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# Derivation of the Landauer conductance formula

David C. Langreth and Elihu Abrahams
*Serin Physics Laboratory, Rutgers University, Piscataway, New Jersey 08854*
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The Landauer formula expresses the conductance of a disordered one-dimensional conductor as the ratio of transmission ($T$) to reflection ($R$) coefficients of scattered waves, $G = (e^2/2\pi\hbar)(T/R)$. We give a rigorous derivation of this formula from linear-response theory (Kubo formula). The generalization to the many-scattering-channel case is given in detail. It is found that only in very special circumstances can the currents in different channels be decoupled in such a way as to give a simple conductance formula.

## I. INTRODUCTION

Some[1-3] recent scaling theories of localization have been based on a formula derived by Landauer[4,1] for the dc conductance $G$ of noninteracting spin-up electrons in a disordered one-dimensional medium:

$$G = (e^2/2\pi\hbar)T/R\,, \tag{1.1}$$

where $T$ and $R$ are the transmission and reflection coefficients of the medium. There has been some question[3,5] as to whether the Landauer result can be obtained from a conventional application of the Kubo formula. In addition, it is of interest to see how to generalize the formula so that it is applicable to real wires: the "many-channel" case.[1,2]

In this paper, we address both these questions. For the one-channel case, we indeed derive the Landauer formula from the linear-response (Kubo) formalism. The calculation is carried out on the basis of the scheme proposed by Economou and Soukoulis[3]: the 1d disordered system, of length $L$ is attached to infinite perfectly conducting leads. This device gives a continuous spectrum and simplifies the zero-frequency limit required for the dc conductance. We show, however, that it is essential to take account of the nature of the leads: it plays a crucial role in the (transient) establishment of the steady state. As the steady state is approached, all time-dependent charge fluctuations must vanish. Thus the current becomes spatially uniform during the final stages of the turning-on process as well as in the steady state. A relevant experiment, and a practically realizable one, may be thought of as driving a given current through the arrangement and measuring the voltage drop across the sample of length $L$ in order to get the conductance.

We have received a communication from Thouless[6] who has also derived the Landauer formula in the single-channel case. His method differs somewhat from ours, but the correct treatment of the field in the leads, as in Sec. III below, is also an essential ingredient of his calculation.

We discuss also the many-channel case. Here we take a rather elementary model in which identical channels are represented by a spinor index. We show in the Appendix that a more physical definition of channels changes neither the physics nor the results. In the perfect conductor leads, the channels do not communicate, while in the sample, they are connected by reflection and transmission amplitudes. We find that only in special cases is a simple generalization[1,2] of the Landauer formula obtainable.

In Sec. II, we generalize the Landauer argument[4] to the many-channel case. We sketch the derivation of the Landauer formula via the Kubo formula in Sec. III where we obtain the one-channel result. It is in this section that the importance of treating the leads correctly is explained. In Sec. IV we give the details of the calculation for the many-channel case. An appendix is devoted to a discussion of how to modify the calculations when channels are defined differently, as, for example, in momentum space as has been done by Fisher and Lee.[5] The final results are the same as those in the text.

## II. MANY-CHANNEL LANDAUER DERIVATION

In the Landauer derivation,[4] one imagines waves incident on the one-dimensional medium. The currents and densities on the left ($L$) and right ($R$) are expressed in terms of transmission and reflection coefficients. By dividing the current by the density gradient, Landauer finds the diffusion constant which is related to the conductance by the Einstein relation.

We generalize the argument to the many-channel model described in Sec. I. We define $P_a^L$, $P_a^R$, as the probability of the incident waves from $L$ and $R$ in channel $a$. Then we may write, for the current $j$ and density $n$ in channel $a$ on the left,

 

$$j_a^L = v_a\Big(P_a^L - \sum_b R_{ab}^L P_b^L - \sum_b T_{ab}^R P_b^R\Big),$$
$$n_a^L = \Big(P_a^L + \sum_b R_{ab}^L P_b^L + \sum_b T_{ab}^R P_b^R\Big), \tag{2.1}$$

where $v_a$ is the speed in channel $a$, $R_{ab}^L$ is the reflection coefficient on the left for the reflection from channel $b$ to $a$, and $T_{ab}^R$ is the transmission probability from channel $b$ on the right to channel $a$ on the left. A similar expression holds on the right. We write it in an obvious matrix notation

