{\kappa_-^2}\frac{\mathcal{T}_-}{\mathcal{T}_+}\left[1-\frac{u\mathfrak{U}_-}{3\mathcal{T}_-^2}(\mathcal{K}_+-\mathcal{K}_-)\right] =\frac{(1-3u^*p_-)}{(1-\epsilon/4)}\frac{u\mathfrak{U}_-(\kappa_-)^{-\epsilon}}{6} , \tag{3.14b}$$

where $w=\frac{2}{3}+O(\epsilon)$, To examine the structure set $u=u^*$

$$-\mathfrak{z}_-\left[1+2\left(\frac{\kappa_+}{\kappa_-}\right)^{1/\nu}\right]=\frac{|v|}{3}(\kappa_-)^{-\alpha/\nu}\frac{(1-3u^*)}{(1-\epsilon/4)} , \tag{3.15a}$$

$$\left(\frac{\kappa_+}{\kappa_-}\right)^{1/\nu}\left[\frac{4}{1+\epsilon}-\left(\frac{\kappa_+}{\kappa_-}\right)^{-\alpha/\nu}\right]\Big/\left[\frac{4}{1+\epsilon}-1\right] =\frac{(1-3u^*)}{(1-\epsilon/4)}\frac{u^*}{6} . \tag{3.15b}$$

The differences in the values of $L, L_\pm$, do not matter at this order. Note again that $4/(1+\epsilon)$ is the rigid-system amplitude ratio to this order. For completeness a few of the other thermodynamic quantities at the $\Delta h(t^*)=0$ match points are listed:

$$\frac{\partial A}{\partial t}=X\left[\frac{\mathfrak{D}M^2}{2}\mathcal{T}-t\mathcal{K}+\frac{\mathcal{T}\kappa^{2-\epsilon}}{4}\frac{u^*p}{w}(f-1)\right] , \tag{3.16a}$$

$$\frac{\partial^2 A}{\partial t^2}(\equiv C_M)=X\mathcal{K}+\frac{X^2\mathcal{T}^2\kappa^{-\epsilon}}{2}\left[1+\frac{u^*p}{2w}(f-1)+\frac{u^*pq}{4}(f+1)+\frac{u^*pw}{2}\right] , \tag{3.16b}$$

$$\frac{\partial^2 A}{\partial M^2}(\equiv\Gamma_2)=\mathfrak{D}\left\{t\mathcal{T}X+\frac{u\mathfrak{U}}{2}\mathfrak{D}M^2+\frac{v\mathcal{T}^2X\mathfrak{D}M^2}{2}+\frac{\kappa^2u^*pqw^2}{2}\left[\left\{1-\frac{u^*p}{w}+\frac{u^*pw}{2}\right\}-\frac{(f+1)}{2}\frac{u^*p}{w}\left\{1-q\frac{w}{2}\right\}\right]\right\}, \tag{3.16c}$$

$$\frac{1}{M}\frac{\partial^2 A}{\partial t\,\partial M}\left(\equiv\frac{\partial}{\partial t}\frac{h}{M}\right)=\mathfrak{D}\mathcal{T}X\left[1+\frac{u^*pw}{2}\left\{1+\frac{u^*p}{w}+\frac{u^*pq}{4}(f+1)+\frac{u^*pw}{2}\right\}\right]. \tag{3.16d}$$

It is simple to check that Γ_2 remains positive in the ordered phase even though the Landau-Ginzburg terms vanish at $t=0$ and are negative for $t>0$. In fact, $\mathfrak{z}$ would have to be roughly twice as negative as its value at the first-order transition before Γ_2 would vanish, signaling the end of the metastable state.

I have shown that systems with first-order transitions can be handled by the same methods which apply to the usual second-order transition. For both the generator approach of Sec. II and the Bruce-Wallace matching procedure of Sec. III the validity of the result throughout the thermodynamic space is achieved by focusing on physical variables and exact nonlinear scaling ideas. The most direct comparison is between the matching used by Bruce and Sak and the Bruce-Wallace approach used here. The former fails to cover the entire t axis because it attempts to utilize an explicit expression in terms of t rather than an implicit expression in terms of κ^2. Now, in the disordered phase $\kappa^2\sim t[1+O(\epsilon)]$ and it might seem that this replacement would be a valid and useful one, say, in the Y functions: $\kappa^\epsilon\sim t^{\epsilon/2}$. However, the instant this is done, the physics of the problem is ruined since κ^2 is not zero at $t=0$. Of course, before $t=0$ is reached the first-order transition intervenes but one could follow the disordered state into a metastable region. For $u=u^*$

$$t=\kappa_+^{1/\nu}\left(1+\frac{v}{3}\frac{\kappa_+^{-\alpha/\nu}-1}{\alpha/\nu}\right). \tag{3.17}$$

The (metastable) disordered state reaches $t=0$ at a finite κ_+^2. On the other hand if the lowest-order expression $\kappa^{1/\nu}\simeq t$ is inserted the equation becomes nonsensical at the equivalent t. Thus, the matched results of Bruno and Sak would be extended in validity simply by leaving the results implicit. A similar analyis applies to the ordered phase. Here $\kappa_+^2\geq 0$ everywhere since it is essentially the inverse susceptibility.