$$j^R/v = T^L P^L + (R-1)P^R,$$
$$n^R = T^L P^L + (R+1)P^R\ . \tag{2.2}$$

In these formulas, the probabilities $T, R$ are squares of amplitudes $t, r$ which are elements of a many-channel $S$ matrix as discussed in Ref. 2. Thus $T_{ab}^L = |t_{ab}^L|^2$, etc.,

$$S = \begin{pmatrix} r^L & t^R \\ t^L & r^R \end{pmatrix},$$

and there are a variety of relations among the $r$ and $t$ matrices which are a consequence of unitarity of $S$. In addition, if time-reversal invariance holds,

$$\tilde{r}^{L,R} = r^{L,R}, \quad \tilde{t}^{L,R} = t^{R,L}, \tag{2.3}$$

where the tilde indicates the transpose. We assume time-reversal invariance in the remainder of this section.

It is a simple matter to eliminate the $P$'s from Eqs. (2.1) and (2.2). We find

$$\tfrac{1}{2}(1+R^L)(j^L/v) - \tfrac{1}{2}T^R(j^R/v) = \tfrac{1}{2}(1-R^L)n^L - \tfrac{1}{2}T^R n^R, \tag{2.4}$$

$$\tfrac{1}{2}(1+R^R)(j^R/v) - \tfrac{1}{2}T^L(j^L/v) = \tfrac{1}{2}T^L n^L - \tfrac{1}{2}(1-R^R)n^R. \tag{2.5}$$

We note that these expressions relate the currents and densities in the perfect conductor leads on the $L$ and $R$ of the medium.

The emf is generated entirely by density gradients in this picture,[4] so that the elements of $n^L$ ($n^R$) are proportional to the emf between the left (right) end of the sample and a point where the induced density change vanishes. These emf's are independent of channel by definition of the problem to be solved. Therefore $n^L$ and $n^R$ are each constant spinors, and $n^L - n^R$ is related to the emf $\mathcal{E}$ across the sample by

$$n^L - n^R = \delta n = \frac{dn}{dE}\, e\mathcal{E} = \frac{1}{\pi\hbar v}\, e\mathcal{E}. \tag{2.6}$$

In these arguments, we have assumed a common $v$ and density of states for the identical channels. The assumption is not necessary; see the Appendix. Since the $n$'s are constant, the right-hand sides of Eqs. (2.4) and (2.5) simplify because unitarity and time-reversal invariance imply

$$1 = \sum_b (R_{ba}^{L,R} + T_{ba}^{L,R})$$
$$= \sum_b (R_{ab}^{L,R} + T_{ab}^{R,L})\ .$$

We obtain $\frac{1}{2}T^R(n^L - n^R)$ and $\frac{1}{2}T^L(n^L - n^R)$ for the right-hand sides of the two equations, respectively. We use (2.6) and finally obtain

$$j_a^L + \sum_b R_{ab}^L j_b^L - \sum_b T_{ab}^R j_b^R = \frac{e}{\pi\hbar}\sum_b T_{ab}^R\, \mathcal{E}, \tag{2.7}$$

$$j_a^R + \sum_b R_{ab}^R j_b^L - \sum_b T_{ab}^L j_b^R = \frac{e}{\pi\hbar}\sum_b T_{ab}^L \mathcal{E}\ . \tag{2.8}$$

In the one-channel case, all $R$ and $L$ quantities in Eqs. (2.7) and (2.8) are equal. In particular, as expected,

$$j^R = j^L = G\,\mathcal{E}/e = (e/2\pi\hbar)(T/R)\mathcal{E}\ .$$

Equations (2.7) and (2.8) give the result. In general it is not possible to disentangle the channels to give a simple formula as proposed in Ref. 1. A simple case is obtained when the current is independent of channel. Then $j_a^R = j_a^L = j/N$ where $j$ is the total current and $N$ the number of channels. We add Eqs. (2.7) and (2.8) and use unitarity and time reversal to find

$$G = \frac{ej}{\mathcal{E}} = \frac{e^2}{2\pi\hbar}\,\frac{\mathcal{T}}{1-\mathcal{T}/N}, \tag{2.9}$$

where $\mathcal{T} = \frac{1}{2}\sum_{ab}(T_{ab}^L + T_{ab}^R)$. In the limit of large $N$, this reduces to the formula used in the scaling theory of Ref. 2. Unfortunately, the constant current case is realized if and only if the $T$ matrices have the property

$$\sum_b T_{ba}^L = \sum_b T_{ba}^R = \text{const}. \tag{2.10}$$

Since we are considering a particular disordered configuration we can hardly expect this condition to be satisfied. Formula (2.9) *is* useful for an estimate of the behavior at large $N$. In fact, in the general case, we expect $j_a = O(1/N)$ from which, by adding Eqs. (2.7) and (2.8), we obtain

$$G \underset{N\to\infty}{\sim} (e^2/2\pi\hbar)\mathcal{T}, \tag{2.11}$$

with $\mathcal{T} = O(1)$. In certain cases, however, even (2.11) will fail as, for example, when the diagonal elements of $T$ are much larger than the off-diagonal ones. This would be the case if the dis-

order were translationally invariant perpendicular to the wire and all the channels were independent.

## III. DERIVATION VIA THE KUBO FORMULA: OVERVIEW

Here we derive the results (1.1), (2.7), and (2.8) directly from linear-response theory. Before presenting the details, we give a rough overview of the calculation in the single-channel case. Although the details of the model are not important to the final results, it is helpful to the discussion to imagine an infinitely long filament containing a one-dimensional noninteracting electron gas. The "sample" consists of a disordered region of finite length near the origin, and the "leads" consist of the remaining portions which are assumed uniform, and hence are assumed to offer no resistance to a dc current. An oscillating electromotive force $\mathcal{E}(\omega) \propto e^{-i(\omega+i\eta)t}$ is now turned on adiabatically (perhaps by connecting the ends at infinity and applying a time varying magnetic flux). The current $j(x)$, a function of position $x$ along the filament, is given by

$$j(x) = \int_{-\infty}^{\infty} dx' \, F(x, x') E(x')$$

$$= \int_{\text{sample}} dx' F(x, x') E(x') + \int_{\text{leads}} dx' \, F(x, x') E(x'), \tag{3.1}$$

where $E(x)$ is the electric field ($E \propto e^{-i(\omega+i\eta)t}$). According to standard linear-response theory, the quantity $F(x, x')$ is given in terms of the matrix elements of the current operator and the excitation energies of the system [see Eq. (4.2) below].

In a real wire (or for interacting electrons in the limit of a large number of channels), the electric field $E(x)$ is determined self-consistently. In a single- (or few-) channel case, or for noninteracting electrons where the electrons cannot produce their own mean field, one could, in principle, apply any field whatever as in Refs. 3 and 5. However, most such fields would not result in the required final steady state without time-dependent charge fluctuations, because our leads are such that the electrons in them move ballistically. We apply a space- and time-dependent electric field in the leads during the turning-on process so as to achieve this steady state thus mimicking the situation in a real wire. (Steady state might also be achieved by driving the electrons around the circuit and through the sample many times; this possibility is not considered here because we take the limit of an infinite system from the beginning.) In addition, we require that the final steady-state field be such that there is no induced charge-density difference from one end of the sample to the other, so that $\int E \, dx$ is the complete electromotive force. The above criteria mean that $E(x)$ is not known *a priori* (even in the noninteracting case), and indeed it will depend on the microscopic details of the sample. Fortunately, however, as the steady-state limit ($\omega \to 0$) is approached, one may rigorously say enough about $E$ to calculate the conductance $G$ of the sample. First, the existence of steady state implies by current continuity that $j(x)$ is the same everywhere. Second, we know that in the leads a long way from the sample the current is given by

$$j = \sigma(\omega) E_{\text{leads}} , \tag{3.2}$$

where $\sigma(\omega)$ is the uniform (one-dimensional) electron gas conductivity. Since $\sigma(\omega) \propto i/(\omega + i\eta)$ as $\omega \to 0$, we surmise that the electric field $E$ in the leads goes to zero with $\omega$ in just such a way as to yield a constant product in (3.2). The function $F(x, x')$ has been evaluated by Economou and Soukoulis[3] (Fisher and Lee[5] for the multichannel case) for $\omega \to 0$ *assuming* that both $x$ and $x'$ are finite. In this case it is found that (henceforth our units are such that $e^2 = \hbar = 1$)

$$F(x, x') \to \frac{1}{2\pi} T \tag{3.3}$$

independently of $x$ and $x'$, where $T$ is the transmission probability for a Fermi surface electron or hole to cross the disordered region (sample). Now as $\omega \to 0$, the field outside the sample vanishes, so that the electromotive force $\mathcal{E}$ is just

$$\mathcal{E} = \int_{\text{sample}} E(x) dx . \tag{3.4}$$

Substitution of (3.2), (3.3), and (3.4) in (3.1) gives

$$j = \frac{1}{2\pi} T \mathcal{E} + \left( \frac{1}{\sigma(\omega)} \int_{\text{leads}} dx' F(x, x') \right) j , \tag{3.5}$$