In physical terms the generator approach and the exact spherical limit suggest that the effective masses should represent the full rather than bare quantities. This is what is implemented by the implicit relationships obtained in the two-loop matching used here. On a technical level, the use of physical "renormalized" quantities permits exact nonlinear scaling. The match point κ's are exact global nonlinear scaling fields[16] and the present approach preserves all the formal scaling properties of an exact solution. Replacing κ^2 with t, as above, violates the global scaling properties. This points out the danger of merely guaranteeing the sensibility of a match point scheme to $O(\epsilon)$; although perturbation theory and the ϵ expansion are limitations, the best results are achieved by respecting the global renormalization-group properties. Thus, for example, the $n=\infty$ and $u=0$ limits of the compressible magnet model are easily obtained exactly, but are recovered from matching only if the implicit form is respected.

At the first-order transition it is easy to see that crossing the mean-field stability line is not the obstacle that it has been considered previously. The theory is sensible as long as the free-energy minima have positive curvature which is easily confirmed from Eq. (3.16). In fact, the perturbation theory itself is on even safer ground since it involves only $u+v/3$ and $\kappa^2=h/M+uM^2/3$ or their renormalized counterparts and these remain of the same order at the stability line. In general,[18] one expects that if the bare Hamiltonian satisfies the stability criterion (here, $u+v>0$) that the perturbation theory will be meaningful since the values of terms such as κ^2 will be determined self-consistently from an implicit equation of state.

ACKNOWLEDGMENTS

The author is grateful to R. K. P. Zia for many useful discussions and to I. D. Lawrie for communication of his work prior to publication. This work was supported by NSF Grant No. DMR-79-10819.

APPENDIX: PERTURBATION SERIES

The free energy is calculated to two-loop order with a propagator $p^2+p^4/\Lambda^2+\kappa^2$, $\kappa^2=t+uM^2/2+vM^2/6$,

and then a scale factor (B_0 of Ref. 16) is removed to conform to the scale of the sharp cutoff of Sec. II.

$$\Delta A = -\frac{1}{2\epsilon}(\kappa^2)^2\Lambda^{-\epsilon}\left[\frac{(\kappa/\Lambda)^{-\epsilon}}{1-\epsilon/4}-1\right]+\frac{1}{2}\left[u+\frac{v}{3}\right](\kappa^2)^2\Lambda^{-\epsilon}\left[\frac{L}{2}\right]^2+\frac{u^2M^2\kappa^2}{8}[(L-1)^2-f]\ , \tag{A1}$$

where $L\equiv\ln\kappa^2/\Lambda^2$ and $f=4+\pi^2-8\lambda$, $\lambda\sim 1.17$, $f\sim 4.5$. It is useful to give explicit expressions for the derivatives of ΔA

$$\frac{\partial\Delta A}{\partial t} = -\kappa^2\Lambda^{-\epsilon}\left[\frac{(\kappa/\Lambda)^{-\epsilon}-1}{\epsilon}\right]+\frac{1}{4}\left[u+\frac{v}{3}\right]\kappa^2\Lambda^{-\epsilon}(L^2+L)+\frac{u^2M^2}{8}[(L-1)^2-f+2(L-1)]\ , \tag{A2}$$

$$\frac{\partial^2\Delta A}{\partial t^2} = \frac{1}{\epsilon}\left[\left[1-\frac{\epsilon}{2}\right]\left[\frac{\kappa}{\Lambda}\right]^{-\epsilon}-1\right]\Lambda^{-\epsilon}+\frac{(u+v/3)}{4}(L^2+3L+1)\Lambda^{-\epsilon}+\frac{u^2M^2}{4\kappa^2}L\ , \tag{A3}$$

$$\begin{aligned}\frac{1}{M}\Delta\frac{\partial A}{\partial M} = -\frac{\Delta h}{M} &= \left[u+\frac{v}{3}\right]\kappa^2\frac{(\kappa/\Lambda)^{-\epsilon}-1}{\epsilon}+\frac{(u+v/3)^2}{4}\kappa^2L(L+1)\\ &+\left[u+\frac{v}{3}\right]\frac{u^2M^2\Lambda^{\epsilon}}{8}[(L-1)^2-f+2(L-1)]+\frac{u^2\kappa^2}{4}[(L-1)^2-f]\ ,\end{aligned} \tag{A4}$$