where $\omega \to 0$ is implied. It is tempting to drop the second term in (3.5) because $1/\sigma(\omega)$ goes to zero. This, however, is incorrect because the integral in (3.5) goes over an infinite interval, so that according to (3.3), an indeterminate form would result for the large parentheses. We will show in the next section that as $\omega \to 0$,

$$\int_{\text{leads}} dx' F(x, x') \to T \sigma(\omega) , \tag{3.6}$$

a result almost obvious by inspection. Upon substituting (3.6) in (3.5) and solving for $j$ one finds

$$\frac{j}{\mathcal{E}} \equiv G = \frac{1}{2\pi} \frac{T}{1 - T} = \frac{1}{2\pi} \frac{T}{R} , \tag{3.7}$$

where $R$ is the reflection probability of the sample. This is the result originally derived by Landauer.[4] We see here that it follows rigorously from linear-response theory.

## IV. DERIVATION VIA THE KUBO FORMULA: DETAILS

We now provide the details of the derivation of (3.6) for the multichannel case, and hence provide a derivation of Eqs. (2.7) and (2.8) directly from linear-response theory. We define channels as in Sec. II, and suppose that at each end of the sample each channel is connected to a lead. The leads are identical, one-dimensional electron gases, which are independent of each other, and noninteracting. Thus the current and electric field operators are channel diagonal in the representation defined by the leads. The leads are perfectly insulated from one another, so that at finite frequency, the electric field may be channel dependent, even though there is a common emf applied across all channels. (The left and right leads can be imagined to be connected across a common generator at infinity.) Because of the essential one-dimensional nature we assume for our sample, the channels are all taken to be in electrical contact within the sample, so that $E=E(x)$ independent of channel as the steady state is reached.

The multichannel version of Eq. (3.1) is therefore

$$j_a(x)=\int_{\text{sample}} dx' \sum_b F_{ab}(x,x')E(x') + \int_{\text{leads}} dx' \sum_b F_{ab}(x,x')E_b(x') , \tag{4.1}$$

where

$$F_{ab}(x,x')=\mathfrak{F}_{ab}(x,x';\omega)+\mathfrak{F}^*_{ab}(x,x';-\omega) \tag{4.2a}$$

and

$$\mathfrak{F}_{ab}(x,x';\omega)=\sum_{\alpha\beta}\frac{J^a_{\alpha\beta}(x)J^b_{\beta\alpha}(x')}{i\omega_{\beta\alpha}(\omega_{\beta\alpha}-\omega-i\eta)} . \tag{4.2b}$$

The sum over $\alpha$ and $\beta$ in (4.2) goes over the exact single-particle states of the system (sample plus leads) with the restriction $\epsilon_\beta>\epsilon_F>\epsilon_\alpha$ is the single-particle energy of the $\alpha$th eigenstate, $\epsilon_F$ is the Fermi energy, and $\omega_{\beta\alpha}=\epsilon_\beta-\epsilon_\alpha$. The quantities $J^a_{\beta\alpha}(x)$ are the matrix elements between the states $\beta$ and $\alpha$ of the operator $J^a(x)$ for the current in channel $a$ at point $x$:

$$J^a(x)=\frac{1}{2m}[\rho\delta(x-q)+\delta(x-q)\rho]|a\rangle\langle a| , \tag{4.3}$$

where $p$ is the momentum operator for an electron at position $q$, and $|a\rangle\langle a|$ projects onto the $a$th channel (as defined by the leads).

We choose our eigenstates $\psi_\alpha$ in analogy with scattering theory to have a single incoming component with channel index $a$ and wave vector $\sigma k$ where $k=\sqrt{2m\epsilon_\alpha}>0$ and $\sigma=\pm1$. Thus, $\alpha=\{a,k,\sigma\}$. We choose our normalization such that

$$(\psi_\alpha,\psi_{\alpha'})=2\pi\delta(k-k')\delta_{a,a'}\delta_{\sigma\sigma'} , \tag{4.4}$$

so that

$$\sum_\alpha \psi_\alpha(x)\psi^*_\alpha(x')\equiv\int_0^\infty\frac{dk}{2\pi}\sum_{a\sigma}\psi_{ak\sigma}(x)\psi^*_{ak\sigma}(x')=\delta(x-x') . \tag{4.5}$$

Since the functions $\psi$ have different analytic forms for waves whose incoming parts travel in the $\pm x$ direction, respectively ($\sigma=\pm1$), we let

$$\psi_{a,k,+1}(x)\equiv u_{ak}(x) , \qquad \psi_{a,k,-1}(x)\equiv v_{ak}(x) , \tag{4.6}$$

The functions $u$ and $v$ have simple asymptotic forms outside the sample.[7] In particular, for $x$ to the left,

$$u_{ak}(x)=e^{ikx}|a\rangle+\sum_b r^L_{ba}e^{-ikx}|b\rangle , \tag{4.7a}$$

$$v_{ak}(x)=\sum_b t^R_{ba}e^{-ikx}|b\rangle . \tag{4.7b}$$

Similarly, for $x$ to the right,

$$u_{ak}(x)=\sum_b t^L_{ba}e^{ikx}|b\rangle , \tag{4.8a}$$

$$v_{ak}(x)=e^{-ikx}|a\rangle+\sum_b r^R_{ba}e^{ikx}|b\rangle . \tag{4.8b}$$

The quantities $r^L$ and $t^L$, $r^R$ and $t^R$ have $k$ dependence which we suppress because only Fermi surface values $k=k_F$ will ultimately occur.

If we let $r^L$, $r^R$, $t^L$, $t^R$ be the operators having the matrix elements $r^L_{ba}$, etc., then (4.4) implies (since delta-function contributions can come only from asymptotic regions) that

$$\begin{aligned}&\bar r^L r^L+\bar t^L t^L=1 ,\\ &\bar r^R r^R+\bar t^R t^R=1 ,\\ &\bar r^L t^R+\bar t^L r^R=0 ,\end{aligned} \tag{4.9}$$

and (4.5) implies that

$$\begin{aligned}&r^L\bar r^L+t^R\bar t^R=1 ,\\ &t^L\bar t^L+r^R\bar r^R=1 ,\\ &r^L\bar t^L+t^R\bar r^R=0 ,\end{aligned} \tag{4.10}$$

where a bar over the quantity means the Hermitian adjoint. Equations (4.9) and (4.10) are equivalent to the unitarity of the corresponding $S$ matrix of Sec. II.

Now consider the evaluation of (4.1). The current outside the sample (in the leads) in each channel is separately independent of $x$. (Henceforth, we use the term sample to mean a spatial region sufficiently large that all wave functions have decayed to their asymptotic values; thus, our sample would be larger than the physical sample.) The term leads refers to the asymptotic region. Thus, we take $x$ in Eq. (4.1) to be in the leads. Now the $x'$ dependence of $\sum_b F_{ab}(x,x')$ in the first term in (4.1) involves only matrix elements of the total current density operator which is a constant in space as $\omega \to 0$. Therefore $x'$ in this term may also be taken in the leads.[5] However, in the second term of (4.1), it will generally turn out that the transient $E$ field produced as steady state is approached is dependent on channel so that the sum over $b$ cannot be done as in the first term. Thus, $F_{ab}(x,x')$ in the second term does depend on $x'$; fortunately the integrals here are already in the leads, but $F$ has a value to the left different from its value to the right.

To determine $F_{ab}(x,x')$, the matrix elements of $J$ between the states (4.7) and (4.8) are needed. These matrix elements involve linear combinations of

$$\begin{aligned} &(k+k')e^{i(k-k')x'} \quad \text{(type I)}, \\ &(k-k')e^{i(k+k')x'} \quad \text{(type II)}, \end{aligned} \tag{4.11}$$

plux complex conjugates of these, where $k>0$, $k'>0$. As $\omega \to 0$, all combinations of type II vanish because $k \to k'$. To see this, first consider the first term in (4.1). If $\omega>0$ and $\omega \to 0$, only the real part of the first term of (4.2a) contributes

$$F_{ab} \to \sum_{\alpha\beta} J^a_{\alpha\beta}(x) J^b_{\beta\alpha}(x') \frac{\pi\delta(\omega-\omega_{\beta\alpha})}{\omega_{\beta\alpha}}. \tag{4.12}$$

The $\omega_{\beta\alpha}$ in the denominator cancels a similar term from the Fermi statistics restriction in the $\alpha\beta$ sums. Thus a factor $\delta(\omega_{\beta\alpha}) \propto \delta(k-k')$ multiplies all the the terms in the sum, forcing type-II terms to vanish.