$$\begin{aligned}\frac{\Delta\partial^2 A}{\partial M^2} = \Delta\Gamma_2 = \frac{\Delta h}{M} &+\left[u+\frac{v}{3}\right]^2M^2\Lambda^{\epsilon}\left\{-\frac{1}{\epsilon}\left[\left[1-\frac{\epsilon}{2}\right]\left[\frac{\kappa}{\Lambda}\right]^{-\epsilon}-1\right]+\frac{(u+v/3)}{4}(L^2+3L+1)+\frac{u^2M^2\Lambda^{+\epsilon}}{4\kappa^2}L\right\}\\ &+\frac{u^2(u+v/3)}{2}M^2\Lambda^{\epsilon}[(L-1)^2-f+2(L-1)]\ ,\end{aligned} \tag{A5}$$

$$\begin{aligned}\Delta\frac{1}{M}\frac{\partial^2 A}{\partial t\,\partial M} &= \left[u+\frac{v}{3}\right]\left\{-\frac{1}{\epsilon}\left[\left[1-\frac{\epsilon}{2}\right]\left[\frac{\kappa}{\Lambda}\right]^{-\epsilon}-1\right]+\frac{(u+v/3)}{4}(L^2+3L+1)+\frac{u^2M^2\Lambda^{\epsilon}}{4\kappa^2}L\right\}\\ &+\frac{u^2}{2}[(L-1)^2-f+2(L-1)]\ .\end{aligned} \tag{A6}$$

These imply ($n=1$)

$$\beta=-\epsilon u+3u^2-\tfrac{17}{3}u^3\ ,\quad \eta=\tfrac{1}{6}u^2\ ,\quad 2-\frac{1}{\nu(u)}\equiv\gamma_2(u)=u-u^2+\tfrac{1}{6}u^2\ ,\quad B(u)=1\ . \tag{A7}$$

The match points for each of the various quantities are determined by the corresponding series equal to zero.

[1]J. Sak, Phys. Rev. B 10, 3957 (1974).

[2]F. J. Wegner, J. Phys. C 7, 2109 (1974); G. Bender, Z. Phys. B 23, 285 (1976); D. J. Bergman and B. I. Halperin, *ibid.* 13, 2145 (1976).

[3]M. E. Fisher, Phys. Rev. 176, 257 (1968).

[4]L. Gunther, D. J. Bergman, and Y. Imry, Phys. Rev. Lett. 27, 558 (1971); Y. Imry, O. Entin-Wohlman, and D. J. Bergman, J. Phys. C 6, 2846 (1973); O. Entin-Wohlman, D. J. Bergman, and Y. Imry, *ibid.* 7, 496 (1974); J. Rudnick, D. J. Bergman, and Y. Imry, Phys. Lett. 46A, 449 (1974).

[5]J. F. Nicoll, T. S. Chang, and H. E. Stanley, in *Magnetism and Magnetic Materials—1975 (Philadelphia)*, edited by J. J. Becker, G. H. Lander, and J. J. Rhyne, AIP Conf. Proc. No. 29 (AIP, New York, 1975), p. 467.

[6]T. S. Chang, J. F. Nicoll, and H. E. Stanley. Lett. Appl. Eng. Sci. 4, 89 (1976).

[7]J. F. Nicoll, T. S. Chang, and H. E. Stanley, Phys. Rev. A 13, 1251 (1976).

[8]J. F. Nicoll and T. S. Chang, Phys. Rev. A 17, 2083 (1978).

[9]J. Bruno and J. Sak, Phys. Lett. 73A, 343 (1979).

[10]J. Bruno and J. Sak, Phys. Lett. 77A, 46 (1980).

[11]J. Bruno and J. Sak, Phys. Rev. B 22, 3302, 3319 (1980).

[12]D. R. Nelson and J. Rudnick, Phys. Rev. Lett. 35, 178 (1975); J. Rudnick and D. R. Nelson, Phys. Rev. B 13, 2208 (1976); D. R. Nelson, *ibid.* 11, 3504 (1975); D. R. Nelson and E. Domany, *ibid.* 13, 236 (1976).

[13]J. Rudnick, Phys. Rev. B 18, 1406 (1978).

[14]L. Schäfer and H. Horner, Z. Phys. B 29, 251 (1978).

[15]A. D. Bruce and D. J. Wallace, J. Phys. A 9, 1117 (1976).

[16]J. F. Nicoll and J. K. Bhattacharjee, Phys. Rev. B 23, 389 (1981).

[17]I. D. Lawrie, J. Phys. A (in press).

[18]H. H. Iacobson and D. H. Amit (unpublished).

[19]W. K. Theumann, Phys. Rev. B 21, 1930 (1980).

[20]The number f is nonuniversal and may be removed by a change of scale of M relative to t. It does not appear in any amplitude ratio at this order.