In the second term of (4.1), however, the condition $k \to k'$ arises for a different reason. Here we must look for a term $\propto\omega^{-1}$ in $F$ so that the product $\int FE$ remains finite as $\omega \to 0$. This can only happen if the integral over $x'$ produces a term $\propto\delta(k-k')$ in the infinite asymptotic region; then the remaining $\omega_{\beta\alpha}$ in the denominator of (4.2b) vanishes, leaving the $\omega^{-1}$ dependence necessary. Note that only contributions $\propto\delta(k-k')$ need to be kept. The integral $\int^{\infty} e^{i(k-k')x}$ (or $\int_{-\infty} e^{i(k-k')x}$) produces such a term, where the finite lower (or upper) limit is irrelevant, while an integral like $\int e^{i(k+k')x}$ does not. Therefore, we can again throw away the type-II terms in $J^b_{\beta\alpha}(x')$. Since the integral over $x'$ effectively sets $k=k'$, we may throw away type-II terms in $J^a_{\alpha\beta}(x)$ as well.

Now consider $J^a_{\beta'\beta}$ for $x$ to the left. We denote this by $J^a_{b'b}(\sigma',\sigma;L)$ where we have suppressed the $k$ dependence. We shall often write, instead of $\sigma = \pm 1$, $\sigma = u$ or $v$ according to Eqs. (4.6)–(4.8). For the current in channel $a$ in the leads to the left, we then have, according to (4.1),

$$\begin{aligned} j^L_a = &\sum_{kk'} \frac{1}{\epsilon'-\epsilon}\pi\delta(\omega+\epsilon-\epsilon') \sum_{cc'b\sigma\sigma'} J^a_{cc'}(\sigma,\sigma';L) J^b_{c'c}(\sigma',\sigma;L)\,\mathcal{E} \\ &-\frac{2}{i(\omega+i\eta)} \sum_{kk'} \frac{1}{\epsilon'-\epsilon} \sum_{cc'\sigma\sigma'b} J^a_{cc'}(\sigma,\sigma';L) \int dx'\, J^b_{c'c}(\sigma',\sigma;x') E_b(x'), \end{aligned} \tag{4.13}$$

where the integral in the second term excludes the sample. In the $k$ sums $\epsilon' = \epsilon_{k'}$, $\epsilon = \epsilon_k$, and $\epsilon' > \epsilon_F > \epsilon$. In the second term only those pieces proportional to $\delta(k-k')$ are to be retained and then the limit $\omega \to 0$ is to be taken.

The calculation of the matrix elements is straightforward. As an example consider $J^b_{c'c}(u,u;x')$ for $x'$ to the left,

$$\begin{aligned} J^b_{c'c}(u,u,x') = &-\frac{i}{2m}\langle u_{c'k'}(x')|b\rangle \frac{\partial}{\partial x'}\langle b|u_{ck}(x')\rangle \\ &-\left(\frac{\partial}{\partial x'}\langle u_{c'k'}(x')|b\rangle\right)\langle b|u_{ck}(x')\rangle, \end{aligned} \tag{4.14}$$

where $u$ is given by (4.7a) because $x'$ is to the left. Upon substitution we get (we include only type-I terms)

$$J^b_{c'c}(u,u;x') = \left(\frac{k+k'}{2m}\right)(\delta_{c,c'}\delta_{c,b}e^{i(k-k')x} - \bar{r}^L_{c'b} r^L_{bc'} e^{-i(k-k')x'}). \tag{4.15}$$

No matter where this term is used, either $k=k'$ so that each exponential is unity *or* it is integrated to yield $\pi\delta(k-k')$. All other matrix elements have the same property. Therefore, we may let

$$J^b_{c'c}(\sigma'\sigma;x') = v_F e^{-i(k-k')x'} K^b_{c'c}(\sigma',\sigma;x'), \tag{4.16}$$

where $K = v_F^{-1} J\big|_{k=k'}$. With this simplification (4.13) becomes

$$j^L_a = \frac{1}{\pi}\mathcal{E}\sum_b S^{LL}_{ab} + \sigma(\omega)\sum_b (S^{LL}_{ab}E^L_b + S^{LR}_{ab}E^R_b), \tag{4.17}$$

where we have used the fact that the quantity $v_F e^{i(k-k')x}$ is the value of $J(\sigma,\sigma,x)$ for a single channel in the leads, assuming the sample is not present. We have written $E_b^{R,L}$ for the respective asymptotic values of the field in channel $b$. Therefore, once the summation in the 2nd term of (4.17) is factored out of the second term of (4.13), the remaining factor is just the Kubo formula for the (frequency-dependent) conductivity $\sigma(\omega)$ of one of the (single-channel) leads. The quantities $S_{ab}^{nm}$ are given by

$$S_{ab}^{nm} = \tfrac{1}{4} \sum_{\substack{cc' \\ \sigma\sigma'}} K_{cc'}^{a}(\sigma,\sigma';n)\, K_{c'c}^{b}(\sigma',\sigma;m)\,, \tag{4.18}$$

where $n$ and $m$ each take on the values $L$ and $R$.

The various reduced matrix elements $K$ are easily worked out as in (4.15). On substitution in (4.18) and use of the unitarity relations (4.9) and (4.10) we find that

$$\begin{aligned} S_{ab}^{LL} &= \tfrac{1}{2}(\delta_{ab} - \hat{R}_{ba}^{L})\,, \\ S_{ab}^{LR} &= \tfrac{1}{2}\hat{T}_{ba}^{L}\,, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} \hat{R}_{ab}^{L} &= \tfrac{1}{2}(|r_{ab}^{L}|^2 + |r_{ba}^{L}|^2)\,, \\ \hat{T}_{ab}^{L} &= \tfrac{1}{2}(|t_{ab}^{L}|^2 + |t_{ba}^{R}|^2)\,. \end{aligned} \tag{4.20}$$

Of course if our sample is time-reversal invariant, the two terms in the right-hand side of each equation in (4.20) are equal and $\hat{R}_{ab}^{L}$ ($\hat{T}_{ab}^{L}$) is identical to the reflection (transmission) probability of Sec. II. Finally, since $\sigma(\omega)E_b^{L,R} \to j_b^{L,R}$, we find that substitution of (4.19) in (4.17) yields

$$j_a^L + \sum_b R_{ba}^L j_b^L - \sum_b T_{ba}^L j_b^R = \frac{1}{\pi}\sum_b T_{ba}^L \mathcal{E}\,, \tag{4.21}$$

which is the same as Eq. (2.7), since $R = \tilde{R}$, $T^L = \tilde{T}^R$. We can of course derive the current on the right, by repeating the same steps, or more simply by interchanging the superscripts $R$ and $L$ in (4.20) and (4.21).

We emphasize that Eqs. (2.7), (2.8), and (4.21) are completely general and rigorous. The extra technical details necessary to treat asymptotic channels that differ from one another are given in the Appendix. Thus, our results may be applied to the spatial channels of standard quasi-one-dimensional conductors, as well as to the transverse momentum channels in an ordinary metallic wire.

An apparent paradox occurs in this latter case if the disorder is such that the current in one transverse momentum channel is different from that in another. It would seem from Eq. (4.17) that we would need a momentum-dependent field $E$ in the asymptotic regions to bring the system to steady state. This is actually not a contradiction. Since velocity times density of states *is* independent of channel, differing currents mean differing densities in the different channels (actually density divided by the density of states is what must vary). In other words, the Fermi surface in the (free electron) leads must change from a sphere to another shape moving with the drift velocity (instead of from a sphere to a sphere moving with the drift velocity) during the transient period when the field is turned on. This implies a different density gradient (in the leads) for differing transverse momentum states during the time when the emf is being turned on. In linear-response theory such density gradients cannot be treated directly as driving forces in the Hamiltonian, and are usually handled by replacing them by equivalent fictitious fields. Our momentum-dependent fields should be thought of in this sense.

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## APPENDIX: DIFFERING ASYMPTOTIC CHANNELS

We consider briefly here the case in which the channels of the leads are different from each other, in the sense of having different Fermi velocities $v_a$, different Fermi wave vectors $k_a$, and therefore different densities of states $\rho_a$, where $\rho_a = (2\pi v_a)^{-1}$. This in no way changes the results, but the derivations are more cumbersome because of the necessity of keeping track of these factors. We employ here a method used by one of us for a similar problem in surface physics.[8]

Because the energy shell contains a differing number states for each channel, it is best to talk of transmission and reflection amplitudes between states of differing $k$ but nearby energy. These are (see Ref. 8)

$$\mathcal{R}_{a'k',ak}^{L} \equiv r_{a'a}^{L}\frac{\delta(\epsilon_{k'}^{a'} - \epsilon_{k}^{a})}{(\rho_a \rho_{a'})^{1/2}}$$

and (A1)

$$\mathcal{T}_{a'k',ak}^{L} \equiv t_{a'a}^{L}\frac{\delta(\epsilon_{k'}^{a'} - \epsilon_{k}^{a})}{(\rho_a \rho_{a'})^{1/2}}\,,$$

where (for example) $2\pi\rho_a = \int_0^\infty dk\, \delta(\epsilon_F - \epsilon_k^a) = (2\pi v_a)^{-1}$. Similar expressions hold for $\mathcal{R}^R$ and $\mathcal{T}^R$. The wave functions $u$ and $v$ have the asymptotic forms as in (4.7). Thus, for $x$ far on the left,

$$u_{ak}(x) = e^{ikx}|a\rangle + \sum_{a'}\int \frac{dk'}{2\pi} \mathcal{R}^L_{a'k',ak} e^{-ik'x}|a'\rangle\,, \quad \text{(A2a)}$$

$$v_{ak}(x) = \sum_{a'}\int \frac{dk'}{2\pi} \mathcal{T}^R_{a'k',ak} e^{-ik'x}|a'\rangle\,. \quad \text{(A2b)}$$

Expressions similar to (4.8) hold for $x \gg 0$. The orthonormality and completeness of these wave functions requires that the matrices $\mathcal{R}$ and $\mathcal{T}$ satisfy unitarity relations like (4.9) and (4.10) on the complete Hilbert space, which in turn implies that $r$ and $t$ satisfy (4.9) and (4.10) in the *reduced* space consisting of only the channel indices (with $k = k' = k_F$). By evaluating the $k'$ integrals, one can see that (for example) $|r^L_{ab}|^2$ is the conventionally defined reflection probability (on the left) from state $b$ to state $a$, that is, the ratio of the outgoing current in channel $a$ to the incoming current in channel $b$. Similar relations hold for the other quantities. One also notes that the quantity $(2\pi i)^{-1}(\rho_a\rho_b)^{-1/2} r^L_{ba}$ is the transition matrix element from a state in channel $a$ (going to the right) to another state in channel $b$ (going to the left) with both states on the energy shell. Therefore, according to the golden rule, the rate at which an initial particle in channel $a$ is reflected to the energy-shell states of channel $b$ is

$$W_{ba}[r^L] \equiv 2\pi \left| \frac{r^L_{ba}}{2\pi i(\rho_a\rho_b)^{1/2}} \right|^2 \rho_b\,. \quad \text{(A3)}$$

Similar relations hold for $r^R$, $t^L$, and $t^R$.

We now use the generalized Landauer method to derive equations (2.7), (2.8), and (4.21). Assume time-reversal invariance to simplify the discussion. Let $Q^n_a$ be the probability per unit energy of an extra incident particle in the $n$th side of the sample ($n = R$ or $n = L$) in channel $a$. Then we may write

$$j^L_a = Q^L_a \rho_a v_a - \sum_b W_{ab}[r^L] Q^L_b \rho_b - \sum_b W_{ab}[t^R] Q^R_b \rho_b\,. \quad \text{(A4)}$$

We substitute (A3) and obtain

$$j^L_a = \frac{Q^L_a}{2\pi} - \sum_b R^L_{ab} \frac{Q^L_b}{2\pi} - \sum_b T^R_{ab} \frac{Q^R_b}{2\pi}\,. \quad \text{(A5)}$$

Similarly, the extra density in channel $a$ to the left is

$$n^L_a = Q^L_a \rho_a + v_a^{-1} \sum_b W_{ab}[r^L] Q^L_b \rho_b + v_a^{-1} \sum_b W_{ab}[t^R] Q^R_b \rho_b\,. \quad \text{(A6)}$$

If we let $\mathcal{E}^L_a$ be the voltage in channel $a$ at the left, then $\mathcal{E}^L_a = n^L_a/\rho_a$ or

$$\frac{\mathcal{E}^L_a}{2\pi} = \frac{Q^L_a}{2\pi} + \sum_b R^L_{ab} \frac{Q^L_b}{2\pi} + \sum_b T^R_{ab} \frac{Q^R_b}{2\pi}\,. \quad \text{(A7)}$$

Equations (A5) and (A7) are completely equivalent to Eqs. (2.1) with $P$ replaced by $Q/2\pi$. When combined with their analogs for the right of the sample, they can be solved as in Sec. II, eliminating $Q$, and Eqs. (2.7) and (2.8) are again obtained. These equations are now proved to hold for channels that have different Fermi wave vectors, Fermi velocities, and densities of states.

It is also a simple matter to repeat the steps of Sec. IV, with wave functions like those of (A2) instead of (4.7). Of course, all the extra factors from differing state densities all cancel out, and hence we still obtain Eqs. (2.7) and (2.8) from the Kubo formula as well.

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